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Incompleteness of Cross-Product Steering and a Mathematical Formulation of Extended-Cross-Product Steering

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Abstract

Cross-product steering, as presented by Battin, is incomplete and cannot achieve the desired results. A further condition on the magnitude of rate of change of velocity is needed to bring the spacecraft in the desired orbit. The new control law is named as *extended-cross-product steering*, which incorporates this additional condition. Mathematical representation using elliptic-astrodynamical-coördinate mesh is presented.

Nomenclature

a) Symbols (in alphabetical order)

Symbol	*Description*
x	Generalized coördinate describing shape of ellipse
m_R	Reduced mass of the two-body system
a	Semi-major axis of the ellipse

(continued on the next page)

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Symbol	*Description*
$\hat{\boldsymbol{a}}_N$	Unit vector normal to the trajectory plane (in the direction of relative angular momentum of the two-body system)
$\hat{\boldsymbol{a}}_{para}$	Unit vector pointing parallel to the positive sense of semi-major axis
$\hat{\boldsymbol{a}}_{perp}$	Unit vector determined by $\hat{\boldsymbol{a}}_{para} \times \hat{\boldsymbol{a}}_{perp} = \hat{\boldsymbol{a}}_N$
b	Semi-minor axis of the ellipse
c	Distance of focus from center of the ellipse
c_0	Speed of light in free space
e	Eccentricity of the ellipse
$\hat{\boldsymbol{e}}_N$	Re-labeling of vector $\hat{\boldsymbol{a}}_N$
$\hat{\boldsymbol{e}}_{para}$	Unit vector tangent to the trajectory curve, pointing in the direction of motion of spacecraft (direction of this vector is different from $\hat{\boldsymbol{e}}_E$)
$\hat{\boldsymbol{e}}_{perp}$	Unit vector, normal in the trajectory plane, determined by $\hat{\boldsymbol{e}}_{para} \times \hat{\boldsymbol{e}}_{perp} = \hat{\boldsymbol{e}}_N$
$\hat{\boldsymbol{e}}_x$	Unit vector in the direction of increasing x coödinate
$\hat{\boldsymbol{e}}_y$	Unit vector in the direction of increasing y coödinate
$\hat{\boldsymbol{e}}_z$	Unit vector in the direction of increasing z coödinate
$\hat{\boldsymbol{e}}_f$	Unit vector in the direction of increasing true anomaly, f
$\hat{\boldsymbol{e}}_x$	Unit vector in the direction of increasing elliptical-shape coödinate, x
$\hat{\boldsymbol{e}}_E$	Unit vector in the direction of increasing eccentric anomaly
E	Eccentric anomaly
$\mathcal{E}$	Energy of the system
f	True anomaly
G	Universal constant of gravitation
H	Hamiltonian of the system
l	Relative angular momentum of the two-body system
L	Lagrangian of the system
m	Mass of the lighter body
M	Mass of the heavier body
$\boldsymbol{p}$	Parameter of the orbit (semi-latus rectum of the ellipse)
p_E	Canonical momentum corresponding to eccentric anomaly, E
p_x	Canonical momentum corresponding to elliptical-shape coödinate, x
r	Radial coordinate
$\boldsymbol{r}$	Radius vector in the inertial coördinate system
$\boldsymbol{r}_2$	Radius vector of desired location
t	Universal time
TYPE	Variable expressing direction of motion of spacecraft relative to earth rotation
$\boldsymbol{v}$	Velocity vector in the inertial coördinate system
$\boldsymbol{v}_{para}$	Velocity vector in the inertial coördinate system parallel to the desired trajectory
$\boldsymbol{v}_{perp}$	Velocity vector in the inertial coördinate system in a plane normal to the desired trajectory
$\boldsymbol{v}_{\boldsymbol{P}}$	Component of $\boldsymbol{v}_{perp}$ in the plane of trajectory (normal in the trajectory plane)
$\boldsymbol{v}_{\boldsymbol{N}}$	Component of $\boldsymbol{v}_{perp}$ normal to the plane of trajectory
x	x coordinate in the inertial system
y	y coordinate in the inertial system
z	z coordinate in the inertial system

b) Compact Notations

In order to simplify the entries,

$$\in \; = \sqrt{1-\boldsymbol{e}^2}\,,\; \ni \; = \sqrt{\frac{1-\boldsymbol{e}}{1+\boldsymbol{e}}}\,,\; m = G\,(m+M)$$

are used in the expressions. A dot above any variable denotes time rate of change. For example, $\dot{E}$ means dE/dt. A double dot means second derivative with respect to time, $\ddot{E}$ represents d^2E/dt^2.

c) Coördinate Systems

The geocentric-inertial-coördinate system $O(\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z)$ is a right-handed cartesian coördinate system fixed at a certain instant $t = t_1$ with the z axis coinciding with the axis of earth, the positive x axis directed from the center of earth towards a point on the surface of earth at the intersection of the equator and the meridian.

The ellipse-based-inertial-coördinate system $O(\hat{\boldsymbol{a}}_{para}, \hat{\boldsymbol{a}}_{perp}, \hat{\boldsymbol{a}}_N)$ is a right-handed coördinate system with origin at the center of ellipse. The positive senses of major and minor axes are determined by the convention that $\hat{\boldsymbol{a}}_{para.} \times \hat{\boldsymbol{a}}_{perp}$ points in the direction of relative angular momentum of the two-body system.

The trajectory-based-noninertial-coördinate system $O(\hat{\boldsymbol{e}}_{para}, \hat{\boldsymbol{e}}_{perp}, \hat{\boldsymbol{e}}_N)$ is a right-handed body coördinate system. The positive sense of $\hat{\boldsymbol{e}}_{perp}$ is determined by the convention that $\hat{\boldsymbol{e}}_{para.} \times \hat{\boldsymbol{e}}_{perp}$ points in the direction of relative angular momentum of the two-body system.

The elliptic-astrodynamical-noninertial-coördinate system $O(\hat{\boldsymbol{e}}_E, \hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_N)$ is described in the paper with drawing (Fig. 1).

The cylindrical--noninertial-coördinate system $O(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_f, \hat{\boldsymbol{e}}_N)$ is the standard coördinate system used in two-body problem.

Introduction

Spacecraft dynamics is involved with correct and timely answers of questions like, where the spacecraft is currently located in space ((*navigation*), in which orbit the spacecraft is desired to be (*gudance*), and, what action is needed to bring the spacecraft to the desired orbit (*control action*). If a spacecraft, or, a satellite is not in its proper orbit, it would not serve its purpose. Hence, it is very important to take the spacecraft to the desired orbit and keep it there, for the entire duration of its flight.

In order to accomplish this, one may need to employ control systems. In the *open-loop control system*, one does not check the output against the reference after taking the control action. Regular of a fan may be cited as an example of such a system. In the *closed-loop control system*, one does check the output against the reference after taking the control action. Voltage stabilizer is a good example of such a system.

Control laws are needed, on the basis of which autopilots are designed. It must be borne in mind that every control law is valid under certain conditions. It is not possible to devise a universal control law.

In this paper validity of a control law, *cross-product steering*, is discussed. The flight of a spacecraft may be considered, mathematically, as a two-point, fixed-transfer-time (fixed-time-of-flight), boundary-value problem. In the Q system a correlated spacecraft is supposed to be following the reference trajectory, having the same transfer time. From the current location of actual spacecraft to the corresponding position in the trajectory of correlated spacecraft, a vector is constructed, which is termed as *velocity-to-be-gained*. This control law is used to drive *velocity-to-be-gained* vector to zero at the end of flight. It is shown that the definition of *cross-product steering* is incomplete and needs an additional condition.

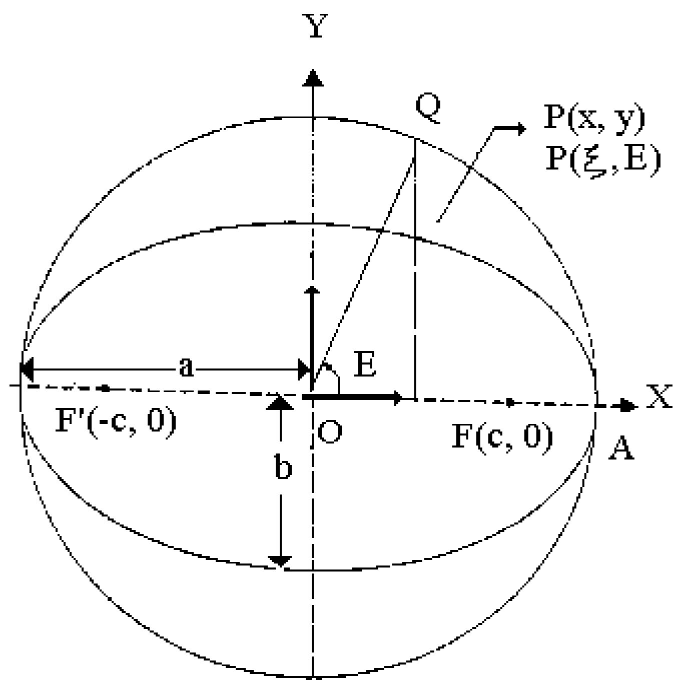

Fig. 1. The elliptic-astrodynamical-coördinate mesh

The Elliptic-Astrodynamical-Coördinate Mesh

Two-body, central force motion is, generally, presented in the plane-polar coördinates, with the polar angle termed as the *true anomaly*, *f*. Combined with the *z* coördinate this represents a cylindrical-coördinate mesh (*r*, *f*, z). Although simple enough, this is unfortunately, not the optimum choice for the bounded keplarian motion, as the orbits are ellipses, in general. The elliptic-astrodynamical-coördinate mesh (Fig. 1), (*x*, *E*, *z*), is adapted from the elliptic-cylindrical-coödinate mesh, well known in the literature. x ¾ a generalized coördinate describing the shape of ellipse ¾ is a function of a (semi-major axis of the elliptical trajectory) and e (eccentricity). E is the eccentric anomaly and z is same as the z coördinate in the cartesian mesh. For an elliptic orbit, x = constant. The lagrangian and the hamiltonian are, therefore, functions of a single variable, E. Appendix A lists coördinate transformations and Appendix B lists transformation of unit vectors for the cartesian-, the cylindrical- and the elliptic-astrodynamical-coördinate meshes.

Lagrangian and Hamiltonian Formulation

Taking the elliptic-astrodynamical coördinates as generalized coördinates, the expressions for lagrangian and hamiltonian are obtained using the following general results, valid for two-body central force motion:

(1*a*) $$L = \tfrac{1}{2} m_R \left|\frac{d\mathbf{r}}{dt}\right|^2 - U|\mathbf{r}|$$

(2*a*) $$H = \sum p_j \dot{q}_j - L$$

If the force law takes the form, $U(|\mathbf{r}|) = -\dfrac{GmM}{|\mathbf{r}|}$, the expression for lagrangian becomes

(1*b*) $$L = \tfrac{1}{2} m_R \left(\dot{r}^2 + r^2 \dot{\theta}^2\right) + \frac{GmM}{r}$$

Applying the transformations ¾ cylindrical to elliptic-astrodynamical coördinates ¾ and rearranging, the above may be written as

(1*c*) $$L = \frac{mMa^2(1-e^2\cos^2 E)}{2(m+M)}\dot{E}^2 + \frac{GmM}{a(1-e\cos E)}$$

The canonical momenta, p_E and p_x may be obtained from this lagrangian,

(3*a*, *b*) $$p_E = \frac{\partial L}{\partial \dot{E}} = \frac{mMa^2(1-e^2\cos^2 E)}{m+M}\dot{E}, \quad p_x = \frac{\partial L}{\partial \dot{x}} = 0$$

The hamiltonian, therefore, may be obtained as

(2*b*) $$H = p_E \dot{E} + p_x \dot{x} - L$$

or, in terms of the elliptic-astrodynamical-coördinate mesh

(2*c*) $$H = \frac{1}{a(1-e\cos E)}\left[\frac{(m+M)\,p_E^2}{2mMa(1+e\cos E)} - GmM\right]$$

a) Constants of Motion

Examining Eq. (2*c*), one notes that the hamiltonian does not contain time, explicitly. Therefore [1, 2]

(4) $$0 = \frac{\partial H}{\partial t} = \frac{dH}{dt}$$

Also, the transformation equations (see Appendix A) do not contain time, explicitly. Therefore, $H = E$, energy of the system. Hence, the first constant of motion is found to be

E (energy of the system)

Since, $\dot{p}_x = 0$, the other constant of motion is

p_x (canonical momentum corresponding to coördinate x)

Also, $\dot{x} = 0$ on an elliptical trajectory. The third constant of motion is

x (elliptical-shape coördinate)

Recall that there were only two constants of motion in the conventional treatment of two-body problem in the plane-polar coördinates, viz., the total energy, E, and the relative angular momentum, l.

b) Rates of Change of Coördinates and Momenta

Using expressions for lagrangian (1*c*) and hamiltonian (2*c*), the rates may be evaluated

(5*a*) $$\frac{\partial L}{\partial E} = \dot{p}_E = -\frac{\partial H}{\partial E} = \frac{e^2 p_E^2 (m+M)\sin 2E}{2Ma^2(1-e^2\cos^2 E)^2} - \frac{GmMe\sin E}{a(1-e\cos E)^2}$$

(5*b*) $$\frac{\partial L}{\partial x} = \dot{p}_x = -\frac{\partial H}{\partial x} = 0$$

(6*a*) $$\dot{E} = \frac{\partial L}{\partial p_E} = \frac{(m+M)p_E}{mMa^2(1-e^2\cos^2 E)^2}$$

(6*b*) $$\dot{x} = \frac{\partial L}{\partial p_x} = 0$$

c) Equation of Motion

Lagrangian equation for the elliptic-astrodynamical coördinate, *E*, is set up to obtain the equation of motion along $\hat{\boldsymbol{e}}_{E}$,

(7) $$\frac{\partial L}{\partial E} - \frac{d}{dt}\frac{\partial L}{\partial(dE/dt)} = 0$$

Using Equations (3*a*), (5*a*) and rearranging, one obtains

(8) $$(1-e^2\cos^2 E)\ddot{E} + ½(e^2\sin 2E)\dot{E}^2 + \frac{me\sin E}{a^3(1-e\cos E)^2} = 0$$

This is a second-order, inhomogeneous differential equation, whose solution must be Kepler's equation. Using Kepler's equation in the form (t is time for pericenter passage)

(9) $$\sqrt{m}(t-t) = a^{3/2}(E - e\sin E)$$

it has been verified that Eq. (9) is a solution of Eq. (8).

d) Transfer-Time Equation

Transfer-time equation between two points having eccentric anomalies E_1 and E_2, (corresponding to times t_1 and t_2, respectively) may be expressed as

(10) $$t_2 - t_1 = \sqrt{\frac{a^3}{m}}[(E_2 - e\sin E_2) - (E_1 - e\sin E_1)][TYPE]$$

The factor *TYPE* has to be introduced because Kepler's equation is derived on the assumption that ***t*** increases with the increase in ***f***. Therefore, the difference

$$[(E_2 - e\sin E_2) - (E_1 - e\sin E_1)]$$

shall come out to be negative for spacecrafts orbiting in a sense opposite to rotation of earth. The factor *TYPE* ensures that the transfer time (which is the physical time) remains positive in all situations by adapting the convention that $TYPE = +1$ for spacecrafts moving in the direction of earth rotation, whereas, $TYPE = -1$ for spacecrafts moving opposite to the direction of earth rotation. This becomes important in computing correct flight-path angles in Lambert scheme.

Cross-Product Steering

Battin remarks in his book [3]: "If you want to drive a vector to zero, it is sufficient to align the time rate of change of the vector with the vector itself. Therefore, components of the vector cross product

$$\boldsymbol{v}_g \times \frac{d\boldsymbol{v}_g}{dt}$$

could be used as the basic autopilot rate signals — a technique that became known as *cross-product steering* ($\boldsymbol{v}_g$ represents velocity-to-be-gained in the Q system)". However, this definition has a condition missing. The complete definition follows.

a) Extended-Cross-Product Steering

In order to drive a vector to zero, it is sufficient to align the time rate of change of the vector with the vector itself provided the time rate of change of the magnitude of this vector is a monotonically decreasing function. This law may be termed as ***extended-cross-product steering***. Let $\boldsymbol{A}$ be a vector, which needs to be driven to zero. Then, we must have

(11) $$\boldsymbol{A} \times \frac{d\boldsymbol{A}}{dt} \to 0, \frac{d|\boldsymbol{A}|}{dt} < 0$$

This definition may be rewritten using elliptic-astrodynamical-coördinate formulation. One notes that Equations (3***a***, ***b***) show that there is no motion along the $\hat{\boldsymbol{e}}_x$ direction (because $p_x = 0$). On the basis of Eq. (6***b***), one concludes that x = constant. This may, also, be written as

(12) $$0=\frac{dx}{dt}=\frac{\partial x}{\partial x}\frac{dx}{dt}+\frac{\partial x}{\partial y}\frac{dy}{dt}+\frac{\partial x}{\partial z}\frac{dz}{dt}=\nabla x\cdot v$$

which implies that $\boldsymbol{v}$ is perpendicular to ∇x. This is the basis of the following control law.

b) Normal-Component-Cross-Product Steering

In order to bring a vehicle to the desired trajectory one needs to align the normal component of velocity with its time rate of change and make its magnitude a monotonically decreasing function of time. By normal component one means the component of velocity in the plane normal to reference trajectory. This plane passes through a point on the reference trajectory, which is closest to current location of center-of-mass of spacecraft. Mathematically,

(13) $$\boldsymbol{v}_{\boldsymbol{perp}}\times\frac{d\boldsymbol{v}_{perp}}{dt}\rightarrow 0,\frac{d\left|\boldsymbol{v}_{\boldsymbol{perp}}\right|}{dt}<0$$

Therefore, components of the vector

$$\boldsymbol{v}_{\boldsymbol{perp}}\times\frac{d\boldsymbol{v}_{\boldsymbol{perp}}}{dt}$$

should be used as the basic autopilot rate signals, where

(14) $$\boldsymbol{v}_{\boldsymbol{perp}}=\boldsymbol{v}_{\boldsymbol{P}}+\boldsymbol{v}_{\boldsymbol{N}}=v_P\hat{\boldsymbol{e}}_{perp}+v_N\hat{\boldsymbol{e}}_N$$

For elliptic-astrodynamical-coördinate formulation, Eq. (14) takes the form

(15) $$\boldsymbol{v}_{\boldsymbol{perp}}=v_x\hat{\boldsymbol{e}}_x+v_z\hat{\boldsymbol{e}}_z$$

To correct for down-range error, one must have

(16*a*) $$\boldsymbol{v}_x\times\frac{d\boldsymbol{v}_x}{dt}\rightarrow 0,\frac{d\left|\boldsymbol{v}_x\right|}{dt}<0$$

To correct for cross-range error, the following could be used as autopilot rate signals

(16*b*) $$\boldsymbol{v}_z\times\frac{d\boldsymbol{v}_z}{dt}\rightarrow 0,\frac{d\left|\boldsymbol{v}_z\right|}{dt}<0$$

Conclusions

Down-range and cross-range errors need to be eliminated to make the spacecraft reach the desired location. All the components of velocity normal to the desired orbit (trajectory) must be driven to zero, in order to accomplish this goal. The undesired components of velocity, $\boldsymbol{v}_x$, and, $\boldsymbol{v}_z$, must be made to vanish using *extended-cross-product steering* (or, more appropriately, *normal-component-cross-product steering*). The desired component is $\boldsymbol{v}_E$, which is responsible for taking the spacecraft to its pre-assigned location. With some modifications, *extended-cross-product steering* may be used for attitude control of satellites. A continuation of this work is presented elsewhere in this volume [4], which may, also, be used to drive the normal (undesired) components of velocity to zero.

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Appendix A: Transformation of Coördinates

a) Cartesian to Cylindrical and Vice Versa

(A1*a*, *b*) $x = r\cos f\,;$ $\qquad r = \sqrt{x^2 + y^2}$

(A1*c*, *d*) $y = r\sin f\,;$ $\qquad f = \tan^{-1}\dfrac{y}{x}$

(A1*e*, *f*) $z = z\,;$ $\qquad z = z$

b) Cylindrical to Elliptic-Astrodynamical and Vice Versa

(A2*a*, *b*) $r = a(1 - e\cos E)\,;$ $\qquad x = \dfrac{1}{2ae}\ln\dfrac{1+\in}{1-\in}$

(A2*c*, *d*) $f = 2\tan^{-1}(\dfrac{1}{\text{э}}\tan\dfrac{E}{2})\,;$ $\qquad E = 2\tan^{-1}(\text{э}\tan\dfrac{f}{2})$

(A2*e*, *f*) $z = z\,;$ $\qquad z = z$

c) Elliptic-Astrodynamical to Cartesian and Vice Versa

(A3*a*, *b*) $x = \dfrac{1}{2ae}\ln\dfrac{1+\in}{1-\in}\,;$ $\qquad x = a(\cos E - e)$

(A3*c*, *d*) $E = \tan^{-1}\dfrac{y}{\in(x + ae)}\,;$ $\qquad y = a \in \sin E$

(A3*e*, *f*) $z = z\,;$ $\qquad z = z$

Appendix B: Transformation of Unit Vectors

a) Cartesian to Cylindrical and Vice Versa

(B1*a*, *b*) $\hat{e}_x = \cos f \hat{e}_r - \sin f \hat{e}_f\,;\qquad \hat{e}_r = \dfrac{x\hat{e}_x + y\hat{e}_y}{\sqrt{x^2 + y^2}}$

(B1*c*, *d*) $\hat{e}_y = \sin f \hat{e}_r + \cos f \hat{e}_f\,;\qquad \hat{e}_f = \dfrac{-y\hat{e}_x + x\hat{e}_y}{\sqrt{x^2 + y^2}}$

(B1*e*, *f*) $\hat{e}_z = \hat{e}_z\,;\qquad \hat{e}_z = \hat{e}_z$

b) Cylindrical to Elliptic-Astrodynamical and Vice Versa

(B2*a*, *b*) $\hat{e}_r = \dfrac{\sinh(2ae)\hat{e}_x - \sin(2E)\hat{e}_E}{H}\,;\qquad \hat{e}_x = \dfrac{a_0\hat{e}_r - a_1\hat{e}_f}{\sqrt{a_0^2 + a_1^2}}$

(B2*c*, *d*) $\hat{e}_f = \dfrac{\sin(2E)\hat{e}_x + \sinh(2aex)\hat{e}_E}{H}\,;\qquad \hat{e}_E = \dfrac{a_1\hat{e}_r + a_0\hat{e}_f}{\sqrt{a_0^2 + a_1^2}}$

(B2*e*, *f*) $\hat{e}_z = \hat{e}_z\,;\qquad \hat{e}_z = \hat{e}_z$

c) Cartesian to Elliptic-Astrodynamical and Vice Versa

(B3*a*, *b*) $\hat{e}_x = \dfrac{\sinh(aex)\cos E\hat{e}_x - \cosh(aex)\sin E\hat{e}_E}{h}\,;\qquad \hat{e}_x = \dfrac{x\in^2 \hat{e}_x + y\hat{e}_y}{ha\in}$

(B3*c*, *d*) $\hat{e}_y = \dfrac{\cosh(aex)\sin E\hat{e}_x + \sinh(aex)\cos E\hat{e}_E}{h}\,;\qquad \hat{e}_E = \dfrac{-y\hat{e}_x + x\in^2 \hat{e}_y}{ha\in}$

(B3*e*, *f*) $\hat{e}_z = \hat{e}_z\,;\qquad \hat{e}_z = \hat{e}_z$

where,

$$h = \sqrt{(\in/e)^2 + (y/a\in)^2} = \sqrt{\sinh^2(aex) + \sin^2 E}\,,\ H = \sqrt{\sinh^2(2aex) + \sin^2(2E)}\,,$$

$$a_0 = r + ae - e^2(ae + r\cos f)\,,\ a_1 = ae\sin f - e^2\sin f(r\cos f + ae)\,,$$

$$a = \frac{ex + \sqrt{x^2 + y^2}}{\in^2} = \frac{1}{2ex}\ln\frac{1+\in}{1-\in}$$

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