Marx's Theory of Value and the Transformation Problem (1977)
in *The Subtle Anatomy of Capitalism,* Jesse Schwartz (ed.), Goodyear Publishing Co.

Mathematical Appendix

I. Use-Value

In a fully-integrated factory system, where production is continuous, at any moment of time all the different stages of the production process - from the initial transformation of the raw materials to the finishing touches on the completed product - will be in operation simultaneously.

Consider the k^{th} industry, in which a complement of N_k workers produce $X_k^{\oplus}$ units of output each <u>hour</u>, using up ($X_{1k}^{\oplus}, \ldots, X_{nk}^{\oplus}$) units of material inputs over the same period of time.*

Let h = length of the working day. Then in one <u>day</u>, N_k workers will produce $X_k = X_k^{\oplus} \cdot h$ units of output and use up $(X_{1k}^{\oplus}, \ldots, X_{nk}^{\oplus}) = (X_{1k}, \ldots, X_{nk}) \cdot h$ units of input.

From the point of view of the whole society, therefore, we may define the following daily quantities (vectors and matrices are in bold type):

I.1A $\quad \mathbf{N} = (N_1, \ldots, N_n)$ = vector of industry employment (number of workers) per day

I.1B $\quad \mathbf{L} = \mathbf{N} \cdot h$ = vector of number of hours worked in each industry, per day

I.2

$$\mathbf{A}(h) = \mathbf{A}^{\oplus} \cdot h = \begin{bmatrix} x_{11}^{\oplus} & \cdots & x_{1k}^{\oplus} & \cdots & x_{1n}^{\oplus} \\ \vdots & & \vdots & & \\ x_{n1}^{\oplus} & \cdots & x_{nk}^{\oplus} & \cdots & x_{nn}^{\oplus} \end{bmatrix}$$

The k^{th} column of A represents the hourly input use of the k^{th} industry, while that of $\mathbf{A}(h)$ represents the corresponding daily use.**

Now let $\mathbf{y}_S$ be the column vector of daily subsistence requirements of the average worker. Then the k^{th} industry, which employs N_k workers per day, will necessitate $\mathbf{y}_S N_k$ commodities in support of its work force. It follows that

I.3 $\quad \mathbf{y}_S\mathbf{N}$ = matrix whose k^{th} column represents the daily subsistence requirements for the work force of the k^{th} industry.***

From the above we may define the matrix $\mathbf{Q}(h)$ as the matrix whose columns represent the overall commodity requirements of each industry, either directly as its means of production or indirectly as means of subsistence of its work force.

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* Properly speaking, N_k workers operate a complement of instruments of production (machines, buildings, etc.), in each hour of whose operation they use up raw materials $(X_{1k}^{\oplus}, \ldots, X_{nk}^{\oplus})$, and produce output X_k . If the system is fully "balanced", however, the number of instruments of production which "die" will be the same from one hour to the next. Consequently, we may consider them as part of the overall vector of material inputs. This is merely a device to avoid completely the problems of turnover, since space does not permit the luxury of this distinction.

** If there are some commodities which do not enter into the production of <u>any</u> commodity, the corresponding rows of $\mathbf{A}(h)$ will contain all zeros.

***For commodities not consumed by workers, the corresponding rows of $\mathbf{y}_S\mathbf{N}$ will be null rows.

I.4 $\mathbf{Q}(h) = \mathbf{A}(h) + \mathbf{y}_S\mathbf{N}$

If there are commodities which serve neither as means of production nor as means of subsistence, the corresponding rows of $\mathbf{Q}$ will be null rows. Following Sraffa, we will denote as "basic" commodities which do enter into $\mathbf{Q}$, and as "non-basic", those which do not.[48] The null rows thus correspond to the non-basic commodities.

Lastly, since $X_k^{\oplus}$ is the hourly output in the k^{th} industry, the daily output is $X_k(h) = X_k^{\oplus} \cdot h$. Denoting the diagonal matrix whose elements are the daily industry outputs as $\langle X_j \rangle$,

I.5 $$\langle X_j(h) \rangle = \begin{bmatrix} X_1 & \cdots & 0 \\ 0 & X_2 \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & X_n \end{bmatrix} = \langle X_j^{\oplus} \rangle h.$$

From this we may construct

I.6 $$\mathbf{q}(h) = \langle X_j(h) \rangle^{-1} \mathbf{Q}(h) = \langle X_j(h) \rangle^{-1} [\mathbf{A}(h) + \mathbf{y}_S\mathbf{N}],$$

in which each element in $q_{ij}(h) = Q_{ij}(h)/X_j(h)$ represents the fraction of industry I's daily output which is required each day by industry J's operation, either as means of production or as means of subsistence.[49] It should be noted here that these fractional requirements are invesely related to the length of the working day.*

Each row sum of $\mathbf{q}(h)$ represents the total fraction of the corresponding commodity output which must be used up either as means of production or means of subsistence, per day, for the system as a whole. Consequently, the properties of $\mathbf{q}(h)$ are intimately related to the capability of the system to produce surplus - products. Though we do not have the space to derive it here, it can be proven that if the system is capable of producing any surplus-product at all (i.e., if the row sums of $\mathbf{q}(h) \leq 1$, with at least one row sum strictly less than one), then the matrix $\mathbf{q}(h)$ has a positive dominant characteristic root less than unity, and a strictly positive associated characteristic vector which is its only non-negative characteristic vector.**

II. Value

In one day, in the k^{th} industry, N_k workers put in $L_k = N_k \cdot h$ hours of work, producing $X_k = X_k^{\oplus} \cdot h$ units of output, and using up $(X_{k1}, \ldots, X_{kn}) = (X_{k1}^{\oplus}, \ldots, X_{kn}^{\oplus}) \cdot h$ units of inputs. If $\lambda_1, \ldots, \lambda_n$ are the unit values of commodities, then the value of the daily product will be the value added by living labor (L_k) plus the Value transformed:

II.1 $$\lambda_k X_k^{\oplus} \cdot h = N_k \cdot h + \sum_{i=1}^{n} \lambda_i X_{ik}^{\oplus} \cdot h$$

Dividing through by $X_k^{\oplus} \cdot h$, we get

$$\lambda_k = \frac{N_k \cdot h}{X_k^{\oplus} \cdot h} + \sum_{i=1}^{n} \lambda_i \frac{X_{ik}}{X_k}$$

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* In expanding out I.6, the first term will be independent of h, but the second term will be inversely related to h.

** This assumes that the "basic" square submatrix of q(h) is indecomposable. "Basic" here includes all commodities that function either as means of production or means of subsistencs.

The first term in the right-hand side represents the number of hours of abstract labor-time added to each unit of the daily product in industry K. We designate this by ℓ_K, noting that it does not depend on the length of the working day.

In the second term of the above expression, we find the familiar input-output coefficient, $X_{ij}^{\oplus}/X_j^{\oplus}$, which we designate by a_{ij}. Since both numerator and denominator are defined over the same period of time, this ratio too is independent of the length of the working day.

In matrix terms, we can define row vectors $\boldsymbol{\lambda} = (\lambda_j)$, $\boldsymbol{\ell} = (\ell_j)$, and the n×n matrix $\mathbf{a} = (a_{ij})$, where i = 1,....,n and j = 1,....,n. Then the vector of unit Values is*

II.2 $\boldsymbol{\lambda} = \boldsymbol{\ell} + \boldsymbol{\lambda}\mathbf{a}$

Since neither ℓ_j nor a_{ij} depend on the length of the working day, it follows that unit Values λ_j are independent of the length of the working day.

Designating the vector of total Values by $\mathbf{W}$, we have

$$\mathbf{W} = \boldsymbol{\lambda}\langle X_j\rangle = \boldsymbol{\ell}\langle X_j\rangle + \boldsymbol{\lambda}\mathbf{a}\langle X_j\rangle$$

The term $\boldsymbol{\ell}\langle X_j\rangle = \mathbf{L}$, the vector of hours worked in each industry per day, hence the vector of Values added by living labor in the various indistries, per day. Since $\mathbf{a}\langle X_j\rangle = \mathbf{A}$, the matrix of total daily input uses, the term $\boldsymbol{\lambda}\mathbf{a}\langle X_j\rangle = \mathbf{C}$, the vector of the Values of material inputs used up each day by the various industries.

Consequently,

II.3 $\mathbf{W} = \mathbf{L} + \mathbf{C}$, where $\mathbf{C} = \boldsymbol{\lambda}\mathbf{A}$.

Assuming a uniform wage rate, as indicated by the subsistence basket $\mathbf{y}_S$, the necessary part of the working day will be the same for each worker: it will be the Value of the subsistence basket

II.4 $\eta = \boldsymbol{\lambda}\mathbf{y}_S$ = necessary part of the working day

Assuming a uniform working day of length h, it follows that the surplus part of each working day is $h-\eta$. The uniform rate of surplus-Value will therefore be $S/V = (h-\eta)/\eta = h/\eta - 1$.

Lastly, since $\mathbf{N}$ is the vector of number of workers employed in the various industries, per day, and $\mathbf{L} = \mathbf{N}h$ is the vector of Values added,

II.5 $\mathbf{V} = \eta\mathbf{N}$ = vector of Values of labor-power employed per day in each industry, and

$\mathbf{S} = \mathbf{L} - \mathbf{V} = (h-\eta)\mathbf{N}$ = vector of industry surplus-Values, per day.

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* The matrix $\mathbf{a}$ is the familiar Leontief matrix of input-output coefficients. But the vector $\boldsymbol{\ell}$ is not the same as the Leontief vector $\mathbf{a}_0$. The units of ℓ_j are worker-hours per unit of output, whereas those of a_{0j} are workers per unit output. Short-run changes in the length of the working day (overtime) and long-run changes (shartening of the working day due to class struggle) make it important to distinguish the two measures.

III. Direct Prices

By definition, direct prices are the Values of commodities relative to the unit Values of the money commodity (gold). Designating the unit Value of gold by λ_g, and using the superscript "o" for money quantities reflecting direct prices, the vector of total direct prices is

$$\text{III.1} \qquad \mathbf{P}^o = \frac{1}{\lambda_g} \cdot \mathbf{W}$$

Similarly, vector of money cost-prices is *

$$\mathbf{C}^o + \mathbf{V}^o = \frac{1}{\lambda_g} (\mathbf{C} + \mathbf{V})$$

But $\mathbf{C} = \boldsymbol{\lambda}\mathbf{A}$ (from II.3), and $\mathbf{V} = \eta\mathbf{N} = \boldsymbol{\lambda}\mathbf{y}_S\mathbf{N}$ (from II.5 and II.4, respectively), so that

$$\mathbf{C}^o + \mathbf{V}^o = \frac{1}{\lambda_g} (\boldsymbol{\lambda}\mathbf{A} + \boldsymbol{\lambda}\mathbf{y}_S\mathbf{N}) = \frac{1}{\lambda_g} \boldsymbol{\lambda} (\mathbf{A} + \mathbf{y}_S\mathbf{N})$$

Since $\mathbf{W} = \boldsymbol{\lambda}\langle X_j \rangle$, we can also write

$$\mathbf{C}^o + \mathbf{V}^o = \frac{1}{\lambda_g} \cdot \mathbf{W} \left[\langle X_j \rangle^{-1} (\mathbf{A} + \mathbf{y}_S\mathbf{N}) \right]$$

The term in the square brackets is of course $\mathbf{q}(h)$ (see I.6), while $\frac{1}{\lambda_g}\mathbf{W} = \mathbf{P}^o$ (from III.1 above). Thus

$$\text{III.2} \qquad \mathbf{C}^o + \mathbf{V}^o = \mathbf{P}^o\mathbf{q}(h)$$

In any industry, profit is that amount of money which is left over after cost-prices are deducted from sales. The vector of direct profits is therefore

$$\text{III.3} \qquad \mathbf{S}^o = \mathbf{P}^o - (\mathbf{C}^o + \mathbf{V}^o) = \mathbf{P}^o\left[\mathbf{I} - \mathbf{q}(h)\right]$$

Obviously, direct profits will be also proportional to surplus-Values: $\mathbf{S}^o = \frac{1}{\lambda_g} \cdot \mathbf{S}$. Substituting $(h - \eta)\mathbf{N}$ for $\mathbf{S}$, from II.5, and solving for $\mathbf{P}^o$, we can write

$$\text{III.4} \qquad \mathbf{P}^o = \frac{1}{\lambda_g}(h - \eta)\mathbf{N}\left[\mathbf{I} - \mathbf{q}(h)\right]^{-1}$$

Lastly, the sum of direct profits will be $S^o = \mathbf{S}^o\mathbf{1}$, while the sum of cost-prices will be $C^o + V^o = (\mathbf{C}^o + \mathbf{V}^o)\mathbf{1}$, where $\mathbf{1}$ is the unit (column) vector. Since the rate of money profit is $r^o = S^o/(C^o + V^o)$, we can utilize III.2 and III.4 to write

$$\text{III.5} \qquad (1 + r^o) = \frac{(\mathbf{C}^o + \mathbf{V}^o)\mathbf{1} + \mathbf{S}^o\mathbf{1}}{(\mathbf{C}^o + \mathbf{V}^o)\mathbf{1}} = \frac{\mathbf{P}^o\mathbf{1}}{\mathbf{P}^o\mathbf{q}(h)\mathbf{1}}$$

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* It will be noted that I have used Marx's notation M' in the text to refer to total money prices, whereas in this appendix I have used P. This is simply to avoid having two sets of superscripts, such as $(M')^o$ for direct prices, $(M')^*$ for prices of production, etc.

IV. Prices of Production

Consider a vector of arbitrary unit prices $\mathbf{p}'$. Then the vector of material costs will be $\mathbf{p}'\mathbf{A}$, the money wage rate will be $\mathbf{p}'\mathbf{y}_S$, and the vector of labor costs $\mathbf{p}'\mathbf{y}_S\mathbf{N}$. The vector of cosr-prices corresponding to this arbitrary price vector will be therefore

$$\mathbf{C}' + \mathbf{V}' = \mathbf{p}'\mathbf{A} + \mathbf{p}'\mathbf{y}_S\mathbf{N} = \mathbf{p}'\mathbf{Q}(h)$$

Since total prices $\mathbf{P}' = \mathbf{p}'\langle X_j \rangle$, and $\mathbf{q}(h) = \langle X_j \rangle^{-1}\mathbf{Q}(h)$,

IV.1 $\quad \mathbf{C}' + \mathbf{V}' = \mathbf{P}'\mathbf{q}(h)$

Money profits, as always, are the difference between sales and cost-prices.

IV.2 $\quad \mathbf{S}' = \mathbf{P}' - (\mathbf{C}' + \mathbf{V}') = \mathbf{P}'\left[\mathbf{I} - \mathbf{q}(h)\right]$

We now consider a very special set of prices, ones such that profits in each industry are proportional to their respective cost-prices.* Let us denote these special prices by p* (the asterisk here does not indicate a footnote), and the constant of propor - tionality — which of course in the uniform rate of profit — by r*. Then, the vector of money profits corresponding to $\mathbf{p}^*$ must satisfy not only IV.2, but also the condition of proportionality

IV.3 $\quad \mathbf{S}^* = \mathbf{P}^* - (\mathbf{C}^* + \mathbf{V}^*) = r^*(\mathbf{C}^* + \mathbf{V}^*)$

Since IV.1 applies to any price vector, it applies here too, so that we can solve the above for P*

IV.4 $\quad \mathbf{P}^* = (1+r^*)\mathbf{P}^*\mathbf{q}(h)$

The above expression tells us that $\mathbf{P}^*$, the vector of total prices of production, is a characteristic vector of the matrix $\mathbf{q}(h)$, and that $\left(\frac{1}{1+r^*}\right)$ is a characteristic root.

It is at this point that the properties of $\mathbf{q}(h)$ alluded to in section I of this Appendix come into play, since they establish that $\mathbf{q}(h)$ has a positive dominant root less than unity and a unique stric-strictly positive characteristic vector corresponding to the dominant root. In terms of the above, this means $\mathbf{P}^*$ will be strictly positive, and $r^* > 0$ (since $0 < \frac{1}{1+r^*} < 1$)

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* We have assumed equal periods of turnover, so that the magnitudes of total capital advanced equals that of total capital used up (cost-price), in each industry. But the units differ, the first being a stock and the second a flow. Properly speaking, even when the two are equal in magnitude, profit relative to capital advanced is the rate of profit, and profit relative to cost-price is the profit margin on costs.

V. The Iteration Procedure

It is well known that a characteristic equation of the type in IV.4 can be solved iteratively for $\mathbf{P}^*$ and r^*, beginning from some arbitrary initial vector and scalar.

In this iteration procedure, however, we begin not from an arbitrary point, but from the vector of direct prices $\mathbf{P}^o$ and the direct rate of profit r^o. These are non-arbitrary starting points in two senses: first, from the point of view of the <u>economic</u> content, Values are fundamentals to the analysis and direct prices represent their direct money expressions; secondly, there is a mathematical relation between direct prices and prices of production,[50] since

$$\mathbf{P}^o = (h-\eta)\mathbf{N}\left[\mathbf{I} - \mathbf{q}(h)\right]^{-1} \qquad \text{(from III.4)},$$

whereas

$$\mathbf{P}^*\left[\mathbf{I} - (1+r^*)\mathbf{q}(h)\right] = 0 \qquad \text{(from IV.4)}$$

There are dimensions to this relation which are beyond the sc scope of this paper. Here, we confine ourselves to the <u>transformation</u> of the direct form of Value into its price of production form.

Marx begins this transformation by forming a set of prices defined as direct cost-prices $\mathbf{C}^o + \mathbf{V}^o$ plus an amount of profit = ~~the~~ average direct rate profit r_o times the cost-prices. That is, Marx's total prices of production (step 1B in Table 4 of this paper) are defined by

$$\text{(V.1)} \quad \mathbf{P}^{(1)} = (\mathbf{C}^o + \mathbf{V}^o)(1 + r^o)$$

Rewriting $\mathbf{C}^o + \mathbf{V}^o$ in terms of III.2 and $(1+r^o)$ in terms of III.5,

$$\text{(V.2)} \quad \mathbf{P}^{(1)} = \mathbf{P}^o\mathbf{q}(h)\frac{\mathbf{P}^o\mathbf{1}}{\mathbf{P}^o\mathbf{q}(h)\mathbf{1}}$$

This procedure automatically keeps the sum of prices constant, since

$$\text{(V.3)} \quad \mathbf{P}^{(1)}\mathbf{1} = \mathbf{P}^o\mathbf{q}(h)\mathbf{1}\frac{\mathbf{P}^o\mathbf{1}}{\mathbf{P}^o\mathbf{q}(h)\mathbf{1}} = \mathbf{P}^o\mathbf{1}$$

The prices of production $P_j^{(1)}$, as we have already noted, will in general differ from direct prices P_j^o. In step 2A of Table 4 where we apply the multipliers $\Psi_j \equiv P_j^{(1)}/P_j^o$, $(j = 1,\dots,n)$ to the cost-prices $\mathbf{C}^o + \mathbf{V}^o$, we in effect replace all total direct prices $\mathbf{P}_j^o$ by new total prices $\mathbf{P}_j^{(1)}$. The vector of cost prices $\mathbf{C}^o + \mathbf{V}^o$ is thus transformed into a new set of cost prices $\mathbf{C}^{(1)} + \mathbf{V}^{(1)}$:

$$\mathbf{C}^o + \mathbf{V}^o = \mathbf{P}^o\mathbf{q}(h) \longrightarrow \mathbf{C}^{(1)} + \mathbf{V}^{(1)} = \mathbf{P}^{(1)}\mathbf{q}(h)$$

But $\mathbf{P}^{(1)}$ is itself related to $\mathbf{C}^o + \mathbf{V}^o$ from VI.2:*

$$\text{(V.4)} \quad \mathbf{C}^{(1)} + \mathbf{V}^{(1)} = \mathbf{P}^{(1)}\mathbf{q} = \mathbf{P}^o\mathbf{q}^2\frac{\mathbf{P}^o\mathbf{1}}{\mathbf{P}^o\mathbf{q}\mathbf{1}}$$

Similarly, the new money rate of profit $r^{(1)}$ is now

$$1 + r^{(1)} = \frac{\mathbf{P}^{(1)}\mathbf{1}}{\mathbf{P}^{(1)}\mathbf{q}\mathbf{1}}$$

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* For the sake of convenience in notation, we will henceforth write $\mathbf{q}(h)$ as simply $\mathbf{q}$.

Since this procedure automatically implies a constant sum of prices (see (V.3) above), $\mathbf{p}^{(1)}\mathbf{1}=\mathbf{p}^{0}\mathbf{1}$; substituting for $\mathbf{p}^{(1)}$ in the denominator, we get

(V.5) $$1 + r^{(1)} = \frac{\mathbf{p}^{0}\mathbf{1}}{\mathbf{p}^{0}\mathbf{q}\,\mathbf{1}}$$

In step 2B we apply this new rate of profit to the new cost-prices of (V.4) to get new money prices of production

(V.6) $$\mathbf{p}^{(2)} = (\mathbf{c}^{(1)} + \mathbf{v}^{(1)})(1 + r^{(1)}) = \mathbf{p}^{0}\mathbf{q}^{2}\frac{\mathbf{p}^{0}\mathbf{1}}{\mathbf{p}^{0}\mathbf{q}^{2}\mathbf{1}}$$

Each successive step repeats the patterns of 1A-B, 2A-B. It is therefore obvious that in general the T^{th} price iterate will be

(V.7) $$\mathbf{p}^{(T)} = \mathbf{p}^{0}\mathbf{q}^{T}\frac{\mathbf{p}^{0}\mathbf{1}}{\mathbf{p}^{0}\mathbf{q}^{T}\mathbf{1}}$$

We turn now to the issue of convergence.

Matrix $\mathbf{q}$ is a semipositive matrix with the dominant characteristic root $1/(1+r^*) < 1$ and a strictly positive principal characteristic vector $\mathbf{p}^*$. In general, it is possible to assume that the n characteristic vectors of matrix $\mathbf{q}$ are linearly independent.* Denoting the characteristic roots by m_k, and the characteristic vectors by $\mathbf{U}_k$, $k= 1,\ldots,n$, and ordering the roots according to their absolute values (so that m_1 is the dominant root and $\mathbf{U}_1$ the principal vector, we can write

(V.8) $$|m_1| > |m_2| > \cdots > |m_n|$$

$$m_1 = \left(\frac{1}{1+r^*}\right), \quad \mathbf{U}_1=\mathbf{p}^*$$

Since the characteristic vectors are linearly independent, they form a basis in Euclidean n-space, so that we can always express $\mathbf{p}^{0}$ as a linear combination of these vectors, for some set of constants α_k, $k=1,\ldots,n$,

(V.9) $$\mathbf{p}^{0} = \sum_{k=1}^{n} \alpha_k \mathbf{U}_k$$

On the strength of this we can write the T^{th} iterate $\mathbf{p}^{(T)}$ (of equation V.7) as

$$\mathbf{p}^{(T)} = \mathbf{p}^{0}\mathbf{q}^{T}\frac{\mathbf{p}^{0}\mathbf{1}}{\mathbf{p}^{0}\mathbf{q}^{T}\mathbf{1}} = \frac{\sum_{k=1}^{n} \alpha_k(\mathbf{U}_k\mathbf{q}^{T})}{\sum_{k=1}^{n} \alpha_k(\mathbf{U}_k\mathbf{q}^{T})\mathbf{1}}\ \mathbf{p}^{0}\mathbf{1}$$

From the definition of the k^{th} characteristic vector $\mathbf{U}_k$, $\mathbf{U}_k\mathbf{q} = m_k\mathbf{U}_k$, where m_k is the k^{th} characteristic root. Post multiplying by $\mathbf{q}$ gives us $\mathbf{U}_k\mathbf{q}^2 = m_k(\mathbf{U}_k\mathbf{q}) = m_k \cdot m_k\mathbf{q} = m_k^2\mathbf{q}$, and post-multiplying again by $\mathbf{q}$ gives us $\mathbf{U}_k\mathbf{q}^3 = m_k^3\mathbf{U}_k$, etc. Thus in general

$$\mathbf{U}_k\mathbf{q}^{T} = m_k^{T}\mathbf{U}_k, \qquad k=1,\ldots,n.$$

The expression for $\mathbf{p}^{(T)}$ can therefore be written as

$$\mathbf{p}^{(T)} = \frac{\alpha_1 m_1^{T}\mathbf{U}_1 + \alpha_2 m_2^{T}\mathbf{U}_2 + \cdots + \alpha_n m_n^{T}\mathbf{U}_n}{[\alpha_1 m_1^{T}\mathbf{U}_1 + \alpha_2 m_2^{T}\mathbf{U}_2 + \cdots + \alpha_n m_n^{T}\mathbf{U}_n]\mathbf{1}}\ \mathbf{p}^{0}\mathbf{1}$$

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* Lancaster (6), Section R5.5, p.287.

Let us now divide the numerator and denominator of the above expression by m_1^T :

$$P^{(T)} = \frac{\alpha_1 U_1 + \alpha_2 \left(\frac{m_2}{m_1}\right)^T U_2 + \dots + \alpha_n \left(\frac{m_n}{m_1}\right)^T U_n}{\left[\alpha_1 U_1 + \alpha_2 \left(\frac{m_2}{m_1}\right)^T U_2 + \dots + \alpha_n \left(\frac{m_n}{m_1}\right)^T U_n\right] 1} \; P^{\circ} 1$$

Since m_1 is the dominant root, $\left|\frac{m_k}{m_1}\right| < 1$ for all $2 \leq k \leq n$. Thus as we continue to iterate, i.e. as $T \to \infty$, $\left|\frac{m_k}{m_1}\right|^T \longrightarrow 0$ for all $2 \leq k \leq n$. In the limit, therefore, as $T \to \infty$

$$\text{(V.10)} \qquad \mathbf{P}^{(T)} \longrightarrow \mathbf{P}^{(\infty)} = \frac{\alpha_1 \mathbf{U}_1}{\alpha_1 \mathbf{U}_1 \mathbf{1}} \mathbf{P}^{\circ} \mathbf{1} = \mathbf{U}_1 \frac{\mathbf{P}^{\circ} \mathbf{1}}{\mathbf{U}_1 \mathbf{1}}$$

But $\mathbf{U}_1$, the principal characteristic vector of $\mathbf{q}$ is in reality $\mathbf{P}^*$, the vector of total prices of production (see V.8). Moreover, since the sum of prices is not affected by this transformation in the form of Value, $\mathbf{P}^* \mathbf{1} = \mathbf{P}^{\circ} \mathbf{1}$, so that as $T \longrightarrow \infty$

$$\text{(V.11)} \qquad \mathbf{P}^{(T)} \longrightarrow \mathbf{P}^{(\infty)} = \mathbf{P}^* \frac{\mathbf{P}^{\circ} \mathbf{1}}{\mathbf{P}^* \mathbf{1}} = \mathbf{P}^*$$

That is to say, the iteration procedure <u>in which Marx's transformation from direct prices to prices of production appears as the first step</u>, will converge on the "correct" prices of production $\mathbf{P}^*$.