The First International Bhurban Conference on Applied Sciences and Technologies
Bhurban, Pakistan.
June 10-15, 2002, pp. 178-184

Dot-Product Steering
A New Control Law for Satellites and Spacecrafts

Syed Arif Kamal [1]
Departments of Mathematics and Computer Science
The NGDS Pilot Project
University of Karachi
Karachi, Pakistan.

Abstract

A control law is formulated, which employs dot products of velocity and time rate of change of velocity. Mathematical representation using elliptic-astrodynamical-coördinate mesh is presented.

Introduction

The *normal-component-cross-product steering* control law [1] put forward in the other paper may be used to derive another law, which can be used to derive normal components of velocity to zero. This law , termed as, *dot-product steering*, is further developed into another control law, *ellipse-orientation steering*, and conditions are derived to determine and, eventually, eliminate down-range and cross-range errors. This paper is a continuation of [1], and, hence, the list of symbols, compact notations and coördinate systems collected in the *Nomenclature* section applies to calculations presented in this paper, as well.

A good overview of orbital dynamics, needed to understand these calculations, may be found in [2]. The elliptic-cylindrical-coördinate mesh [3, 4] is adapted to deal with the

[1] ***Homepage:*** http://ngds-ku.org/kamal; ***e-mail***: kamal(at the rate of)ngds-ku.org; ***paper mail:*** Dr SA Kamal, Project Director, the NGDS Pilot Project, University of Karachi, University Road, Box 8406, Karachi 75270 (Sindh), PAKISTAN.

bounded keplarian orbits, as elliptic-astrodynamical-coördinate mesh. References [3, 4] illustrate another adaptation of these coördinates ¾ the cardiac-coördinate mesh, which is used to model surface anatomy of the human heart.

In this paper mathematical formulations of *dot-product steering*, *normal-component-dot-product steering* (as the special case of *dot-product steering*) and *ellipse-orientation steering* are presented. In Appendix A, trajectory computed under the assumption of constant g (parabola) is shown to be the limiting case of elliptical trajectory, when its semi-minor axis, b, is very small as compared to its semi-major axis, a. In Appendix B, equation of ellipse written in the form,

$$r = \frac{p}{1 + e\cos f}\,;\;\; 0 \le f \le 180^0$$

is shown to be equivalent to the form, traditionally, recognized:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Dot-Product Steering

Dot-product steering is a control law, which involves dot products of the vector, and its time rate of change. In order to derive a vector $\boldsymbol{A}$ to zero, one needs to derive the factor $|\boldsymbol{A}|\left|\frac{d\boldsymbol{A}}{dt}\right|(1+\cos q)$ to zero, where q is the angle between $\boldsymbol{A}$ and $\frac{d\boldsymbol{A}}{dt}$. In other words

(1) $$\boldsymbol{A}\,.\,\frac{d\boldsymbol{A}}{dt} + |\boldsymbol{A}|\left|\frac{d\boldsymbol{A}}{dt}\right| \to 0$$

In the trajectory problems, it is customary to require the normal component of velocity to vanish. Hence, one may develop a special case of *dot-product steering*.

Normal-Component-Dot-Product Steering

In order to bring a vehicle to the desired trajectory one needs to derive the factor $|\boldsymbol{v}_{perp}|\left|\frac{d\boldsymbol{v}_{perp}}{dt}\right|(1+\cos f)$ to zero, where f is the angle between $\boldsymbol{v}_{perp}$ and $\frac{d\boldsymbol{v}_{perp}}{dt}$. The above condition may be expressed, mathematically, as

(2) $$\boldsymbol{v}_{perp}\,.\,\frac{d\boldsymbol{v}_{perp}}{dt} + |\boldsymbol{v}_{perp}|\left|\frac{d\boldsymbol{v}_{perp}}{dt}\right| \to 0$$

Proof: From *normal-component-cross-product steering*, the condition to drive a vector to zero is vanishing of the product

$$\boldsymbol{v}_{perp} \times \frac{d\boldsymbol{v}_{perp}}{dt}$$

This means that the direction of $\boldsymbol{v}_{perp}$ may not change. However, its magnitude should change. The vector $\boldsymbol{v}_{perp}$ should go to zero if $\frac{d|\boldsymbol{v}_{perp}|}{dt} < 0$. This is possible only if $\frac{d\boldsymbol{v}_{perp}}{dt}$ makes an angle of 180^0 with $\boldsymbol{v}_{perp}$, that is

$$\frac{\boldsymbol{v}_{perp} \cdot \frac{d\boldsymbol{v}_{perp}}{dt}}{|\boldsymbol{v}_{perp}|\left|\frac{d\boldsymbol{v}_{perp}}{dt}\right|} = \cos 180^0 = -1$$

which reduces to

(3) $$\boldsymbol{v}_{perp} \cdot \frac{d\boldsymbol{v}_{perp}}{dt} + |\boldsymbol{v}_{perp}|\left|\frac{d\boldsymbol{v}_{perp}}{dt}\right| = 0$$

This completes the proof of *normal-component-dot-product steering*.

Examples: Examples of rectilinear, circular and elliptical trajectories are worked out to illustrate this control law.

a) **Straight Line:** To simplify the calculations, *x* axis is chosen along the trajectory. $\boldsymbol{v}_{para} = \boldsymbol{v}_x = v_x \hat{\boldsymbol{e}}_x$, $\boldsymbol{v}_{perp} = \boldsymbol{v}_y + \boldsymbol{v}_z = v_y \hat{\boldsymbol{e}}_y + v_z \hat{\boldsymbol{e}}_z$. The conditions for *normal-component-dot-product steering* become

$$\boldsymbol{v}_x \cdot \frac{d\boldsymbol{v}_x}{dt} + |\boldsymbol{v}_x|\left|\frac{d\boldsymbol{v}_x}{dt}\right| \to 0 \ ; \ \boldsymbol{v}_y \cdot \frac{d\boldsymbol{v}_y}{dt} + |\boldsymbol{v}_y|\left|\frac{d\boldsymbol{v}_y}{dt}\right| \to 0$$

b) **Circle:** *xy* plane is chosen so that the circular trajectory lies entirely in it, with center of circle coinciding with the origin of the coördinate system. In cylindrical-coördinate mesh, (r, j, z), the components of velocity are $\boldsymbol{v}_{para} = \boldsymbol{v}_j = v_j \hat{\boldsymbol{e}}_j$, $\boldsymbol{v}_{perp} = \boldsymbol{v}_r + \boldsymbol{v}_z = v_r \hat{\boldsymbol{e}}_r + v_z \hat{\boldsymbol{e}}_z$. Therefore, *normal-component-dot-product steering* law takes the form

$$\boldsymbol{v}_r \cdot \frac{d\boldsymbol{v}_r}{dt} + |\boldsymbol{v}_r|\left|\frac{d\boldsymbol{v}_r}{dt}\right| \to 0; \ \boldsymbol{v}_z \cdot \frac{d\boldsymbol{v}_z}{dt} + |\boldsymbol{v}_z|\left|\frac{d\boldsymbol{v}_z}{dt}\right| \to 0$$

c) **Ellipse:** In order to write this condition for elliptic-astrodynamical-coördinate mesh, one notes that $\boldsymbol{v}_{para} = \boldsymbol{v}_E = v_E \hat{\boldsymbol{e}}_E$, $\boldsymbol{v}_{perp} = \boldsymbol{v}_x + \boldsymbol{v}_z = v_x \hat{\boldsymbol{e}}_x + v_z \hat{\boldsymbol{e}}_z$. One notes that, $\boldsymbol{v}_P = \boldsymbol{v}_x = v_x \hat{\boldsymbol{e}}_x$ shall contribute to down-range error and $\boldsymbol{v}_N = \boldsymbol{v}_z = v_z \hat{\boldsymbol{e}}_z$ to cross-range error. Therefore, the steering law may be expressed as

$$\boldsymbol{v}_x \cdot \frac{d\boldsymbol{v}_x}{dt} + |\boldsymbol{v}_x|\left|\frac{d\boldsymbol{v}_x}{dt}\right| \to 0, \ \boldsymbol{v}_z \cdot \frac{d\boldsymbol{v}_z}{dt} + |\boldsymbol{v}_z|\left|\frac{d\boldsymbol{v}_z}{dt}\right| \to 0$$

Ellipse-Orientation Steering

The condition for no down-range error is

(4*a*) $$(\hat{\boldsymbol{a}}_{para}.\boldsymbol{r} + ae)(\hat{\boldsymbol{a}}_{para}.\boldsymbol{v})(e^2 - 1) = (\hat{\boldsymbol{a}}_{perp}.\boldsymbol{r})(\hat{\boldsymbol{a}}_{perp}.\boldsymbol{v})$$

and the condition for no cross-range error is

(4*b*) $$\hat{\boldsymbol{a}}_N.\boldsymbol{v} = 0$$

These equations describe another control law, termed as e*llipse-orientation steering*.

Proof: Expressing v_x in terms of v_x and v_y

$$v_x = v_x \sinh(aex)\cos E + v_y \cosh(aex)\sin E$$

and substituting the values of

$$\sinh(aex) = \frac{y}{ae\sin E},\ \cosh(aex) = \frac{x+ae}{ae\cos E},\ E = \tan^{-1}\frac{y}{\in(x+ae)}$$

one gets, for no down-range error

$$v_x = \frac{v_x(x+ae)\in^2 + yv_y}{ae\in} \to 0$$

which can be, immediately, generalized to (4*a*). Similarly, for no cross-range error, $v_z \to 0$, which is generalized to (4*b*).

Cross-Range Error Detection

For no cross-range error velocity of the spacecraft must lie in the plane containing $\boldsymbol{r}\times\boldsymbol{r}_2$. In other words

(5) $$\boldsymbol{v}.\boldsymbol{r}\times\boldsymbol{r}_2 = 0$$

In order to detect cross-range error present, v_x is first expressed in terms of *x*, *y*, *z* (body-coördinate mesh), and then in terms of inertial system to be able to obtain a condition to eliminate down-range error. Similarly, v_z is expressed in terms of inertial-coördinate mesh so that conditions may be obtained to eliminate cross-range error.

Down-Range Error and Cross-Range Error Elimination

To eliminate down-range error, the following condition should hold

(6*a*) $$\frac{v_x}{v_y} \propto \tan E$$

and to eliminate cross-range error

(6*b*) $$v_z = 0$$

This can be easily proved using the expressions for v_x, v_E and Eq. (5).

Conclusions

Normal-component-dot-product steering, involves, dot products of normal component of velocity and its time rate of change. It is derived from *extended-cross-product steering*. The formulation presented in [1] and this paper may be useful in satellite dynamics. Using this formulation a satellite-launch vehicle (SLV) may be constructed, which can inject satellites into the desired orbits. Suitable autopilots may be designed for attitude control of the satellites.

It is imperative to develop similar formulations for parabolic and hyperbolic orbits, and compare the results to this formulation.

Acknowledgements

The author would like to express his heartfelt thanks to National Center for Physics, in particular, the Organizing Committee of IBCAST for making my stay at Bhurban enjoyable. This work was made possible, in part, by *Dean's Research Grant* awarded by University of Karachi.

Appendix A: Equivalence of a Projectile Trajectory in Constant Gravity (Parabola) and a Trajectory Computed From Kepler's Equation (Ellipse)

A projectile is a freely falling body. Its trajectory, as determined in elementary physics under the assumption of constant gravity, comes out to be a parabola in free space. The projectile is bound to the earth's gravitational field and, therefore, comes back to the surface of earth. Table 1 shows that the trajectory of a bound projectile (the potential energy larger than the kinetic energy) must be either an ellipse or a circle. A parabolic orbit is possible only when total energy of the projectile is zero. In other words, the potential energy must be numerically equal to the kinetic energy (they have opposite signs). If this condition is satisfied at the surface of earth for a vertical lunch, velocity of the projectile becomes equal to the escape velocity to get out of the earth's gravitational field.

This contradiction may be resolved if one looks at the parabolic trajectory as the limiting case of an elliptical trajectory ¾ the semi-major axis being very large as compared to the semi-minor axis. In this case the eccentricity, ***e***, shall approach unity.

Table 1. Orbits for Two-Body, Central-Force Motion

Energy *(E)*	*Eccentricity* *(e)*	*Shape of the Orbit*	*Type of the Orbit*	*System (Bound/Free)*	*Number of Turning Points*
$E < E_{\min}$	$e < 0$	Not allowed	—	—	—
$E = E_{\min}$	$e = 0$	Circle	Closed	Bound	∞
$E_{\min} < E < 0$	$0 < e < 1$	Ellipse	Closed	Bound	2
$E = 0$	$e = 1$	Parabola	Open	Free	1
$E > 0$	$e > 1$	Hyperbola	Open	Free	1

Table 2. Features of Orbits

Energy (E)	*Features*
$E < E_{min}$	$E_{min} = V(r)\big\vert_{min}$
$E = E_{min}$	$\dot{r} = 0$ everywhere, and r = constant
$E_{min} < E < 0$	r_{min} = pericenter, r_{max} = apocenter, which are the turning points
$E = 0$	r_{min} is the turning point
$E > 0$	r_{min} is the turning point

Mathematically,

$$b^2 = a^2(1-e^2)^2 \Rightarrow e = \sqrt{1-\frac{b^2}{a^2}}$$

which gives,

$$\lim_{a\to\infty} e = \lim_{a\to\infty}\sqrt{1-\frac{b^2}{a^2}} = 1$$

In fact, this is, exactly, what happens in the case of constant g. The assumption of constant g (acceleration due to gravity) implies that the range is very small as compared to the circumference of earth (altitude is small). Since one of the foci lies at the center of earth the semi-major axis is of the order of radius of earth. The semi-minor axis, however, is of the order of range. Therefore, parabolic trajectory is, actually, the limiting case of elliptical trajectory.

It is interesting to note that for a bound system the number of turning points is greater than one, the energy is negative and the orbit is closed.

Appendix B: Equivalence of Ellipse Equations

An equation of ellipse in the cartesian-coördinate mesh with center at (h, k)

(B1) $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is shown to be equivalent to an equation of ellipse in polar coördinates:

(B2) $$r = \frac{p}{1 + e\cos f}\ ;\ 0 \le f \le 180^0$$

To do so one notes that (B2) refers to an ellipse with origin at one of the foci. The coördinates of center must, therefore, be taken as (–*ae*, 0) in (B1), that is *h* = *ae* and *k* = 0. The semi-major axis, *a*, and the semi-minor axis, *b*, are related by $b^2 = a^2(1 - e^2)$. Therefore, perimeter of the orbit ((semi latus rectum of the ellipse), *p*, may be found by noting that it is the value of *r*, when $f = 90^0$ (or, the value of *y*, when *x* = 0). Substituting, *x* = 0, *y* = *p*, in (B1) and rearranging, one gets

(B3) $$p = a(1 - e^2)$$

Substituting, $b^2 = a^2(1 - e^2)$, $x = r\cos f$, $y = r\sin f$, $h = -ae$, $k = 0$, in (B1), and using (B3), one obtains a quadratic equation:

(B4) $$(1 - e^2\cos^2 f)r^2 + (2ep\cos f)r - p^2 = 0$$

whose roots are

$$r_1 = -\frac{p}{1 - e\cos f}, r_2 = \frac{p}{1 + e\cos f}$$

The solution r_1 is unphysical because it may give negative values of *r* (*r*, being the radius vector, is always positive). The solution r_2 is the required form.

References

1. S. A. Kamal, "Incompleteness of Cross-Product Steering and a Mathematical Formulation of Extended-Cross-Product Steering", *Proceedings of the First International Bhurban Conference on Applied Sciences and Technologies*, Bhurban (Pakistan), June 10-15, 2002, pp. 167-177

2. R. Deusch, *Orbital Dynamics of Space Vehicles*, Prentice Hall, Englewood Cliffs, New Jersey, USA (1963)

3. S. A. Kamal & K. A. Siddiqui, "Modeling of Heart Function", *Proceedings of Symposium on Trends in Physics*, Department of Physics, University of Karachi, ed. By K. A. Siddiqui, 1992, pp. 107-112

4. S. A. Kamal & K. A. Siddiqui, "The Human Heart as a System of Standing Waves", *Kar. U. J. Sc.* (to be published)