Cross-Range Error in the Lambert Scheme

Syed Arif Kamal[*]

University of Karachi, Karachi, Pakistan

Abstract

The determination of an orbit, having a specified transfer time (time-of-flight) and connecting two position vectors, frequently referred to as the Lambert problem, is fundamental in astrodynamics. Of the many techniques existing for solving this two-body, two-point, time-constrained orbital boundary-value problem, Gauss' and Lagrange's methods are combined to obtain an elegant algorithm based on Battin's work. This algorithm includes detection of cross-range error. A variable *TYPE*, introduced in the transfer-time equation, is flipped, to generate the inverse-Lambert scheme. The Lambert scheme could be useful in steering a satellite-launch vehicle (SLV) as well as constructing the control system of a passenger craft traveling in a ballistic trajectory.

Keywords: Lambert scheme, inverse-Lambert scheme, two-body problem, transfer-time equation, orbital boundary-value problem

---

Nomenclature

a) Symbols (in alphabetical order)

a	Semi-major axis of the ellipse
a_m	Semi-major axis of the minimum-energy orbit
c	Length of cord of the arc connecting the points corresponding to radial coördinates r_1 and r_2
e	Eccentricity of the ellipse
E	Eccentric anomaly
E_1	Eccentric anomaly corresponding to the launch point
E_2	Eccentric anomaly corresponding to the final destination
f	True anomaly
G	Universal constant of gravitation
m	Mass of spacecraft
M	Mass of earth
n	Unit vector, indicating normal to the trajectory plane
p	Parameter of the orbit
g	Flight-path angle
r	Radial coordinate
r_1	Radial coördinate of launch point
r_2	Radial coördinate of final destination
r	Radius vector in the inertial coördinate system

r_2	Radius vector of final destination
t	Universal time
t_1	Launch time
t_2	Time of reaching final destination
TYPE	Variable expressing direction of motion of spacecraft relative to earth's rotation
v	Velocity vector in the inertial coördinate System
m	Product of universal constant of gravitation and sum of masses of spacecraft and earth
t	Time of pericenter passage
q	Transfer angle
f	Angle of inclination of velocity vector and normal to the plane

Astrodynamical terminologies and relationships are given in Appendices A and B, respectively. Appendix C contains a proof of the relationship connecting true and eccentric anomalies, providing a justification of positive sign in front of the radical.

b) Compact Notations

In order to simplify the entries, $\in = \sqrt{1-e^2}$, $\ni = \sqrt{\dfrac{1-e}{1+e}}$, $m = G\,(m + M)$, are used in the expressions and equations.

[*] Convener, National Curriculum Revision Committee (Mathematics), Higher Education Commission; Chairman, Department of Mathematics and Inderdepartmental Faculty, Departments of Physics and Computer Science, Institute of Space and Planetary Astrophysics (ISPA), University of Karachi, Visiting Faculty, Department of Aeronautics and Astronautics, Institute of Space Technology, Islamabad; *paper mail:* Box 8406, Karachi 75270; *e-mail:* kamal(at the rate of)ngds-ku.org; *homepage:* http://www.ngds-ku.org/kamal. Member AIAA

1. Introduction

Determination of trajectory is an important problem in astrodynamics. For a spacecraft moving under the influence of gravitational field of earth in free space (no air drag) the trajectory is an ellipse with the center of earth lying at one of the foci of the ellipse. This constitutes a standard two-body-central-force problem, which has been treated, in detail, in many standard textbooks [1, 2]. The trick is to first reduce the problem to two dimensions by showing that the trajectory always lies in a plane perpendicular to the angular-momentum vector. Then the problem is set up in plane-polar coordinates. Angular momentum is conserved and the problem, effectively, reduces to one-dimensional problem involving only the variable r [3].

A problem famous in astrodynamics, called "the Lambert Problem", is based on the Lambert theorem [4, 5]. According to this theorem the orbital-transfer time depends only upon the semi-major axis, the sum of the distances of the initial and the final points of the arc from the center of force as well as length of the line segment joining these points. Based on this theorem a problem called the Lambert problem is formulated. This problem deals with determination of an orbit having a specified flight-time and connecting the two position vectors. Battin [6, 7] has set up the Lambert problem involving computation of a single hypergeometric function. Since transfer time (time-of-flight) computation is done on-board, it is desirable to use an algorithm involving as few computation steps as possible. The use of polynomials instead of actual expression and reduction of the number of degrees of freedom contribute towards the same goal.

In this paper an elegant Lambert algorithm, presented by Battin, is scrutinized and omissions/oversights in his calculations pointed out. Battin's formulation, which highlighted the main principles involved, was developed and expanded to present a set of formulae suitable for coding in the assembly language to be used as a practical scheme outside the atmosphere for steering the satellite-launch vehicle (SLV). These formulae may be used to compute the velocity and the flight-path angle required at any intermediate time to be compared with the initial velocity and flight-path angle of the spacecraft. A spacecraft cannot reach the desired location if cross-range error is present. Battin's original work does not address this issue. In this paper a mathematical formulation is given to detect cross-range error. Algorithms have been developed and tested,

which indicate cross-range error. In order to correct cross-range error velocity vector should be perpendicular to normal to the desired trajectory (*i. e.*, the velocity must lie, entirely, in the desired trajectory plane).

A variable *TYPE* is introduced in the transfer-time equation to incorporate direction of motion of the spacecraft. This variable can take on 2 values, +1 (for spacecrafts moving in the direction of rotation of earth) and −1 (for spacecrafts moving opposite to the direction of rotation of earth). In the inverse-Lambert scheme, *TYPE* is flipped, whereas all other parameters remain the same.

For an efficient trajectory choice, a transfer time close to minimum-energy orbital transfer time should be selected. The paper highlights a procedure for finding the minimum-energy orbital transfer time. Additionally, formulae are given to compute the orbital para-meters in which the SLV must be locked in at a certain position, at a time, *t*, based on the Lambert scheme.

2. Statement of the Problem

In order to choose a particular trajectory on which the spacecraft could be locked so as to reach a certain point one must select a certain parameter to fix this trajectory out of the many possible ones connecting the two points. The parameter to be chosen in the Lambert problem is transfer time, which is fixed between the two points. One is, therefore, interested to put the Kepler equation in such a form so as to make it computationally efficient utilizing hypergeometric functions or quadratic functions instead of circular functions (sine or cosine, etc.). This equation should express the transfer time between these two points in terms of a series or a polynomial, and another formula should be available to compute velocity vector (speed and flight-path angle), corresponding to this transfer time. Velocity desirable for a particular trajectory may, then, be computed on-board using this formula and compared with velocity of the spacecraft obtained from integration of acceleration information, which is available from on-board accelerometers and rate gyroscopes.

3. The Lambert Theorem

In 1761 Johann Heinrich Lambert, using a geometrical argument, demonstrated that the time taken to traverse any arc (now called transfer time), $t_2 - t_1$, is a function, only, of the major axis, a, the

sum of radii vectors to the end of the specified arc, $(r_1 + r_2)$, and length of chord of the arc, c, for elliptical orbits, i. e.,

$$t_2 - t_1 = f(r_1 + r_2, c, a)$$

The symbol, f, is used to express functional relationship in the above equation (do not confuse with true anomaly). Therefore, one notes that the transfer time does not depend on true or eccentric anomalies of the launch point or the final destination. Fig. 1 illustrates geometry of the problem. Mathematically, transfer time for an elliptical trajectory may be shown to be [6]

$$\sqrt{\frac{m}{a^3}}(t_2 - t_1) = (a - \sin a) - (b - \sin b)$$

where, $\sin^2 \frac{a}{2} = \frac{r_1 + r_2 - c}{4a}$, $\sin^2 \frac{b}{2} = \frac{r_1 + r_2 + c}{4a}$.

In our case, $m = G(m + M) \cong GM$, because $m \ll M$. Based on this theorem a formulation to calculate transfer time and velocity vector at any instant during the boost phase is developed. This formulation is termed as the Lambert scheme.

4. The Lambert Scheme

Suppose a particular elliptical trajectory is connecting the points P_1 and P_2. Let t_1 and t_2 be the times, when the spacecraft passes the points P_1 and P_2, respectively, the radial coördinates being r_1 and r_2. The standard Kepler equation (t is time of pericenter passage)

$$\sqrt{m}(t - t) = a^{3/2}(E - e\sin E) \qquad (1)$$

may be expressed as

$$\sqrt{m}(t_2 - t_1) = 2a^{3/2}(y - \sin y \cos c) \qquad (2)$$

where $y = \frac{1}{2}(E_2 - E_1)$, $\cos c = e\cos \frac{1}{2}(E_2 + E_1)$.

This equation may be used to calculate transfer time between 2 points by iterative procedure. However, it is unsuitable for on-board computation, because computing time is large owing to the presence of circular functions. In order to put this in a form involving power series, one introduces

$$2a_m = s = \frac{1}{2}(r_1 + r_2 + c) \qquad (3)$$

$$\Lambda s = \sqrt{r_1 r_2}\cos\frac{q}{2} \qquad (4)$$

where a_m is the semi-major axis of the minimum-

energy orbit and q the transfer angle. From the geometry, one has

$$c = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos q} \qquad (5)$$

$$y = \sqrt{1 - \Lambda^2(1 - x^2)} \qquad (6)$$

where $x = \cos\frac{1}{2}(y + c)$. Further, introducing

$$h = y - \Lambda x \qquad (7)$$

$$S_1 = \frac{1}{2}(1 - \Lambda - xh) \qquad (8)$$

and a Q function expressible in terms of a hypergeometric function

$$Q = \frac{4}{3}F(3,1;\frac{5}{2};S_1) \qquad (9)$$

Transfer time may be expressed as

$$\sqrt{\frac{m}{a_m^3}}(t_2 - t_1) = h^3 Q + 4\Lambda h \qquad (10)$$

This involves a hypergeometric function instead of a circular function, and hence it is easy to evaluate. The magnitude of velocity, v, and the flight-path angle, g, may be evaluated using the expressions [4, 6]

$$v = \frac{1}{h}\sqrt{\frac{m}{a_m}}\sqrt{[\frac{2\Lambda a_m}{r_1} - (\Lambda + xh)]^2 + \frac{r_2}{r_1}\sin^2\frac{q}{2}} \qquad (11a)$$

$$\cos^2 g = \frac{r_2(1 - \cos q)}{2r_1 v^2} \qquad (11b)$$

The hypergeometric function in equation (9) is given by the continued-fraction expression (this expression is needed to reduce on-board computing time)

$$F(3,1;\frac{5}{2};S_1) = \cfrac{3}{3 - \cfrac{18S_1}{5 + \cfrac{6S_1}{7 - \cfrac{40S_1}{9 - \cfrac{4S_1}{11 - \cfrac{70S_1}{13 - \cfrac{18S_1}{15 - \cfrac{108S_1}{17 - \ldots}}}}}}}}$$

About 100 terms are needed to get an accuracy of 10^{-4}.

To compute the transfer time corresponding to the minimum energy orbit connecting the current position and the final destination one uses the transfer-time equation, with $x = 0$, corresponding to $a = a_m$ and solves it using Newton-Raphson method. Transfer time in the Lambert algorithm must be set close to this time. Fig. 2 shows the flow chart of Lambert algorithm.

5. Cross-Range-Error Detection (Mathematical Model)

For no cross-range error, velocity of the space-craft must lie in the plane containing $r \times r_2$ (the trajectory plane). In other words, the velocity vector must make an angle of 90^0 with normal to the trajectory plane. This is equivalent to [8]

$$v \cdot (r \times r_2) = 0 \qquad (13)$$

which says that v (current velocity), r (current position) and r_2 (position of destination) are co-planar. For no cross-range error, the angle of inclination, f, between the velocity vector, v, and normal to the trajectory, n, must be 90^0. In the Lambert scheme a subroutine computes deviation of f from 90^0. Extended-cross-product steering [9] and dot-product steering [10] could be used to eliminate cross-range error.

6. The Inverse-Lambert Scheme

Transfer-time equation between two points having eccentric anomalies E_1 and E_2, (corresponding to times t_1 and t_2, respectively) may be expressed as [11-13]

$$t_2 - t_1 =$$
$$\sqrt{\frac{a^3}{m}}[(E_2 - e\sin E_2) - (E_1 - e\sin E_1)][TYPE] \qquad (14)$$

The variable $TYPE$ has to be introduced because the Kepler equation is derived on the assumption that t increases with the increase in f. Therefore, the difference

$$[(E_2 - e\sin E_2) - (E_1 - e\sin E_1)]$$

shall come out to be negative for spacecrafts orbiting in a sense opposite to rotation of earth. The variable $TYPE$ ensures that the transfer time (which is the physical time) remains positive in all situations by adapting the convention that $TYPE = +1$ for spacecrafts moving in the direction of earth's rotation, whereas, $TYPE = -1$ for space-crafts moving opposite to the direction of earth's rotation. This becomes important in computing correct flight-path angles in Lambert scheme.
If one wants to put another spacecraft in the same orbit as the original spacecraft, but moving in the opposite direction, one can use the inverse-Lambert scheme. This can be accomplished by

flipping sign of the variable, $TYPE$, whereas all the other parameters remain the same. Burns and Sherock [14] try to accomplish the same objective by a three-degree-of-freedom interceptor simulation designed to rendezvous with a ballistic target: position and velocity matching, no flipping of $TYPE$. The strategy presented in this paper is simpler: orbit matching, flipping of $TYPE$. The inverse-Q system has, also, been proposed by the author to accomplish the same objective [15]. Do not confuse Q system with the symbol Q introduced in eq. (9).

7. Conclusions

The Lambert scheme is applicable in free space, in the absence of atmospheric drags, for burnout times large as compared to on-board computation time (for example, if the burnout time for a given flight is 18 second and the computation time is 1 second, there may not be enough time to utilize this scheme). This is needed to allow sufficient time for the control decisions to be taken and implemented before the rocket runs out of fuel. Detection of cross-range error is incorporated in this formulation. It is assumed that rocket is fired in the vertical position so as to get out of the atmosphere with minimum expenditure of fuel. Later, in free space this scheme is applied to correct the path of rocket. Since the rocket remains in free space for most of the time, this method may be useful in calculating the desired trajectory.

The Lambert scheme is an explicit scheme, which generates a suitable trajectory under the influence of an inverse-square-central-force law (gravitational field of earth) provided one knows the latitudes and the longitudes of launch point and of destination as well as the transfer time (time spent by the spacecraft to reach destination).

Appendix A: Astrodynamical Terminologies

Down-range error is the error in the range assuming that the vehicle is in the correct plane; cross-range error is the offset of the trajectory from the desired plane. An unwanted pitch movement shall produce down-range error; an unwanted yaw

movement shall produce cross-range error.

For an elliptical orbit true anomaly, f, is the polar angle measured from the major axis ($\angle PFX$ in Fig. 3). Through the point P (current position of spacecraft, $m\overline{PF} = r$, the radial coördinate) erect a perpendicular on the major axis. Q is the intersection of this perpendicular with a circumscribed auxiliary circle about the orbital path. $\angle QOF$ (cf. Fig.3) is called the eccentric anomaly, E.

For this orbit, pericenter, the point on the major axis, which is closest to the force center (point A in Fig. 3), is chosen as the point at which $f = 0$. Apocenter is the opposite point on the major axis, which is farthest from the force center (point A in Fig. 3). The line joining the pericenter and the apocenter is called the line of apsides.

Appendix B: Astrodynamical Relationhips

Some useful relationships among radial coördinate, eccentric anomaly, eccentricity and semi-major axis for an elliptical orbit are listed below:

$$r = a(1 - e\cos E) \qquad (B1a)$$

$$r\cos f = a(\cos E - e) \qquad (B1b)$$

$$r\sin f = a\in \sin E \qquad (B1c)$$

$$\sqrt{r}\cos\frac{f}{2} = \sqrt{a(1-e)}\cos\frac{E}{2} \qquad (B1d)$$

$$\sqrt{r}\sin\frac{f}{2} = \sqrt{a(1+e)}\sin\frac{E}{2} \qquad (B1e)$$

The following may be useful in converting circular functions involving true anomalies to those involving eccentric anomalies and vice versa.

$$\cos f = \frac{\cos E - e}{1 - e\cos E} \qquad (B2a)$$

$$\cos E = \frac{\cos f + e}{1 + e\cos f} \qquad (B2b)$$

$$\sin f = \frac{\in \sin E}{1 - e\cos E} \qquad (B2c)$$

$$\sin E = \frac{\in \sin f}{1 + e\cos f} \qquad (B2d)$$

$$\ni \tan\frac{f}{2} = \tan\frac{E}{2} \qquad (B2e)$$

In Appendix C, the last relation is proved and a justification is given for the positive sign taken in front of the square root appearing in the expression for ∍.

Appendix C: Relation Connecting Eccentric Anomaly to True Anomaly

In Fig. 3, semi-minor axis of the ellipse, b, is related to a by $b = a\in$. Do not confuse the point Q in Fig. 3 with the quantity Q defined in eq. (9).

Using $r = \dfrac{p}{1 + e\cos f}$ and $p = a(1 - e^2)$, one may write, $r(1 + e\cos f) = a(1 - e^2)$. This may, in turn, be written as

$$er\cos f = a(1 - e^2) - r$$

Adding er to both sides and using (B1a) on the right-hand side, one gets

$$r(1 + \cos f) = a(1 - e)(1 + \cos E) \qquad (C1)$$

Subtracting er from both sides and using (B1a) on the right-hand side, one gets

$$r(1 - \cos f) = a(1 + e)(1 - \cos E) \qquad (C2)$$

Dividing (C2) by (C1)

$$\frac{1 - \cos f}{1 + \cos f} = \frac{(1 + e)(1 - \cos E)}{(1 - e)(1 + \cos E)}$$

Using the identities

$$1 - \cos f = 2\sin^2\frac{f}{2}$$

$$1 + \cos f = 2\cos^2\frac{f}{2}$$

with similar results for $(1 - \cos E)$ and $(1 + \cos E)$, one sees that the above equation reduces to

$$\tan^2\frac{f}{2} = \frac{1 + e}{1 - e}\tan^2\frac{E}{2}$$

which implies

$$\tan\frac{f}{2} = \pm\sqrt{\frac{1 + e}{1 - e}}\tan\frac{E}{2}$$

Below, it is justified that only positive sign with the radical gives the correct answer. Consider Δ ORF (cf. Fig. 3). One notes that,

$$-p \le E \le p \Rightarrow -\frac{p}{2} \le \frac{E}{2} \le \frac{p}{2}$$

Further, $E \ge 0 \Rightarrow f \ge 0$; $E < 0 \Rightarrow f < 0$.

Therefore, $\dfrac{f}{2}$ and $\dfrac{E}{2}$ have the same sign. When

$-\dfrac{p}{2} \le \dfrac{E}{2} < 0$, $\tan\dfrac{E}{2} < 0$, $\tan\dfrac{f}{2} < 0$, which implies that positive sign with the radical should be chosen. Similarly,

$$0 \leq \frac{E}{2} \leq \frac{p}{2} \; , \; \tan\frac{E}{2} \geq 0 \; , \; \tan\frac{f}{2} \geq 0$$

and, hence, positive sign with the radical is the correct choice.

References

[1]   Goldstein H. *Classical Mechanics*, 2nd ed., Addison Wesley, Reading, Ma., 1981; pp. 70-102

[2]   Marion JB. *Classical Dynamics of Particles and Systems*, 2nd ed., Academic Press, New York, 1970; pp. 243-278

[3]   Wu JT. Orbit determination by solving for gravity parameters with multiple arc data. *Journal of Guidance, Control and Dynamics* 1992; **15 (2)**: 304-313

[4]   Battin RH. *An Introduction to the Mathematics and the Methods of Astrodynamics*, AIAA, Washington, DC, 1987; pp. 276-342

[5]   Deusch R. *Orbital Dynamics of Space Vehicles*, Prentice Hall, Englewood Cliffs, New Jersey, USA, 1963; pp. 20-22

[6]   Battin RH. Lambert's problem revisited. *AIAA Journal*, 1977; **15 (5)**: 707-713

[7]   Battin RH, Vaughan RM. An elegant Lambert algorithm. *Journal of Guidance and Control*, 1984; **7 (6)**: 662-670

[8]   Kamal SA. Ellipse-orientation steering: a control law for spacecrafts and satellite-launch vehicles (SLV). *Space Science and Challenges of the Twenty-First Century*, ISPA-SUPARCO Collaborative Seminar (in connection with World Space Week), University of Karachi, 2005 (invited lecture)

[9]   Kamal SA. Incompleteness of cross-product steering and a mathematical formulation of extended-cross-product steering. *Proceedings of the First International Bhurban Conference on Applied Sciences and Technologies*, National Center for Physics, Bhurban, 2002; pp 167-177

[10] Kamal SA. Dot-product steering: a new control law for satellites and spacecrafts. *Proceedings of the First International Bhurban Conference on Applied Sciences and Technologies*, National Center for Physics, Bhurban, 2002; pp 178-184

[11] Kamal SA. Planetary-orbit modeling based on astrodynamical coördinates. *The Pakistan Institute of Physics International Conference*, Lahore, 1997; pp 28-29

[12] Kamal SA. Use of astrodynamical coördinates to study bounded-keplarian motion. *The Fifth International Pure Mathematics Conference*, the Quaid-é-Azam University & the Pakistan Mathematical Society, Islamabad, 2004

[13] Kamal SA. Strong Noether's theorem: applications in astrodynamics. *The Second Symposium on Cosmology, Astrophysics and Relativity*, Center for Advanced Mathematics and Physics, NUST & National Center for Physics, Rawalpindi, 2004; p 2

[14] Burns SP, Scherock JJ. Lambert-guidance routine designed to match position and velocity of ballistic target. *Journal of Guidance, Control and Dynamics* 2004; 27 (6): 989-996

[15] Kamal SA, Mirza A. The multi-stage-Q System and the inverse-Q system for possible application in satellite-launch vehicle (SLV). *The Fourth International Bhurban Conference on Applied Sciences and Technologies*, National Center for Physics, Bhurban, 2005; pp 5
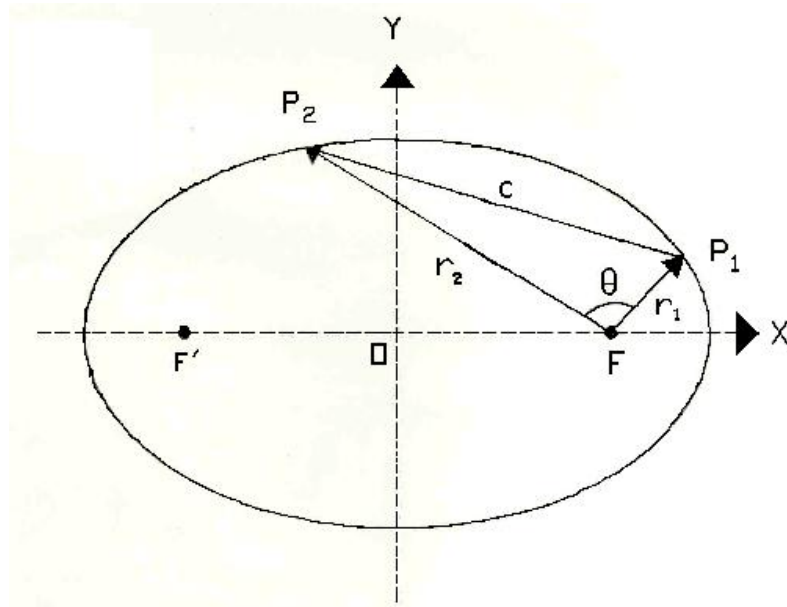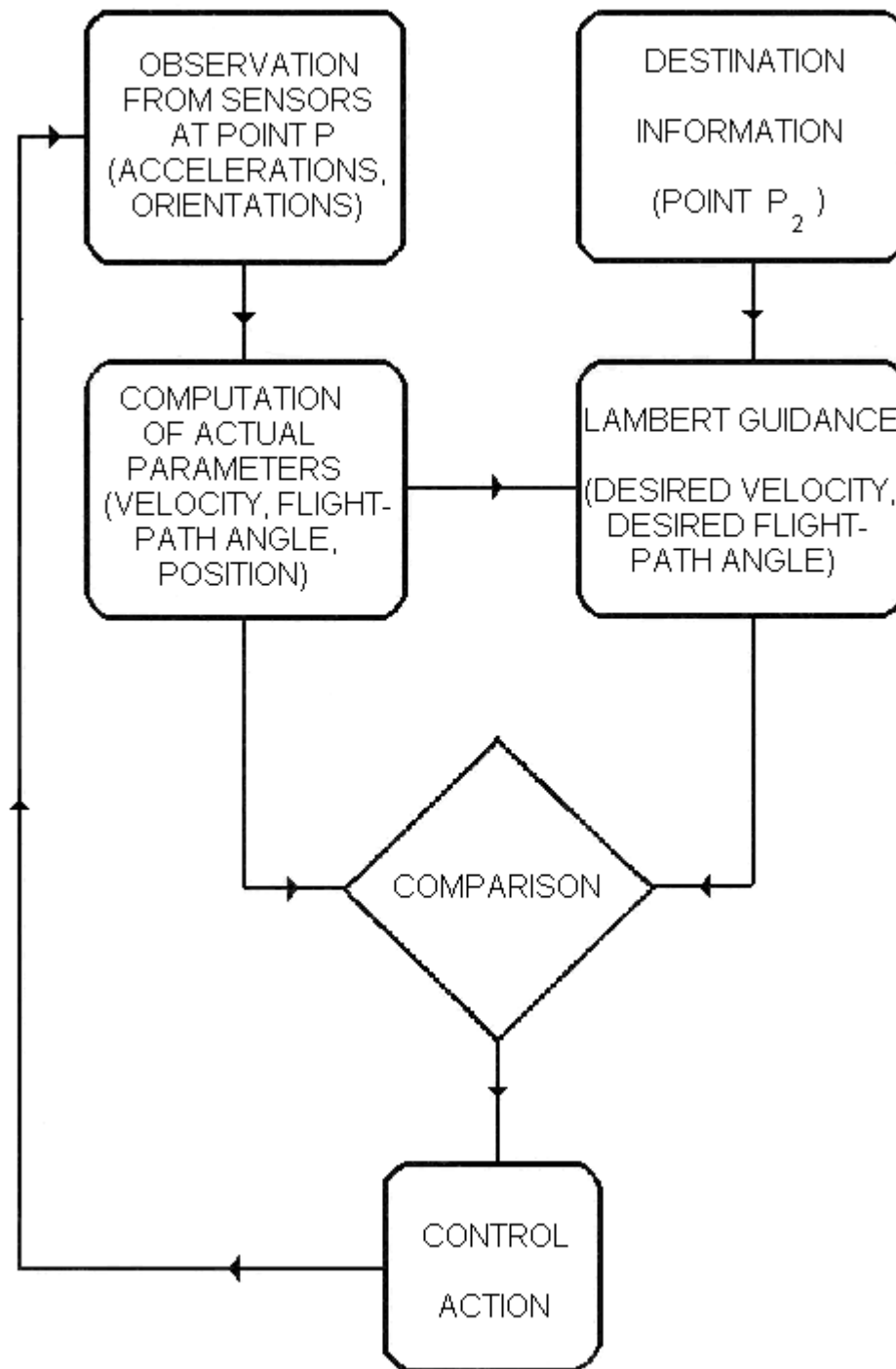
Fig. 1 Geometry of the boundary-value problem
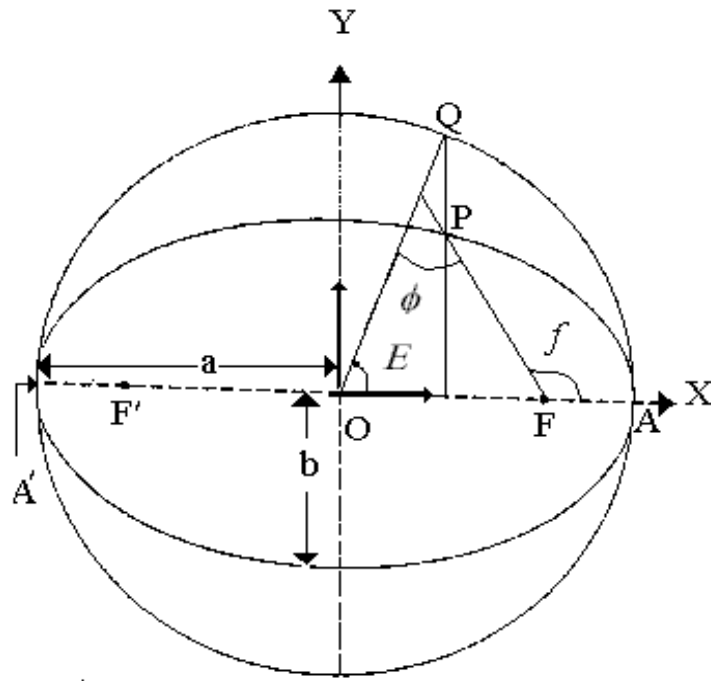
Fig. 2 Flow chart of the Lambert scheme

Fig. 3 Justification of the positive sign in '