Standards for Representation in Robotic Systems

Promoting Interoperability amongst Autonomous Intelligent Systems

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Annex A
Derivation of Rotation Matrix from Quaternions

This annex works through the derivation of a Rotation Matrix from a representation with unit quaternions.

(A.1) $$q = (q_0, < q_1 q_2 q_3 >)$$

(A.2) $$q = (q_0, \mathbf{q})$$

(A.3) $$q = q_0 + q_1 i + q_2 j + q_3 k$$

A unit quaternion is defined such that the quaternions must lie on a unitary 4-dimensional hypersphere:

(A.4) $$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

Given a rotation representation in unit quaternions, it is possible to convert this

(A.5) $$x = Rx'$$

(A.6) $$x' = (x'_0, < x'_1 x'_2 x'_3 >)$$

(A.7) $$x' = (0, \mathbf{x}')$$

(A.8) $$\overline{q} = (q_0, < -q_1 - q_2 - q_3 >)$$

(A.9) $$\overline{q} = (q_0, -\mathbf{q})$$

(A.10) $$Rx' = q \circ x' \circ q^{-1} = q \circ x' \circ \overline{q}$$

(A.11) $$q \circ x' = (q_0 x'_0 - \mathbf{q} \bullet \mathbf{x}', q_0 \mathbf{x}' + x'_0 \mathbf{q} + \mathbf{q} \times \mathbf{x}')$$

We can substitute a new quaternion p for the product of the quaternion q and the vector x'. This simplifies the computation of the second quaternion multiplication.

(A.12) $$p = q \circ x' = (q_0 x'_0 - \mathbf{q} \bullet \mathbf{x}', q_0 \mathbf{x}' + x_0 \mathbf{q} + \mathbf{q} \times \mathbf{x}')$$

(A.13) $$p = (p_0, < p_1 p_2 p_3 >) = (p_0, \mathbf{p})$$

(A.14) $$p \circ \overline{q} = (p_0 q_0 - \mathbf{p} \bullet (-\mathbf{q}), q_0 \mathbf{p} + p_0(-\mathbf{q}) + \mathbf{p} \times (-\mathbf{q}))$$

(A.15) $$p \circ \overline{q} = (p_0 q_0 + \mathbf{p} \bullet \mathbf{q}, q_0 \mathbf{p} + p_0(-\mathbf{q}) + \mathbf{p} \times (-\mathbf{q}))$$

(A.16) $$p_0 = (q_0 x'_0 - \mathbf{q} \bullet \mathbf{x}') = (0 - \mathbf{q} \bullet \mathbf{x}') = -\mathbf{q} \bullet \mathbf{x}'$$

(A.17) $$p_0 q_0 = -q_0(\mathbf{q} \bullet \mathbf{x}')$$

(A.18) $$\mathbf{p} = q_0 \mathbf{x}' + x_0 \mathbf{q} + \mathbf{q} \times \mathbf{x}' = q_0 \mathbf{x}' + \mathbf{q} \times \mathbf{x}'$$

(A.19) $$-\mathbf{p} \bullet (-\mathbf{q}) = (q_0 \mathbf{x}' + \mathbf{q} \times \mathbf{x}') \bullet (\mathbf{q}) = q_0 \mathbf{x}' \bullet \mathbf{q} + \mathbf{q} \times \mathbf{x}' \bullet \mathbf{q}$$

(A.20) $$q_0 \mathbf{p} = q_0(q_0 \mathbf{x}' + \mathbf{q} \times \mathbf{x}')$$

(A.21) $$x_0(-\mathbf{q}) = 0$$

(A.22) $$\mathbf{p} \times (-\mathbf{q}) = (q_0 \mathbf{x}' + \mathbf{q} \times \mathbf{x}') \times (-\mathbf{q})$$

(A.23) $$(q_0 \mathbf{x}' + \mathbf{q} \times \mathbf{x}') \times (-\mathbf{q}) = (q_0 \mathbf{x}' \times -\mathbf{q}) + \mathbf{q} \times \mathbf{x}' \times -\mathbf{q})$$

$$p \circ \overline{q} = (-q_0(\mathbf{q} \bullet \mathbf{x}') + q_0 \mathbf{x}' \bullet \mathbf{q} + \mathbf{q} \times \mathbf{x}' \bullet \mathbf{q}, q_0(q_0 \mathbf{x}' + \mathbf{q} \times \mathbf{x}') + (q_0 \mathbf{x}' \times -\mathbf{q}) + (\mathbf{q} \times \mathbf{x}' \times -\mathbf{q}) + \mathbf{q}(\mathbf{q} \bullet \mathbf{x}'))$$
(A.24)

(A.25) $$\mathbf{x} = R\mathbf{x}' = q_0(q_0\mathbf{x}' + \mathbf{q}\times\mathbf{x}') + (q_0\mathbf{x}'\times -\mathbf{q}) + (\mathbf{q}\times\mathbf{x}'\times -\mathbf{q}) + \mathbf{q}(\mathbf{q}\bullet\mathbf{x}')$$

(A.26) $$\mathbf{x} = R\mathbf{x}' = q_0^2\mathbf{x}' + q_0\mathbf{q}\times\mathbf{x}' + q_0\mathbf{x}'\times -\mathbf{q} + (\mathbf{q}\times\mathbf{x}'\times -\mathbf{q}) + \mathbf{q}(\mathbf{q}\bullet\mathbf{x}')$$

Useful cross product identities are applied to the solution at this point to put it in a more convenient form.

(A.27) $$q_0\mathbf{x}'\times -\mathbf{q} = -q_0\mathbf{x}'\times\mathbf{q}$$

(A.28) $$\mathbf{q}\times\mathbf{x}'\times -\mathbf{q} = -\mathbf{q}\times\mathbf{x}'\times\mathbf{q}$$

(A.29) $$-q_0(\mathbf{x}'\times\mathbf{q}) + q_0(\mathbf{q}\times\mathbf{x}') = 2q_0(\mathbf{q}\times\mathbf{x}')$$

(A.30) $$-\mathbf{q}\times\mathbf{x}'\times\mathbf{q} = -((\mathbf{q}\bullet\mathbf{q})\mathbf{x}' - (\mathbf{q}\bullet\mathbf{x}')\mathbf{q}) = (\mathbf{q}\bullet\mathbf{x}')\mathbf{q} - (\mathbf{q}\bullet\mathbf{q})\mathbf{x}'$$

The above identities are substituted back into the equation to reduce the complexity of the solution:

(A.31) $$\mathbf{x} = q_0^2\mathbf{x}' + 2q_0(\mathbf{q}\times\mathbf{x}') + (\mathbf{q}\bullet\mathbf{x}')\mathbf{q} - (\mathbf{q}\bullet\mathbf{q})\mathbf{x}' + \mathbf{q}(\mathbf{q}\bullet\mathbf{x}')$$

(A.32) $$\mathbf{x} = (q_0^2 - \mathbf{q}\bullet\mathbf{q})\mathbf{x}' + 2q_0(\mathbf{q}\times\mathbf{x}') + 2(\mathbf{q}\bullet\mathbf{x}')\mathbf{q}$$

The dot and cross products are expanded into equivalent matrix form:

(A.33) $$(q_0^2 - \mathbf{q}\bullet\mathbf{q})\mathbf{x}' = (q_0^2 - q_1^2 - q_2^2 - q_3^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}'$$

(A.34) $$2q_0(\mathbf{q}\times\mathbf{x}') = 2q_0\begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}\mathbf{x}'$$

(A.35) $$2(\mathbf{q}\bullet\mathbf{x}')\mathbf{q} = 2\begin{bmatrix} q_1^2 & q_1q_2 & q_1q_3 \\ q_1q_2 & q_2^2 & q_2q_3 \\ q_1q_3 & q_2q_3 & q_3^2 \end{bmatrix}\mathbf{x}'$$

$$\mathbf{x} = \left((q_0^2 - q_1^2 - q_2^2 - q_3^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2q_0 \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} + 2 \begin{bmatrix} q_1^2 & q_1 q_2 & q_1 q_3 \\ q_1 q_2 & q_2^2 & q_2 q_3 \\ q_1 q_3 & q_2 q_3 & q_3^2 \end{bmatrix} \right) \mathbf{x}' \tag{A.36}$$

$$\mathbf{x} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & -2q_0 q_3 + 2q_1 q_2 & 2q_0 q_2 + 2q_1 q_3 \\ 2q_0 q_3 + 2q_1 q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & -2q_0 q_1 + 2q_2 q_3 \\ -2q_0 q_2 + 2q_1 q_3 & 2q_0 q_1 + 2q_2 q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \mathbf{x}' \tag{A.37}$$

Therefore the rotation matrix can be derived from a unit quaternion representation.

$$\mathbf{R} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & -2q_0 q_3 + 2q_1 q_2 & 2q_0 q_2 + 2q_1 q_3 \\ 2q_0 q_3 + 2q_1 q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & -2q_0 q_1 + 2q_2 q_3 \\ -2q_0 q_2 + 2q_1 q_3 & 2q_0 q_1 + 2q_2 q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \tag{A.38}$$