Nonlinear Dynamics and Pseudo-Production Functions

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I. Introduction*

The aggregate production function (APF) is a fundamental neoclassical construct. It is widely used at a theoretical level, in micro- and macroeconomics, in growth theory, in labor economics, in international trade theory, indeed, in virtually every branch of economic analysis (Solow, 1987, p. 15; Felipe and Adams, 2001, pp. 2-3). It is equally ubiquitous at the empirical level, where it is used to analyze the determinants of growth, technical change, productivity growth, capacity utilization, and so on. Almost half a century after Solow's celebrated article (Solow, 1957), it still remains *the* method of accounting for the determinants of national and international growth

And yet it has been known for a long time that the theoretical foundations of this construct are extraordinarily shaky, because it simply cannot be grounded in any plausible micro-foundations (Samuelson 1962, 1966, 1979; Garegnani 1970; Fisher 1971a, 1971b, 1987, 1992; Harcourt, 1972, 1976, 1994; Solow 1987, p. 25; McCombie 2000-2001, p. 268; Felipe and Holz 1999, Felipe and Adams 2001). It is curious, therefore, that a tradition which insists on the necessity of micro-foundations should also continue to rely so heavily on a construction which cannot be derived from micro-foundations. How is this justified?

Defenders claim that aggregate production functions are worth retaining because they possess important virtues in themselves, and because they appear to 'work' at an empirical level. Paul Douglas, the very originator of this tradition, expresses this sentiment most openly.

> A considerable body of independent work tends to corroborate the original Cobb-Douglas formula, but, more important, the approximate coincidence of the estimated coefficients with the actual shares received also strengthens the competitive theory of distribution and disproves the Marxian (Douglas, 1976, p. 914, cited in McCombie and Dixon, 1991, p. 24).

Robert Solow, by far the most important contributor to this tradition, takes a more nuanced position. But in the end he too comes down on the same side.

> The current state of play with respect to the estimation and use of aggregate production functions is best described as Determined Ambivalence. We all do it and we all do it with a bad conscience…One or more aggregate production functions is an essential part of every complete macro-econometric model …It seems inevitable…There seems no practical alternative… [Yet, n]obody thinks there is such a thing as a 'true' *aggregate* production function. Using an estimate of a relation that does not exist is bound to make one uncomfortable (Solow 1987, p. 15).

In spite of these misgivings, Solow (1987) returns to the central theme: aggregate production functions continue to be used because, at least sometimes, they appear to 'work'. And here, he refers not only to their ability to provide "a practical way of representing the relation between the availability of inputs and the capacity to produce output" (p. 16), but also to "their ability to reproduce the distributional facts". He notes that this latter aspect "does reinforce the marginal productivity theory … of distribution", just as Douglas had himself claimed, even though such a claim "is far from airtight [since] … the coincidence could occur in many other ways" (p. 17).

It is worth emphasizing that a 'good' empirical fit between aggregate output and variables such as capital, labor and time can arise from a wide variety of functional forms, ranging from ones with entirely fixed input-output coefficients to those with smoothly variable ones. But even in the latter case, a good fit does not necessarily imply support for neoclassical theory. Two further things are required. First, that the smoothly varying coefficients be embodied in a functional form consistent with a 'well- behaved' neoclassical aggregate production function (Cobb-Douglas, CES, Translog, etc.). And second, that the fitted function yield estimated output elasticities which correspond to observed wage and profit (factor) shares. As Solow once remarked early on, "had Douglas found [parameters which implied] labor's share to be 25 per cent and capital's 75 per cent, we should not now be talking about aggregate production functions" (McCombie 2000-2001, p. 269, footnote 1, quoting a remark by Solow to Fisher, cited in Fisher 1971a).

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And so we are led to the central questions underlying the debate around the use of neoclassical aggregate production functions. Is it really true that aggregate production functions generally 'work' in the sense of giving not only a good fit but also the 'right' coefficients? When they do work, can this be taken as evidence of support for the neoclassical theory of production and income distribution? And finally, can they be used in any case to provide reliable measures of technical change and a decomposition of the sources of growth?

To address the issues involved, we will make use of two very different data sets. The first set is the actual data for the US economy, consisting of real output (real GDP), total employment, total real private fixed capital, and corresponding measures of real wages, profit rates, labor productivity and the capital-labor ratio. The second set is a simulated data series derived from a slightly modified version of Goodwin's formalization of Marx's theory of persistent unemployment. This latter model is decidedly non-neoclassical, since it assumes a fixed coefficient technology at any moment of time (so that actual marginal products cannot even be defined) and Harrod-Neutral technical change over time (so that not even 'surrogate' marginal products, in the sense of Samuelson, can be found). So we have one data set whose provenance is the object of considerable debate, and a control group whose provenance is clear and strictly non-neoclassical.

The two data sets turn out to look very similar. In both cases, the wage shares are roughly stable, so that the appropriate neoclassical production function to test is the Cobb-Douglas. In both cases, the standard fitted functions do *not* work well in the sense outlined by Douglas and Solow. This, as we shall see, is a common outcome in actual data. Nonetheless, in both cases one can *always* create 'appropriate' modifications which will appear to make a neoclassical aggregate production function appear to work well in the standard sense. This is because output, capital, and labor are themselves ineluctably linked to wage and profit rates through an accounting identity, and this identity can always be manipulated to make a neoclassical production function 'come out right' (Phelps-Brown 1957, Simon and Levy 1963, Shaikh 1974, 1980, 1987; McCombie 2000-2001; Felipe and Holz 1999, Felipe and Adams 2001). As we shall see, this is true even for data whose true generating function is entirely non-neoclassical.

The next section will introduce the fundamental issue: the difficulty of distinguishing between a neoclassical aggregate production function, which may or may not exist even in proximate form, and an accounting identity which always exists and is always exactly true. Since the issues raised there will require empirical investigation, Section III will introduce two data sets, one from the US National Income and Product Accounts (NIPA), and the other from a simulation run of a Goodwin-type model. Section IV will then (econometrically) interrogate both sets of data, to see what they appear to tell us. Section V will show that it is always possible to transform a fitted production function that does not work well into one that appears to do so, even when such a procedure entirely misrepresents the true forms of production and technical change. Section VI will take up the implications of these results for the standard neoclassical measures of technical change, and will develop an alternate Samuelson-Sraffa measure of technical change which does not require the existence any sort of aggregate production function, be it actual or 'surrogate'.

II. The significance of the accounting identity

If we define Y_t, L_t , K_t, and w_t as real output, labor, capital, and the real wage respectively, then as a matter of definition the observed profit rate r_t = profits/capital = $(Y_t - w_t A L_t)/K_t$. This means that the variables in question are linked through an accounting identity that is linear in Y, K, L, that always 'adds up', and that is always true.

1) $Y_t = w_t A L_t + r_t A K_t$

On the other hand, if we now also posit some general production relation of the form

2) $Y_t = F(L_t , K_t)$

then we know that it can represent many different underlying conditions. It can take the form a fixed-coefficient technology corresponding a single technique that dominates all others in wage-profit (factor-price) space, as is implicit in Harrod, Goodwin, and many others (Shaikh 1987). It can take the form of jumpy input-output relations representing a wage-profit frontier with kinks at switch points from one technique to another (Michl 1999, p.196, Figure 1). Or it can take the form of a set of smoothly varying coefficients, either because the wage-profit frontier corresponds to an infinite spectrum of fixed coefficient methods of production (Garegani 1970) or because it represents the aggregation of micro-level production functions (Fisher 1971a, 1987, 1993). In none of these cases is the functional form $Y = f(K, L)$ necessarily 'well-behaved' in the traditional neoclassical sense. On the contrary, even when the coefficients are smoothly varying, one can get aggregate relations which appear to be horrendously ill-behaved (Garegnani 1970, p. 430, Figure 7). As Fisher (op. cit.) has repeatedly emphasized, it does not even help to begin by assuming well-behaved microeconomic production functions, because the conditions needed to produce a satisfactory aggregate relation are impossibly stringent. It has therefore become widely accepted at a theoretical level that any aggregate production relation need not be smooth, that aggregate marginal productivity conditions will not generally hold, and that even if they do appear to do so, the causation cannot got from the quantity of factors to their corresponding prices (Samuelson 1966; Harcourt 1969, 1972, 1994).

But suppose that we press ahead, as is so often done, and simply posit the existence of an (approximate) aggregate production function in which factor prices equal corresponding marginal products, and in which constant returns to scale obtain (so as to ensure that that the factor-price-weighted sum of inputs 'adds up' to total output). These additional assumptions then superimpose on equation 2 the further conditions

3) $\partial Y_t / \partial L_t \equiv MPL_t = w_t$
4) $\partial Y_t / \partial K_t \equiv MPK_t = r_t$
5) $Y_t = MPL_t\, AL_t + MPK_t\, A\, K_t$ [from the assumption of constant returns to scale]

Equations 2-5 embody the standard neoclassical assumptions about aggregate production. Taken together, they immediately imply that

6) $Y_t = w_t A\, L_t + r_t A\, K_t$

The trouble is that this relation already holds in the form of the accounting identity (equation 1), *quite independently* of any specification of production or distribution relations. It follows that the imposition of the standard neoclassical assumptions about aggregate production make *it impossible to distinguish the neoclassical argument from a mere tautology*. Since the accounting identity is always true, the only function of these assumptions is merely to interpret the identity. They cannot provide any support whatsoever for the theory itself.

In his response to Shaikh (1974) concerning to his celebrated growth accounting procedure Solow makes just this point.

> The factor-share device of my 1957 article is in no sense a *test* of aggregate production functions or marginal productivity theory or anything else. It merely shows how one goes about interpreting given time series if one starts by *assuming* that they were generated from a production function and that competitive marginal-product relations apply (Solow 1974, p. 121).

But to stop there would reduce the most fundamental construct of neoclassical macroeconomics to a mere article of faith (Ferguson 1971). Therefore, Solow goes on to specify what is actually needed to test the standard neoclassical hypotheses.

> When someone claims that aggregate production functions work, he means a) that they give a good fit to input-output data *without* the intervention of data deriving from factor shares b) that the function so fitted has partial derivatives that closely mimic observed factor prices… [and c) since] technical change is always represented by a smooth function of time (or something else) … part of the test is whether the residuals are well-behaved (Solow, 1974, p. 121 and footnote 1).

So this is the task. To see if aggregate production functions do indeed work in this sense, and to learn what is implied if they do. But first, we need to address the issue of the data.

III. Two aggregate data sets: actual and simulated

Solow tells us that aggregate production functions 'work' when they fit the data well, when their coefficients yield marginal products which mimic factor shares, and when the implied pattern of technical change appears plausible. These are certainly necessary conditions, and we will assess their viability in the next section. But it is important to note that they are *not sufficient* conditions, because it is possible that the existence of the accounting identity might account for these very outcomes. After all, it was precisely because the standard neoclassical assumptions are functionally indistinguishable from the tautologically-true identity that Solow's conditions are necessary. What we also need to know, therefore, is whether they are also sufficient to distinguish between neoclassical and non-neoclassical production relations. In other words, we need a control group against which to apply our tests.

The first data set used in this paper is actual data derived directly from U.S. Bureau of Economic Analysis (BEA) National Income and Product Accounts (NIPA) and from corresponding wealth stocks. Details are provided in Appendix A. It consists of aggregate output Y = real GDP, employment L = full time equivalent employees, capital stock K = real private net stock at the beginning of year, and w = the real (product) wage per worker = employee compensation per unit employee deflated by the GDP price-deflator. In keeping with the accounting identity, the rate of profit r = (Y - wAL)/K. In addition, we can also define productivity y = Y/L, the capital-labor ratio k = K/L, the wage share u = w/y = wAL/Y, and the employment ratio v = employment/labor force = 1 – the unemployment rate expressed in decimals.

The control data set is generated from a simulation run of a slightly modified version of the Goodwin model. The original Goodwin model is, as Solow (1990, pp. 35-36) justly observes, a "beautiful paper" which "does its business clearly and forcefully". Its essential dynamics turn on the interactions between the wage share, the rate of growth, and the employment ratio. Two changes are made here; the model is extended by allowing for a savings rate less than one (Goodwin originally assumed that all profits are saved); and Goodwin's original real wage Phillips curve is modified by allowing for an "employer resistance" drag on real wage growth as the wage share rises (the rate of profit falls). This latter modification is made in order to produce a version of the model which is stable in the presence of stochastic shocks[1].

There are two parts to the logic of the Goodwin model. The first has to do with the nature of the technology and its change over time. Like Harrod, Goodwin assumes that the economy is moving along its warranted path, so that output is equal to capacity. At any moment of time, a single linear fixed-coefficients technology dominates the wage rate-profit rate (factor-price) frontier, whose intercepts can be characterized by the productivity of labor and by the capacity-capital ratio. Over time, it is assumed that technical change is embodied in new technologies with higher capital-labor ratios that yield higher labor productivities, with both rising at the same rate so that the capacity-capital ratio remains unchanged (this is Harrod Neutral technical change). The assumption that coefficients are fixed at any moment of time means that marginal products cannot even be defined for any given technology. And the assumption of Harrod-Neutral technical change means that the choice of technique is invariant to the distribution of income, so that an incremental change in (say) the wage rate cannot even be associated with some corresponding change in labor productivity or in the capital-labor ratio. This excludes not only smooth "surrogate" correlations between real wages and the incremental productivity of labor (Samuelson 1962, 1966) but also any lumpy ones as well (Michl 1999, pp. 196). The assumed technological structure thus excludes both actual and surrogate marginal productivity

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[1] In Goodwin's original formulation, the rate of change of real wages take the form of a real wage Phillips curve, and the resulting model is a quasi-stable center in continuous form and unstable in discrete form. Adding the negative influence of a high wage share to account for increased employer resistance to real wage increases as the wage share itself rises (and the profit rate correspondingly falls). I thank Duncan Foley for suggesting this modification.

conditions. *It follows that the technological structure of this control group model is entirely non-neoclassical.* Figure 1 illustrates this aspect of the model, as taken from Shaikh (1987)[2].

Figure 1: Fixed-Coefficient Technology with Harrod Neutral Technical Change
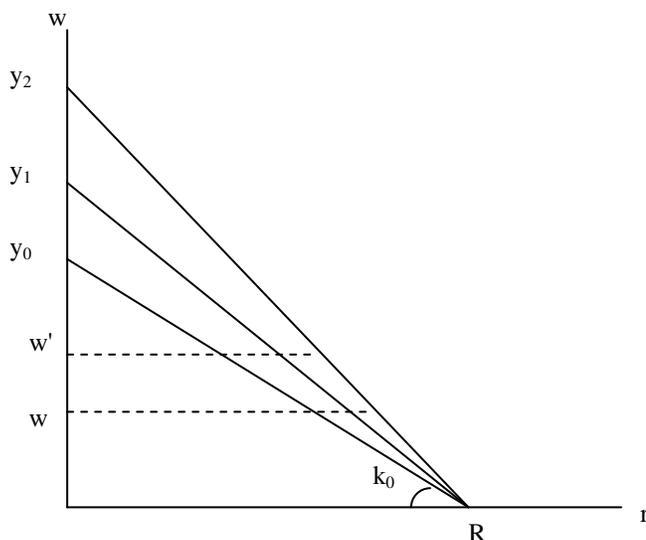
The second part of the model has to do with the dynamical interaction between the wage share and the employment ratio. The movements of the wage share $u = w/y$ are influenced by two factors: the rate of growth of labor productivity which is taken to be some constant (α); and the rate of growth of real wages which depends positively on the rate of employment (v) as in the original model and here also negatively on the (squared) level of the wage share. The movements of the employment ratio $v = L/N = Y/yAN$ in turn depend on three factors: the rate of growth of the labor force N which is taken to be equal to some constant (β); on the rate of growth of labor productivity (previously specified as some constant α) ; and the rate of growth of real output (Y). The employment ratio and the wage rate are then linked by the fact that the wage share influences the profit rate, which influences the rate of growth of capital and hence the growth rate of real output[3]. Since the model has is stable, in the absence of shocks it the growth rate of output converges to the natural growth rate ($\alpha+\beta$), the wage share converges to some constant level u* , and the employment share to some constant level v*.

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[2] In Figure 1, the vertical axis represents the real wage, and the horizontal axis the profit rate. Each technology is characterized by a linear trade-off between the wage rate and the profit rate, with limits arising from the given productivity of labor (y) which is also the maximum real wage, and a given capacity-capital ratio (R) which is also the maximum rate of profit. The slope of each such line is the capital-labor ratio corresponding to that particular technology. The productivity of labor rises over time, but the capacity-capital ratio is constant. Thus at any given real wage w, the latest technology is dominant. Changing the real wage from w to w', for instance, will not change the technology affect the chosen technology and hence will not affect labor productivity or the capital-labor ratio.
[3] Beginning from the short run equilibrium condition that investment equals savings (I = S), and assuming that savings are proportional to profits (P) because workers do not save, we have I = sAP. Dividing by the capital stock yields $I/K \equiv K'/K = P/K \equiv r$, where K'/K stands for the rate of growth of capital. But the profit rate r = P/K = (P/Y)A(Y/K) can be further decomposed by noting that the profit share P/Y = (Y - wAL)/Y = 1- w/y = 1 – u, where y = Y/L = labor productivity, and u = w/y = the wage share. Along the warranted path, output = capacity, and in the presence of Harrod Neutral technical change, the capacity-capital ratio R = Y/K = constant. Thus the rate of growth of output (Y'/Y) = the rate of growth of capacity = the rate of growth of capital (K'/K) = sA r = sA r = sARA(1-u).
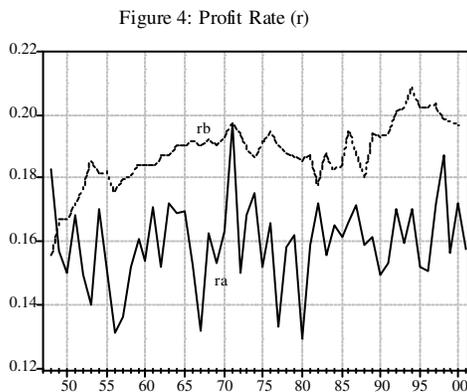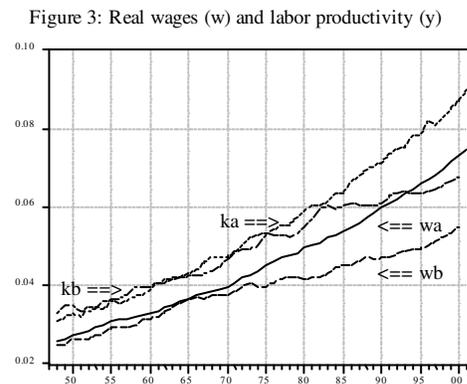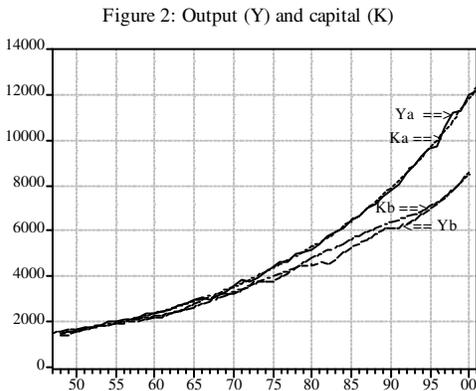
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We can then summarize this modified Goodwin model by the following non-linear system of equations. Further details are provided in Appendix B.

7) $u = w/y$ [u = wage share = real wage/labor productivity]

8) $v = Y/yAN$ [v = employment ratio = real output/(labor productivityAlabor force]

9) $\ln(y_t) = \ln(y_{t-1}) + \alpha$ [constant rate of growth of labor productivity]

10) $\ln(N_t) = \ln(N_{t-1}) + \beta$ [constant rate of growth of the labor force]

11) $\ln(w_t) = \ln(w_{t-1}) - \gamma + \rho A\, v_{t-1} - \rho_1 A\, (u_{t-1})^2$ [real wage growth function]

12) $\ln(Y_t) = \ln(Y_{t-1}) + sARA(1-u_{t-1})$ [output growth rate = savings rate A profit rate]

In the end, we have two data sets, both of which satisfy the accounting identity of equation 1. In what follows, they are labeled sets A and B without any further identification until the end of Appendix B. In the meantime, the reader is invited to keep an open mind.

Figure 1 displays paths of output (Q) and capital (K). Figure 2 depicts real wages (w) and productivity (y), the latter implicitly a display of the path of employment. Figure 3 the profit rate (r). And Figure 4 plots the wage share (u) and the employment ratio (v).

Figure 2: Output (Y) and capital (K)

Figure 3: Real wages (w) and labor productivity (y)

Figure 4: Profit Rate (r)

Figure 5: Wage Share (u) and Employment Ratio (v)

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IV. Do aggregate production functions 'work' at an empirical level?

As we can see from Figure 4, the wage shares in data sets A and B are roughly stable, with means of $u_a \approx 0.84$ and $u_b \approx 0.81$, respectively. This means that a Cobb-Douglas function is an appropriate starting point. For our purposes, it is sufficient to work with the standard form in which technical change is assumed to be neutral ($Y_t = A_t A L_t^b A K_t^c$). In general, coefficients b and c represent the putative factor shares, and their sum represents the degree of returns to scale. However, if we wish to impose the further restriction of constant returns to scale (b + c = 1), then we can divide through by labor to get the per employee form ($y_t = A_t A k_t^c$), in which the coefficient c represents the profit share implied by the marginal productivity theory of distribution.

For the purpose of empirical estimation, we express both of the preceding forms in levels and also in growth rates. As is standard, the technical change parameter is expressed as a log-linear function of time (although we will return to this issue in the next section). This gives us four regressions overall, and two data sets for each. The relation between the estimated coefficients and the observed wage and capital shares will be of particular interest. All regressions are OLS, as is customary in this literature.

13) $\ln(Y_t) = a_0 + a_1 At + bA\ln(L_t) + cA \ln(K_t)$
14) $\Delta\ln(Y_t) = a_1 + bA\Delta\ln(L_t) + cA \Delta\ln(K_t)$
15) $\ln(y_t) = a_0 + a_1 At + cA\ln(k_t)$
16) $\Delta\ln(y_t) = a_1 + cA\Delta\ln(k_t)$

The results are reported in Table 1. Each equation is run on both sets of data. The relevant dependent variable is listed at the top (e.g. $\ln(Y_t)$ for equation 13), estimated coefficients with t-statistics in parentheses are listed in the row of the appropriate dependent variable, and adjusted R^2 and Durban-Watson statistics are listed after these. The last three rows compare the implied wage and profit shares with the actual ones (the latter being listed in parentheses below).

Table 1: Cobb-Douglas production functions fitted to actual and simulated aggregate data (OLS)

Dependent Variable	ln(Y_t)		Δln(Y_t)		ln(y_t)		Δln(y_t)	
	Data A	Data B	Data A	Data B	Data A	Data B	Data A	Data B
Constant	-4.656*	-0.279	0.108*	0.026*	-3.442*	-2.109*	0.0205*	0.015*
	(-2.787)	(-0.147)	(4.266)	(3.267)	(-9.768)	(-4.561)	(6.871)	(6.340)
Time	0.0132	0.0133*			0.020*	0.009*		
	(1.520)	(2.798)			(9.705)	(4.488)		
ln(L_t)	0.988*	0.471*						
	(9.7445)	(2.465)						
ln(K_t)	0.176	0.341*						
	(0.765)	(2.354)						
Δln(L_t)			1.002*	0.948*				
			(10.631)	(9.648)				
Δln(K_t)			-2.212*	-0.271				
			(-3.483)	(-1.062)				
ln(k_t)					0.022	0.395*		
					(0.219)	(2.929)		
Δln(k_t)							-.0029	0.063
							(-0.280)	(0.636)
Adj. R^2	0.999	0.995	0.696	0.673	0.999	0.977	-0.018	-0.012
D.W.	2.117	0.219	2.377	1.941	2.036	0.185	2.974	1.930
Implied Wage Share	0.988	0.471	1.002	0.948	0.978	0.605	1.0029	0.937
(Actual Wage Share)	(0.840)	(0.810)	(0.840)	(0.810)	(0.840)	(0.810)	(0.840)	(0.810)
Implied Profit Share	0.176	0.341	-2.212	-0.271	0.022	0.395	-.0029	0.063
(Actual Profit Share)	(0.160)	(0.190)	(0.160)	(0.190)	(0.160)	(0.190)	(0.160)	(0.190)
Implied Returns to Scale	1.164	0.812	-1.210	0.677	-	-	-	-
t-statistics are listed below estimated coefficients, except for implied wage and profit shares where actual shares are listed below. Starred coefficients imply significance at 5% or better.								

The first two forms of regressions do not assume constant returns to scale, so the sum of the labor and capital coefficients are not restricted in advance. When run in levels, the overall fit is excellent, and the labor coefficient is significant and large for both data sets. In set A the time trend and capital coefficients are not significant but the overall D.W. statistic is quite good (2.117), while in set B the time trend and capital coefficients are significant but the D.W. is not good (0.219). But in neither set are the implied shares close to the actual, or returns to scale close to constant. When run in rates of growth, the overall fits is quite good for both sets of data, the time trends are significant, labor coefficients close to one and highly significant, and D.W.'s are good. But in both cases the capital coefficient is negative, so that the implied shares are very different from actual ones.

The second two forms restrict the coefficients to sum to one (i.e. they assume constant returns to scale), so the relevant variables are per employee output and capital. When run in levels, the overall fit is once again excellent, and the constants and time trend are highly significant. But in set A the coefficient of the capital-labor ratio is small and not statistically significant, while the overall D.W. is quite good; while in set B, the coefficient of the capital-labor ratio is relatively large but the D.W. quite low. Once again, however, the estimated capital coefficient is not even close to the actual profit share in either set. And finally, when run in growth rates, only the constant are significant (which in growth rate form implies significant positive rates of neutral technical change), while all other results are generally quite bad.

On the whole, therefore, even though the wage shares are roughly stationary in both data sets, none of the fitted forms of the Cobb-Douglas aggregate production function 'work well'. Keeping in mind that one of the sets represents actual U.S. data, what are we to make of these results? Are these results somehow atypical?

Douglas certainly seems to make such a claim he says that "a considerable body of independent work tends to corroborate the original Cobb-Douglas formula" (Douglas 1976, p. 914, op. cit.). But Samuelson (1979, p. 924) points out that Douglas' own original regression did not include any term for technical change, and Felipe and Adams (2001, p. 6) show that when a term for neutral technical change is incorporated into the Cobb-Douglas (< la Solow), then Douglas's original data set yields a "coefficient of the index of capital which is negative and insignificant".

Solow too initially emphasized the importance of the overlap between Douglas's estimated parameters and actual factor shares (Fisher 1971, in McCombie 2000-2001, p. 269). And he repeated this sentiment in his first response to Shaikh (1974). Having found that Shaikh's constructed data yields a "point estimate of log k [which] is negative" and not statistically significant, he says that if "this were the typical outcome with real data, we would not now be having this discussion" (Solow 1974, p. 121). However, McCombie comments that "it is surprising that Solow did not seek to [similarly] 'test' the Cobb-Douglas function using his own data". When McCombie returns to Solow's original data, he finds that when the Cobb-Douglas is run in levels "the coefficient of capital term is not statistically significant from zero", and when it is run in ratios "the coefficient of the capital-labor term is negative, but not statistically insignificant". This prompts him to remark that we "can only speculate whether Solow's (1957) paper would have had such a dramatic impact if these regressions had also been reported" (McCombie 2000-2001, p. 281-283).

None of this should be surprising. In point of fact, it has been shown that estimated aggregate production functions generally do *not* 'work well' (Sylos-Labini 1995). Moreover, negative capital coefficients are a fairly *common* finding (Felipe and Adams, 1999, p.6; Reati 2001, pp. 324-324). Nonetheless, aggregate production functions do appear to work on occasion. So at least in these particular instances, can we say that this provides some evidence on the underlying production structure, and perhaps on the viability of the marginal productivity theory of distribution? This is the issue to which we turn next.

V. How to make aggregate production functions always work (even when they are completely inappropriate)

The purpose of this section is to show that one can always turn a dysfunctional estimated aggregate production function into a perfectly functional one. To see this, let us return to the constant returns to scale functions of equations 15-16, whose fitted results appear in the last two columns of Table 1. Both of these embody the assumption of a log-linear time trend for the technical change variable A_t . Of these two regression forms, the growth form performs particularly poorly.

But suppose that we instead consider a log-nonlinear time path for the technical change variable, as illustrated in Figure 6 (A_t) and Figure 7 ($\Delta \ln (A_t)$). Note that both trends in Figure 6 are fairly smooth, and should therefore qualify as 'well behaved' in the traditional sense (Solow, 1974, p. 121 and footnote 1). The rule by which these path can be derived will be developed shortly. For now, we will focus on their properties.

Figure 6: Technical Change Parameters A(t)
(log scale)
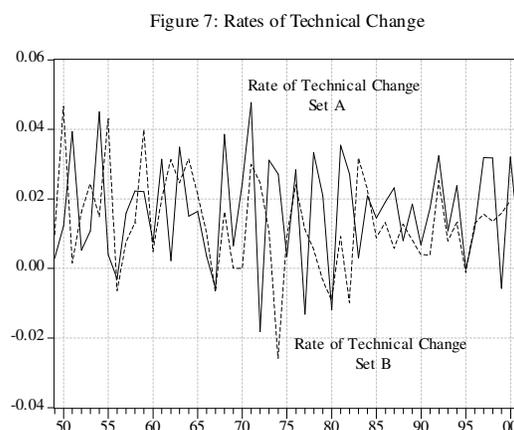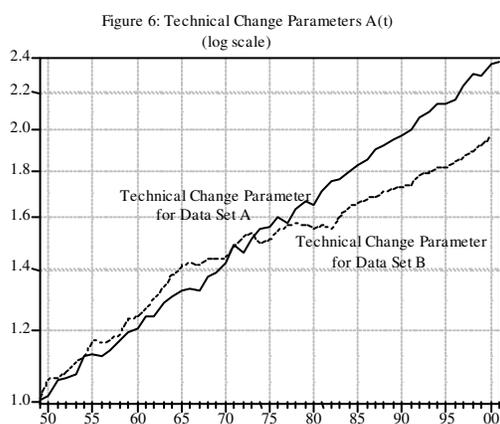
Figure 7: Rates of Technical Change

Table 2 presents the results of re-estimating equations 15-16, which represent the level and growth forms of the constant returns to scale Cobb-Douglas production function with neutral technical change. The only difference between the current estimates and the previous ones is that we have broadened the assumed functional form of the technical change parameter A_t from a log-linear time trend to the log-nonlinear time trend depicted in Figure 6.

Table 2: Constant returns Cobb-Douglas functions with variable time trends for technical change (OLS)

Dependent Variable	$\ln(y_t)$		$\Delta\ln(y_t)$	
	Data A	Data B	Data A	Data B
Constant	-2.932*	-2.825*	-0.000158*	.0000638*
	(-245.72)	(-550.96)	(-2.093)	(1.198)
$\ln(A_t)$	1.021*	1.007*		
	(244.31)	(544.43)		
$\ln(k_t)$	0.156*	0.201*		
	(45.321)	(137.045)		
$\Delta\ln(A_t)$			1.027*	1.012*
			(392.366)	(421.009)
$\Delta\ln(k_t)$			0.158*	0.193*
			(81.209)	(114.295)
Adj. R^2	0.9999	0.9999	0.9997	0.9997
D.W.	0.311	0.286	1.834	1.515
Implied Wage Share	0.844	0.799	0.842	0.807
(Actual Wage Share)	(0.840)	(0.810)	(0.840)	(0.810)
Implied Profit Share	**0.156**	**0.201**	**0.158**	**0.193**
(Actual Profit Share)	**(0.160)**	**(0.190)**	**(0.160)**	**(0.190)**

In Table 2, for regressions in both levels and growth rates, the fits are fantastic, neutral technical change is strongly supported, and the estimated capital coefficients are equal to the observed capital shares (1-u). Aggregate production functions and marginal productivity theory thus appear to have a strong empirical basis. And all of this by merely allowing the technical change parameter to take a log-nonlinear time path.

The results depicted in Table 2 are essentially perfect, for both data sets. *And therein lies the rub*, for we already know that one of the data sets is generated from an entirely non-neoclassical (Goodwin) model with fixed-coefficient technology undergoing Harrod-Neutral technical change. Even the stability of the long run wage share derives from the classical feedback between *persistent* unemployment and its effects on real wages and the rate of growth. Neither actual nor surrogate marginal products, nor any theory of wages linked to them, can be defined within this framework. Nonetheless, aggregate production functions and marginal productivity theory appear to work perfectly even here. How is this possible?

The secret lies in the accounting identity of equation 1, expressed below in per employee form (equation 17). Taking the rates of growth of both sides gives us the growth form of this accounting identity (equation 18), where as before y = the productivity of labor, w = the real wage, u = w/y = the wage share, and r = the profit rate.

17) $y_t = w_t + r_t A k_t$

18) $\Delta\ln(y_t) = \Delta\ln(A_t) + (1-u_t)A\Delta\ln(k_t)$

in which $\Delta\ln(A_t) \equiv u_t A\Delta\ln(w_t) + (1-u_t)A\Delta\ln(r_t) \equiv$ the share weighted average of the rates of change of real wages and the profit rate, respectively, is nothing but the famous *Solow residual* (Shaikh 1974). Note that in the growth rate form of the accounting identity the wage share u_t appears as a time variable. But if, as in the present data sets, the wage share is stable over the long run ($u_t \approx u^*$), equation 18 is transformed from an *exact* identity to an *identity-approximation'* whose power depends precisely on the stability of the wage share. In that latter case, we get

19) $\Delta\ln(y_t) \approx \Delta\ln(A_t) + (1-u^*)A\Delta\ln(k_t)$

20) $\ln(y_t) \approx a_0 + \ln(A_t) + (1-u^*)A\ln(k_t)$

It should now be obvious that there is a strong correspondence between the 'identity-approximations', and the growth and level forms of a constant returns to scale Cobb-Douglas with neutral technical change (equations 16 and 15 respectively). The sole difference between the two sets had to do with the standard simplifying assumption that technical change can be well approximated by a log-linear time path, i.e. that $\ln(A_t) \approx a_0 + a_1 At$ and hence $\Delta\ln(A_t) \approx a_1$. It is precisely this simplifying assumption that produced such bad results in the earlier regressions in Table 1. And it is precisely its replacement by the actual time path of A_t which renders the very same regression forms perfect, as shown in Table 2. But this is an accounting perfection, not a technological one. *As we have seen, it tells us nothing about the underlying technology* (Shaikh 1974, 1980, 1987; McCombie and Dixon 1991, pp. 27-29; McCombie 2000-2001, pp. 284-288; Felipe and Adams, 20001, pp. 15-20).

Once the preceding point has been absorbed, it then becomes clear that one can always make an aggregate production function 'work well' by deriving a good approximation to the time path A_t. In data sets for which this can be accomplished through some simple curvilinear function of time, the resulting aggregate production function will appear to spontaneously fit well. But if not, better approximations can always be found. Alternately, the measures of the inputs (L, K) can be adjusted in a variety of ways , such as by adjusting for "for the intensity of their use over the business cycle" (McCombie 2000-2001, pp. 285-288). Such adjustments will change the measures of the share-weighted growth rates of the real wage and the profit rate (the Solow Residual), and this will in turn improve the workings of aggregate production functions in so far as the resulting path of the altered measures of A_t is simplified.

We know, of course, that the time path of A_t can be approximated to any desired degree of precision by means of a Fourier series. What constitutes a 'good' approximation is then determined by how well it performs in making the aggregate production function appear to 'work' – even when it is entirely inappropriate (Shaikh 1980, Felipe and Adams 2001, pp. 16-17). Table 3 reports the results of using an approximation $\Delta\ln(A_t')$ for the more difficult growth form of the constant returns to scale Cobb-Douglas. Details of the approximations are provided in Appendix B. It is evident that both work quite well.

Dependent Variable	$\Delta\ln(y_t)$	
	Data A	Data B
Constant	not significant	not significant
$\Delta\ln(A_t')$	1.0144*	1.0223*
(std. error)	(0.0481)	(0.1062)
(t-statistic)	(21.111)	(9.6287)
$\Delta\ln(k_t)$	0.143*	0.161*
	(0.0366)	(0.0698)
	(3.903)	(2.300)
Adj. R^2	0.801	0.360
D.W.	1.234	1.672
Implied Wage Share	0.857	0.839
(Actual Wage Share)	(0.840)	(0.810)
Implied Profit Share	**0.143**	**0.161**
(Actual Profit Share)	**(0.160)**	**(0.190)**

Finally, we note that we have focused so far on the Cobb-Douglas function because it happens to be the appropriate one for the two data sets under consideration. But McCombie and Dixon (1991, p. 31-34) have shown that the argument can be generalized to other aggregate production functions as well. As we can see from the identity in equation 18, if the wage share happens to be changing over time, "then a more flexible functional form is needed such as the CES or translog function" (ibid, p. 32) to capture this effect also. But once again, it is merely a matter of approximating the identity by approximating its two central components, the Solow residual and the wage share. This will appear to extend the domain of aggregate production functions and marginal productivity theory. But in fact it will merely extend the body domain of effective 'identity-approximation'.

VI. Assessing technical change

A central implication of the foregoing results is that one cannot generally interpret the famous Solow Residual $\Delta\ln(A_t)$ as a measure of the rate of technical change. To see this, it is worth noting that this measure can be expressed into two equivalent ways, both of which follow purely from accounting identities. In the derivation of equation 18, we saw that the Solow Residual (SR) is simply the share-weighted average of the growth rates of the real wage and profit rate.

20) $SR \equiv \Delta\ln(A_t) \equiv u_t A\Delta\ln(w_t) + (1-u_t)A\Delta\ln(r_t) = \Delta\ln(y_t) - (1-u_t)A\Delta\ln(k_t)$

But using the definition of the output-capital ratio $R \equiv Y/K = y/k$, we can reorder the last expression for SR to get

21) $SR \equiv u_t A\Delta\ln(y_t) + (1-u_t)A\Delta\ln(R_t)$

Thus, purely as a matter of accounting, the Solow Residual is also the share-weighted average of the growth rates of the productivity of labor and the output-capital ratio.

Neoclassical economics attempts to go beyond these accounting identities by taking two further steps. First, it attempts to interpret the aggregate output-capital ratio (R) as the 'aggregate productivity of capital'. For this interpretation to be valid, it is necessary that there exist a well-behaved neoclassical aggregate production function which would allow one to attribute some portion of the change in aggregate real output to the 'contribution' of real aggregate capital. And second, it attempts to reduce the real wage and the profit rate, and hence the corresponding 'factor' share, to physical attributes of this same technology. Only if both of these conditions hold can the two accounting expressions for the Solow Residual be interpreted as (equivalent) measures of technological change. This is precisely what Solow does assume in his original growth accounting procedure (Solow 1957).

But we have seen such assumptions are neither derived from micro-foundations (Harcourt 1972, Fisher 1993)[4] nor supported by empirical evidence.

The essential difficulty faced by the aggregate production function approach arises from its attempt to 'physicalize' the distribution of income. There is, however, another way to proceed, which is to think of technical change in terms of its effects on the competitiveness of individual firms. This is a micro-foundation common to Sraffa (1963), Samuelson (1962), and Okishio (1961), and it gives rise to the notion of a wage-profit frontier. Any given collection of particular methods of production, one for each sector, can be called an aggregate production technique. At any given moment of time there may be many possible techniques available. Of these, any given real wage the one

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[4] Indeed, as Garegnani (1970, Figure 7, p. 430) showed long ago, the changes in the aggregate output-capital ratio as one moves along a wage-profit frontier would inevitably involve relative price effects whose influence would be inseparable from changes in quantities. They could, moreover, produce complex and distinctly 'ill-behaved' associations between output per worker and output per unit capital. The problem lies not in measuring (the value of) capital, but rather in reducing it to some quantity index which must also satisfy neoclassical postulates at an aggregate level.

which will be put into operation through competitive processes will be the one which yields a higher expected rate of return. In terms of Figure 1 of section III, if techniques 0 and 1 happened to coincide at some moment, then at the particular wage rate w the dominant one would be technique 1. It at some later moment a new technique (2) became possible, then it would assume the dominant role. Of course Figure 1 was confined to the illustration to the particular instance of Harrod-Neutral technical change, in which case one single technique would be dominant at any time. But the principle is more general than this, and can be applied to any range of techniques coexisting at a moment of time and varying over time (Michl, 1999, pp. 193-198).

According to this particular 'choice of technique' principle, it is the change in the aggregate (normal capacity) rate of profit at a common real wage which is the appropriate measure of the competitive impact of technical change[5]. From the accounting identity of equation 17, which we can write in the form $r_t = (y_t - w_t)/k_t$, the rate of change of the profit rate at a given real wage can be written as[6]

22) $\tau \equiv [\Delta\ln(r_t)]_{\Delta w = 0} = \Delta\ln(y_t)A(1/(1-u_t)) - \Delta\ln(k_t) = \Delta\ln(y_t)A(u_{t-1}/(1-u_{t-1})) + \Delta\ln(R_t) = SR/(1-u_{t-1})$

Both the technical impact measure τ and the Solow Residual (SR) are share-weighted averages of the rates of change of y and R[7]. But in the Solow Residual the shares too are essentially technological, in keeping with marginal productivity theory, whereas in technical impact measure technology and distribution mutually affect each other, but neither is reducible to the other. There is plenty of room, in other words, for history, economics and politics.

Two Measures Involving Technical Change
(U.S. Data)

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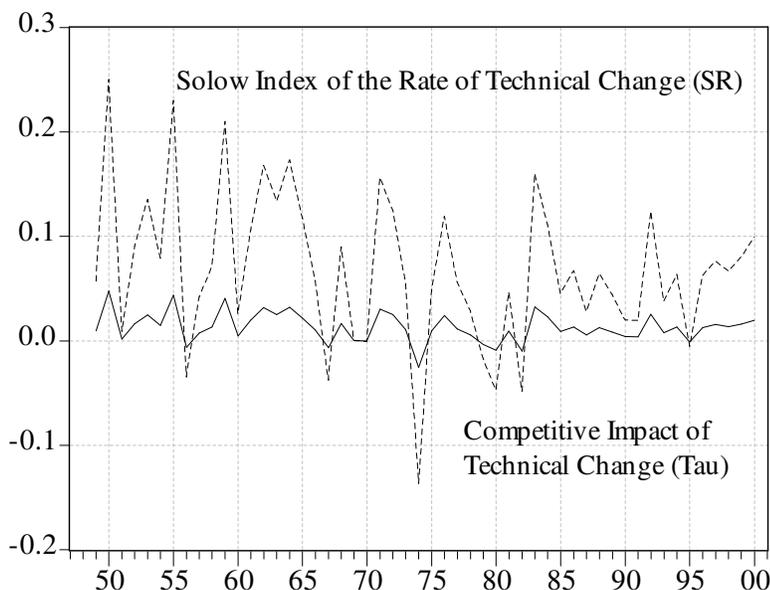
[5] Shaikh (1978, 1999) argues for a somewhat modified form of this choice of technique criterion.
[6] From the identity $r_t = (y_t - w_t)/k_t$, holding the real wage constant, $\Delta r_t/r_{t-1} = \Delta y_t/(y_{t-1} - w_{t-1}) - \Delta k_t/k_{t-1} = (\Delta y_t/y_{t-1})A(1/(1-u_{t-1})) - \Delta k_t/k_{t-1} = (\Delta y_t/y_{t-1})A(u_{t-1}/(1-u_{t-1})) - (\Delta R_t/R_{t-1})$, since $R = Y/K = y/k$.
[7] Dumenil and Levy (1993) have developed a stochastic version of the path generated by this notion of choice of technique.

Appendix A

The actual data used in this paper is taken from the U.S. Bureau of Economics (BEA) website, from the U.S. National Income and Product Accounts (NIPA). It consists of output Y = real GDP in billions of 1992-$, employment L = full time equivalent employees in thousands, capital stock K = beginning of year real private net stock in billions of 1992 chained-$, and w = the real (product) wage per worker = employee compensation per full-time equivalent employee deflated by the GDP price-deflator, in millions of 1992-$ per employee. In keeping with the accounting identity, the rate of profit r = (Y - wAL)/K. In addition, we can also define productivity y = Y/L , in millions of 1992-$ per employee, the capital-labor ratio k = K/L in millions of 1992 chained-$ per employee, the wage share u = w/y = wAL/Y, and the employment ratio v = employment/labor force = 1 – the unemployment rate (expressed in decimals).

Appendix B

The modified Goodwin model used in this paper was summarized by the following non-linear system of equations reproduced from the text. The parameter values used to generate the data are listed below the equations, and small random shocks were added to the equations during the simulation run.

7) $u = w/y$ [u = wage share = real wage/labor productivity]
8) $v = Y/yAN$ [v = employment ratio = real output/(labor productivityAlabor force]
9) $\ln(y_t) = \ln(y_{t-1}) + \alpha$ [constant rate of growth of labor productivity]
10) $\ln(N_t) = \ln(N_{t-1}) + \beta$ [constant rate of growth of the labor force]
11) $\ln(w_t) = \ln(w_{t-1}) - \gamma + \rho A v_{t-1} - \rho_1 A (u_{t-1})^2$ [real wage growth function]
12) $\ln(Y_t) = \ln(Y_{t-1}) + sARA(1-u_{t-1})$ [output growth rate = savings rate A profit rate]

$\alpha = 0.02$, $\beta = 0.02$, $\gamma = 0.10$, $\rho = 0.335$, $\rho_1 = 0.28$, $s = 0.25$, $R = 1$.

The approximations to the Solow Residual A_t over the interval 1949-2000 were developed through a Fourier series of the form

$A_t' = a_0 + 3a_i Acos(n_i A\theta + b_i)$

where θ = a variable constructed to range between -2π and 2π over the interval 1949-2000.

Data Set A			Data Set A		
$a_0 = 0.0167$			$a_0 = 0.0132$		
$a_1 = 0.0040$	$n_1 = 9$	$b_1 = -3.6426$	$a_1 = 0.0042$	$n_1 = 2$	$b_1 = 0.0077$
$a_2 = -0.0072$	$n_2 = 10$	$b_2 = -0.7974$	$a_2 = -0.0037$	$n_2 = 4$	$b_2 = 0.0525$
$a_3 = -.0061$	$n_3 = 11$	$b_3 = 1.1093$	$a_3 = 0.0069$	$n_3 = 6$	$b_3 = 16.843$
$a_4 = 0.0039$	$n_4 = 13$	$b_4 = -0.7849$	$a_4 = 0.0032$	$n_4 = 7$	$b_4 = -15.383$
$a_5 = 0.0037$	$n_5 = 14$	$b_5 = 0$	$a_5 = -0.0037$	$n_5 = 8$	$b_5 = 8.768$
$a_6 = 0.0073$	$n_6 = 15$	$b_6 = -0.5724$	$a_6 = 0.0035$	$n_6 = 9$	$b_6 = -1.798$
$a_7 = 0.0107$	$n_7 = 16$	$b_7 = -0.3630$			
$a_8 = -0.0081$	$n_8 = 18$	$b_8 = -0.8962$			
$a_9 = 0.0067$	$n_9 = 19$	$b_9 = 1.6030$			
$a_{10} = 0.0035$	$n_{10} = 20$	$b_{10} = 0.9249$			
Adj. R^2 = 0.801, D.W. = 1.234			Adj. R^2 = 0.299, D.W. = 2.002		

Finally, the simulated data was labeled set A and the actual labeled set B.

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