A Compendium of BBP-Type Formulas for Mathematical Constants

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February 13, 2011

Abstract

A 1996 paper by the author, Peter Borwein and Simon Plouffe showed that any mathematical constant given by an infinite series of a certain type has the property that its n-th digit in a particular number base could be calculated directly, without needing to compute any of the first $n-1$ digits, by means of a simple algorithm that does not require multiple-precision arithmetic. Several such formulas were presented in that paper, including formulas for the constants π and $\log 2$. Since then, numerous other formulas of this type have been found. This paper presents a compendium of currently known results of this sort, both proven and conjectured. Experimentally obtained results which are not yet proven have been checked to high precision and are marked with a $\overset{?}{=}$. Fully established results are as indicated in the citations and references below.

1 Introduction

This is a collection of formulas for various mathematical constants that are of the form similar to that first noted in the "BBP" paper [12]. That article presented the following formula for π (which was discovered using Ferguson's PSLQ integer relation finding algorithm [20, 13]):

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{1}$$

It was shown in [12] that this formula permits one to calculate the n-th hexadecimal or binary digit of π, without computing any of the first $n-1$ digits, by means of a simple algorithm that does not require multiple-precision arithmetic. A more recent paper by

Borwein, Galway and Borwein showed that there are no other degree-1 BBP-type for π, except those, such as Formula 1, whose "base" is a power of two. However, as we shall see below, in the case of π^2 there is both a base-2 formula (see Formula 21) and a base-3 formula (see Formula 68).

Further, as shown in [12], several other well-known constants also have this individual digit-computation property. One of these is $\log 2$, based on the following centuries-old formula:

$$\log 2 \;=\; \sum_{k=1}^{\infty} \frac{1}{k2^k}. \tag{2}$$

In general, any constant C that can be written in the form

$$C \;=\; \sum_{k=0}^{\infty} \frac{p(k)}{q(k)b^k},$$

where p and q are integer polynomials, $\deg(p) < \deg(q)$, and $q(k)$ is nonzero for nonnegative k, possesses this individual digit-computation property. Note that Formula 1 can be written in this form, since the four fractions can be combined into one, yielding

$$\pi \;=\; \sum_{k=0}^{\infty} \frac{47 + 151k + 120k^2}{16^k(15 + 194k + 712k^2 + 1024k^3 + 512k^4)}.$$

Since the publication of [12], other papers have presented formulas of this type for various constants, including several constants that arise in quantum field theory [17, 18, 14]. More recently, interest in BBP-type formulas has been heightened by the observation that the question of the statistical randomness of the digit expansions of these constants can be reduced to the following hypothesis regarding the behavior of a particular class of chaotic iterations [14]:

Hypothesis A (from the paper [14]). Denote by $r_n = p(n)/q(n)$ a rational-polynomial function, i.e. $p, q \in Z[X]$. Assume further that $0 \le \deg p < \deg q$, with r_n nonsingular for positive integers n. Choose an integer $b \ge 2$ and initialize $x_0 = 0$. Then the sequence $x = (x_0, x_1, x_2, \ldots)$ determined by the iteration

$$x_n \;=\; (bx_{n-1} + r_n) \bmod 1$$

either has a finite attractor or is equidistributed in $[0, 1)$.

Assuming this hypothesis, it is shown in [14] that any BBP-type constant is either normal to base b (i.e., any n-long string digits appears in the base b expansion with limiting frequency b^{-n}), or else it is rational. No proof of Hypothesis A was presented in [14], and indeed it is likely that Hypothesis A is rather difficult to prove. However, it should be emphasized that even particular instances of Hypothesis A, if established, would have interesting consequences. For example, if it could be established that the specific iteration given by $x_0 = 0$, and

$$x_n \;=\; (2x_{n-1} + \frac{1}{n}) \bmod 1$$

is equidistributed in $[0, 1)$, then it would follow that $\log 2$ is normal to base 2. In a similar vein, if it could be established that the iteration given by $x_0 = 0$ and

$$x_n = \left(16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right) \bmod 1$$

is equidistributed in $[0, 1)$, then it would follow that π is normal to base 16 (and thus to base 2 also).

One additional impetus for the study of BPP-type constants comes from a recent paper by Lagarias [23], who demonstrates a connection to G-functions and to a conjecture of Furstenberg from ergodic theory. Lagarias' analysis suggests that there may be a special signficance to constants that have BBP-type formulas in two or more bases — say both a base 2 and a base 3 formula.

This paper is a compendium of the growing set of BBP-type formulas that have been found by various researchers. Part of these formulas are collected here from previously published sources. In other cases, formulas whose existence has been demonstrated in the literature are presented here explicitly for the first time. Still others are new, having been found using the author's PSLQ program [13] in the course of this research.

The PSLQ integer relation algorithm [20] or one of its variants [13] can be used to find formulas such as those listed in this paper as follows. Suppose, for example, that it is conjectured that a given constant α satisfies a BBP-type formula of the form

$$\alpha = \frac{1}{r} \sum_{k=0}^{\infty} \frac{1}{b^k} \left(\frac{a_1}{(kn+1)^s} + \frac{a_2}{(kn+2)^s} + \cdots + \frac{a_n}{(kn+n)^s}\right),$$

where r and a_k are unknown integers, for a specified selection of the parameters b, s and n. To apply PSLQ, first calculate the vector $(\sum_{k \geq 0} 1/(b^k(kn+j)^s), \ 1 \leq j \leq n)$, as well as α itself, to very high precision, then use this $(n + 1)$-long vector (including α at the end) as input to a PSLQ program. If a solution vector (a_j) is found with sufficiently high numerical fidelity, then

$$\alpha = \frac{-1}{a_{n+1}} \sum_{k=0}^{\infty} \frac{1}{b^k} \left(\frac{a_1}{(kn+1)^s} + \frac{a_2}{(kn+2)^s} + \cdots + \frac{a_n}{(kn+n)^s}\right)$$

(at least to the level of numeric precision used).

This compendium is not intended to be a comprehensive listing of all such formulas. In most cases a formula is not listed here if it is merely

1. a telescoping sum.

2. a formal rewriting of another formula on the list.

3. a straightforward formal manipulation starting with another formula in the list.

4. an integer linear combination of two or more formulas already in the list.

Item 1 refers to a summation such as

$$S = \sum_{k=1}^{\infty} \frac{1}{b^k} \left(\frac{b^2}{k} - \frac{1}{k+2} \right),$$

which, if split into two summations, has the property that the terms of the first series cancel with offset terms of the second series, so that S reduces to a rational number (in this example, $S = b + 1/2$). Item 2 refers to the fact that a formula with base b and length n can be rewritten as a formula with base b^r and length rn. Item 4 refers to the fact that the rational linear sum of two BBP series can, in many cases, be written as a single BBP series. This is clear if the two individual series have the same base b. If one has base b^r and the other has base b^s, their sum can be written as a single BBP series with base $b^{\mathrm{lcm}(r,s)}$ [14]. Along this line, many of the formulas listed below possess variants that can be obtained by adding to the listed formula a rational multiple of one of the zero relations listed in Section 11.

The formulas are listed below using a notation introduced in [14]:

$$P(s, b, n, A) = \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{n} \frac{a_j}{(kn+j)^s}. \tag{3}$$

where s, b and n are integers, and $A = (a_1, a_2, \cdots, a_n)$ is a vector of integers. For instance, using this notation we can write formulas 1 and 2 more compactly as follows:

$$\pi = P(1, 16, 8, (4, 0, 0, -2, -1, -1, 0, 0)) \tag{4}$$
$$\log 2 = \frac{1}{2} P(1, 2, 1, (1)). \tag{5}$$

In most cases below, the representation shown using this notation is a translation from the original source. Also, in some cases the formula listed here is not precisely the one mentioned in the cited reference — an equivalent one is listed here instead — but the original discoverer is given due credit. In cases where the formula has been found experimentally (i.e., by using the PSLQ integer relation finding algorithm), and no formal proof is available, the relation is listed here with the $\overset{?}{=}$ notation instead of an equal sign.

The P notation formulas listed below have been checked using a computer program that parses the LaTeXsource of this document, then computes the left-hand and right-hand sides of these formulas to 2000 decimal digit accuracy.

Additional contributions to this compendium are welcome — please send a note to the author at `dhbailey@lbl.gov`.

2 Logarithm formulas

Clearly $\log n$ can be written with a binary BBP formula (i.e. a formula with $b = 2^m$ for some integer m) provided n factors completely using primes whose logarithms have binary BBP formulas — one merely combines the individual series for the different primes

into a single binary BBP formula. We have seen above that $\log 2$ possesses a binary BBP formula, and so does the $\log 3$, by the following reasoning:

$$
\begin{aligned}
\log 3 &= 2\log 2 + \log\left(1 - \frac{1}{4}\right) = 2\sum_{k=1}^{\infty}\frac{1}{k2^k} - \sum_{k=1}^{\infty}\frac{1}{k4^k} \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{4^k}\left(\frac{2}{2k+1} + \frac{1}{2k+2}\right) - \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{4^k}\left(\frac{2}{2k+2}\right) \\
&= \sum_{k=0}^{\infty}\frac{1}{4^k}\left(\frac{1}{2k+1}\right) = P(1,4,2,(1,0)).
\end{aligned}
\tag{6}
$$

In a similar manner, one can show, by examining the factorization of $2^n + 1$ and $2^n - 1$, where n is an integer, that numerous other primes have this property. Harley [21] further extended this list of primes by writing

$$
\mathrm{Re}\left(\log\left(1 \pm \frac{1+i}{2^n}\right)\right) = \left(\frac{1}{2} - n\right)\log 2 + \frac{1}{2}\log(2^{2n-1} \pm 2^n + 1),
$$

where Re denotes the real part. He first noted that the Taylor series of the left-hand side can be written as a binary BBP-type formula. He then applied Aurefeuille's factorization formula

$$
2^{4n-2} + 1 = (2^{2n-1} + 2^n + 1)(2^{2n-1} - 2^n + 1)
$$

to the right-hand side. More recently, Jonathan Borwein has observed that both of these sets of results can be derived by working with the single expression

$$
\mathrm{Re}\left(\log\left(1 \pm \frac{(1+i)^k}{2^n}\right)\right).
$$

A preliminary list of primes p such that $\log p$ has a binary BBP formula was given in [12]. This list has now been augmented by the author to the following:

$$
\begin{aligned}
&2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 29,\ 31,\ 37,\ 41,\ 43,\ 61,\ 73,\ 109,\ 113,\ 127,\ 151, \\
&241,\ 257,\ 337,\ 397,\ 683,\ 1321,\ 1613,\ 2113,\ 2731,\ 5419,\ 8191, 43691,\ 61681, \\
&87211,\ 131071,\ 174763,\ 262657,\ 524287,\ 2796203, 15790321,\ 18837001, \\
&22366891,\ 4278255361,\ 4562284561,\ 2932031007403,\ 4363953127297, \\
&4432676798593.
\end{aligned}
\tag{7}
$$

This list is certainly not complete, and it is unknown whether or not all primes have this property, or even whether the list of such primes is finite or infinite. The actual formulas for $\log p$ for the primes above are generally straightforward to derive and are not shown here.

One can also obtain BBP formulas in non-binary bases for the logarithms of certain integers and rational numbers. One example is given by the base ten formula 76 below, which was used in [12] to compute the ten billionth decimal digit of $\log(9/10)$.

3 Arctangent formulas

Shortly after the original BBP paper appeared in 1996, Adamchik and Wagon observed that [10]

$$\arctan 2 = \frac{1}{8}P(1, 16, 8, (8, 0, 4, 0, -2, 0, -1, 0)). \tag{8}$$

More recently, binary BBP formulas have been found for $\arctan q$ for a large set of rational numbers q. These experimental results, which were obtained by the author using the PSLQ program, coincide exactly in the cases studied so far with the set of rationals given by $q = |\mathrm{Im}(T)/\mathrm{Re}(T)|$ or $|\mathrm{Re}(T)/\mathrm{Im}(T)|$, where

$$T = \prod_{k=1}^{m} \left(1 \pm \frac{(1+i)^{u_k}}{2^{v_k}}\right)^{w_k}. \tag{9}$$

The arctangents of these q clearly possess binary BBP formulas, because $\mathrm{Im}(\log T)$ decomposes into a linear sum of terms, the Taylor series of which are binary BBP formulas. The author is indebited to Jonathan Borwein for this observation. See also [15, pg. 344]. Alternatively, one can write Formula 9 as

$$T = \prod_{k=1}^{m} \left(1 \pm \frac{i}{2^{t_k}}\right)^{u_k} \left(1 \pm \frac{1+i}{2^{v_k}}\right)^{w_k} \tag{10}$$

for various m-long nonnegative integer vectors t, u, v, w and choices of signs as shown. For example, setting $t = (1, 1)$, $u = (1, 1)$, $v = (1, 3)$, $w = (1, 1)$, with signs $(1, -1, -1, 1)$, gives the result $T = 25/32 - 5i/8$, which yields $q = 4/5$. Indeed, one can obtain the formula

$$
\begin{aligned}
\arctan\left(\frac{4}{5}\right) =\ & \frac{1}{2^{17}}P(1, 2^{20}, 40, (0, 2^{19}, 0, -3 \cdot 2^{17}, -15 \cdot 2^{15}, 0, 0, 5 \cdot 2^{15}, 0, 2^{15}, 0, \\
& -3 \cdot 2^{13}, 0, 0, 5 \cdot 2^{10}, 5 \cdot 2^{11}, 0, 2^{11}, 0, 2^{10}, 0, 0, 0, 5 \cdot 2^{7}, 15 \cdot 2^{5}, 128, 0, \\
& -96, 0, 0, 0, 40, 0, 8, -5, -6, 0, 0, 0, 0)).
\end{aligned} \tag{11}
$$

In this manner, it can be seen that binary BBP formulas exist for the arctangents of the following rational numbers. Only those rationals with numerators $<$ denominators $\le$

50 are listed here:

$$1/2,\ 1/3,\ 2/3,\ 1/4,\ 3/4,\ 1/5,\ 2/5,\ 3/5,\ 4/5,\ 1/7,\ 3/7,\ 4/7,\ 6/7,$$
$$1/8,\ 7/8,\ 1/9,\ 2/9,\ 7/9,\ 8/9,\ 3/10,\ 2/11,\ 3/11,\ 7/11,\ 8/11,\ 10/11,$$
$$1/12,\ 5/12,\ 1/13,\ 6/13,\ 7/13,\ 9/13,\ 11/13,\ 3/14,\ 5/14,\ 1/15,\ 4/15,$$
$$8/15,\ 1/16,\ 11/16,\ 13/16,\ 15/16,\ 1/17,\ 6/17,\ 7/17,\ 11/17,\ 15/17,$$
$$16/17,\ 1/18,\ 13/18,\ 4/19,\ 6/19,\ 7/19,\ 8/19,\ 9/19,\ 11/19,\ 17/19,$$
$$1/21,\ 16/21,\ 3/22,\ 7/22,\ 9/22,\ 19/22,\ 2/23,\ 4/23,\ 6/23,\ 7/23,$$
$$11/23,\ 14/23,\ 15/23,\ 7/24,\ 11/24,\ 23/24,\ 13/25,\ 19/25,\ 21/25,$$
$$7/26,\ 23/26,\ 5/27,\ 11/27,\ 2/29,\ 3/29,\ 15/29,\ 17/29,\ 24/29,\ 28/29,$$
$$17/30,\ 1/31,\ 5/31,\ 8/31,\ 12/31,\ 13/31,\ 17/31,\ 18/31,\ 22/31,\ 27/31,$$
$$1/32,\ 9/32,\ 31/32,\ 1/33,\ 4/33,\ 10/33,\ 14/33,\ 19/33,\ 31/33,\ 32/33,$$
$$7/34,\ 27/34,\ 13/35,\ 25/36,\ 5/37,\ 9/37,\ 10/37,\ 16/37,\ 29/37,\ 36/37,$$
$$1/38,\ 5/38,\ 13/38,\ 21/38,\ 20/39,\ 23/39,\ 37/39,\ 9/40,\ 3/41,\ 23/41,$$
$$27/41,\ 28/41,\ 38/41,\ 11/42,\ 19/42,\ 37/42,\ 6/43,\ 19/43,\ 23/43,$$
$$32/43,\ 33/43,\ 7/44,\ 23/44,\ 27/44,\ 3/46,\ 9/46,\ 17/46,\ 35/46,\ 37/46,$$
$$1/47,\ 13/47,\ 14/47,\ 16/47,\ 19/47,\ 27/47,\ 19/48,\ 3/49,\ 8/49,\ 13/49,$$
$$18/49,\ 31/49,\ 37/49,\ 43/49,\ 29/50,\ 49/50. \tag{12}$$

It was recently noted in a paper by Borwein, Galway and Borwein [16], which includes a significant structure theory for BBP-type arctangent formulas, that $\arctan(1/6) = \arctan(1/5) - \arctan(1/31)$, so that $\arctan(1/6)$ also has a binary BBP formula. Similarly, $\arctan(5/6) = \arctan(1) - \arctan(1/5) + \arctan(1/9) - \arctan(1/255)$, and $\arctan(1/11) = \arctan(1) - \arctan(5/6)$. By performing PSLQ over the set of arctangents of the above list, augmented by $1/6$ and $5/6$, one finds that most are linearly dependent on the others. Indeed, by eliminating redundant elements, one can reduce the list to the following set whose arctangents appear to be linearly independent:

$$1/2,\ 1/3,\ 2/3,\ 1/4,\ 2/5,\ 4/5,\ 1/6,\ 5/6,\ 7/8,\ 3/10,\ 4/15,\ 1/16,\ 6/19,$$
$$32/33,\ 13/38. \tag{13}$$

¿From the basis set (13), by performing PSLQ, one can find relations for all of the rationals in the list (12), and, in addition, relations for the following additional rationals:

$$5/7,\ 1/11,\ 4/13,\ 7/16,\ 4/17,\ 9/17,\ 3/19,\ 9/20,\ 13/21,\ 15/21,$$
$$20/21,\ 21/22,\ 9/23,\ 10/23,\ 2/25,\ 8/26,\ 19/26,\ 8/27,\ 14/27,\ 19/27,$$
$$23/27,\ 3/28,\ 17/28,\ 11/29,\ 14/29,\ 26/29,\ 25/31,\ 7/32,\ 13/33,$$
$$19/34,\ 12/35,\ 19/35,\ 11/36,\ 31/36,\ 20/37,\ 22/37,\ 9/38,\ 31/38,$$
$$2/39,\ 25/39,\ 13/40,\ 1/41,\ 13/41,\ 24/41,\ 27/41,\ 31/42,\ 1/43,\ 2/43,$$
$$15/43,\ 36/43,\ 35/44,\ 7/45,\ 11/45,\ 41/45,\ 43/46,\ 23/47,\ 25/47,$$
$$29/47,\ 29/48,\ 2/49,\ 10/49,\ 1/50,\ 41/50. \tag{14}$$

Even after merging our list (12) by this new list (14), note that not all "small" rationals appear. For instance, it is not known whether arctan(2/7) possesses a binary BBP formula.

One can obtain BBP formulas in non-binary bases for the arctangents of certain rational numbers by employing appropriate variants of Formulas 9 and 10.

4 Other degree 1 binary formulas

We present here some additional degree 1 binary BBP-type formulas (in other words, in the P notation defined in equation 3 above, $s = 1$, and $b = 2^m$ for some integer $m > 0$).

$$\pi = \frac{1}{4}P(1, 16, 8, (8, 8, 4, 0, -2, -2, -1, 0)) \tag{15}$$

$$\pi = P(1, -4, 4, (2, 2, 1, 0)) \tag{16}$$

$$\pi\sqrt{2} = \frac{1}{8}P(1, 64, 12, (32, 0, 8, 0, 8, 0, -4, 0, -1, 0, -1, 0)) \tag{17}$$

$$\pi\sqrt{3} = \frac{9}{32}P(1, 64, 6, (16, 8, 0, -2, -1, 0)) \tag{18}$$

$$\sqrt{2}\log(1 + \sqrt{2}) = \frac{1}{8}P(1, 16, 8, (8, 0, 4, 0, 2, 0, 1, 0)) \tag{19}$$

$$\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{8}P(1, 16, 8, (8, 0, -4, 0, 2, 0, -1, 0)) \tag{20}$$

Formula 15 was first found by Ferguson [20], while 16, which is the alternating sign equivalent of 15, was found independently by Hales and by Adamchik and Wagon [10]. Technically speaking, these formulas can be obtained from the original BBP formula for π (formula 1) by adding 1/4 times relation 103 of Section 12, but they are included here for historical interest, since their discovery predated the discovery of relation 103. Formula 17 appeared in [12]. Formulas 18, 19 and 20 are due to Knuth [22, pg. 628].

5 Degree 2 binary formulas

Here are some degree 2 binary formulas (i.e., $s = 2$, and $b = 2^m$ for some integer $m > 0$). The constant G here is Catalan's constant, namely $G = 1 - 1/3^2 + 1/5^2 - 1/7^2 + \cdots =$

$0.9159655941\ldots$:

$$\pi^2 = P(2,16,8,(16,-16,-8,-16,-4,-4,2,0)) \tag{21}$$

$$\pi^2 = \frac{9}{8}P(2,64,6,(16,-24,-8,-6,1,0)) \tag{22}$$

$$\log^2 2 = \frac{1}{6}P(2,16,8,(16,-40,-8,-28,-4,-10,2,-3)) \tag{23}$$

$$\log^2 2 = \frac{1}{32}P(2,64,6,(64,-160,-56,-40,4,-1)) \tag{24}$$

$$G - \frac{1}{8}\pi\log 2 = \frac{1}{16}P(2,16,8,(8,8,4,0,-2,-2,-1,0)) \tag{25}$$

$$\pi\log 2 = \frac{1}{256}P(2,2^{12},24,(2^{12},-2^{13},-51\cdot 2^9,15\cdot 2^{10},-2^{10},39\cdot 2^8,0,$$
$$45\cdot 2^8,37\cdot 2^6,-2^9,0,3\cdot 2^8,-64,0,51\cdot 2^3,45\cdot 2^4,16,196,0,$$
$$60,-37,0,0,0)) \tag{26}$$

$$\pi\sqrt{3}\log 2 = \frac{1}{128}P(2,2^{12},24,(9\cdot 2^9,-27\cdot 2^9,-9\cdot 2^{11},27\cdot 2^9,0,81\cdot 2^7,$$
$$9\cdot 2^6,45\cdot 2^8,9\cdot 2^8,0,0,9\cdot 2^6,-72,-216,9\cdot 2^5,9\cdot 2^6,0,162,$$
$$-9,72,-36,0,0,0)) \tag{27}$$

$$\pi\sqrt{3}\log 2 = \frac{9}{2^{10}}P(2,2^{12},24,(2^{11},-3\cdot 2^{11},0,-2^{11},2^9,0,2^8,2^{10},0,3\cdot 2^7,2^6,0,$$
$$-2^5,-3\cdot 2^5,0,-2^6,-2^3,0,-2^2,2^3,0,6,-1,0)) \tag{28}$$

$$\sqrt{3}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \frac{9}{5\cdot 2^{10}}\,P(2,2^{12},24,(2^{11},-2^{12},0,-2^{10},2^9,0,2^8,3\cdot 2^8,0,2^8,2^6,$$
$$0,-2^5,-2^6,0,-3\cdot 2^4,-2^3,0,-2^2,2^2,0,2^2,-1,0)) \tag{29}$$

$$G = \frac{1}{2^{10}}P(2,2^{12},24,(2^{10},2^{10},-2^9,-3\cdot 2^{10},-256,-2^{11},-256,$$
$$-9\cdot 2^7,-5\cdot 2^6,64,64,0,-16,64,8,-72,4,-8,4,-12,5,$$
$$4,-1,0)) \tag{30}$$

$$G = \frac{3}{2^{12}}P(2,2^{12},24,(2^{11},-2^{11},-2^{11},0,-2^9,-2^{10},-2^8,0,-2^8,-2^7,$$
$$2^6,0,-2^5,2^5,2^5,0,2^3,2^4,2^2,0,2^2,2,-1,0)) \tag{31}$$

Formulas 21, 22, 24 and 25 were presented in [12] (although 25 appeared in a 1909 book by Nielsen [24, pg. 105]). Formulas 23 and 27 were found experimentally by the author, using the PSLQ program. Formula 23 was subsequently proved by Kunle Adegoke [4]. Formulas for $\pi\log 2$ and G were first derived by Broadhurst, although the specific explicit formulas given here (26 and 30) were found experimentally by the author using PSLQ. Formula 31 was found by Kunle Adegoke [1], based on a formula of Broadhurst. Formulas 28 and 29 are due to Kunle Adegoke [9]. Here Cl_2 is a Clausen function (see section 9). Formula 30 = Formula 31 $+5/2^{11}\times$ Relation 115 $+1/2^{12}\times$ Relation 116. Formula 27 = Formula 28 $+27/2^9\times$ Relation 115 $-9/2^{10}\times$ Relation 116. Formula 26 = $8\times$ (Formula 31 $-$ Formula 25) $-1/2^6\times$ Relation 115 $-1/2^8\times$ Relation 116.

6 Degree 3 binary formulas

$$\zeta(3) = \frac{1}{7 \cdot 2^8} P(3, 2^{12}, 24, (3 \cdot 2^{11}, -21 \cdot 2^{11}, 3 \cdot 2^{13}, 15 \cdot 2^{11}, -3 \cdot 2^9, 3 \cdot 2^{10},$$
$$3 \cdot 2^8, 0, -3 \cdot 2^{10}, -21 \cdot 2^7, -192, -3 \cdot 2^9, -96, -21 \cdot 2^5, -3 \cdot 2^7, 0,$$
$$24, 48, -12, 120, 48, -42, 3, 0)) \tag{32}$$

$$\log^3 2 = \frac{1}{256} P(3, 2^{12}, 24, (0, 3 \cdot 2^{13}, -27 \cdot 2^{12}, 3 \cdot 2^{14}, 0, 93 \cdot 2^9, 0, 3 \cdot 2^{14}, 27 \cdot 2^9,$$
$$3 \cdot 2^9, 0, 75 \cdot 2^6, 0, 3 \cdot 2^7, 27 \cdot 2^6, 3 \cdot 2^{10}, 0, 93 \cdot 2^3, 0, 192, -216,$$
$$24, 0, 3)) \tag{33}$$

$$\pi^2 \log 2 = \frac{1}{32} P(3, 2^{12}, 24, (0, 9 \cdot 2^{11}, -135 \cdot 2^9, 9 \cdot 2^{11}, 0, 99 \cdot 2^8, 0, 27 \cdot 2^{10}, 135 \cdot 2^6,$$
$$9 \cdot 2^7, 0, 45 \cdot 2^6, 0, 9 \cdot 2^5, 135 \cdot 2^3, 27 \cdot 2^6, 0, 396, 0, 72, -135, 18, 0, 0)) \tag{34}$$

$$\pi \log^2 2 = \frac{1}{2^{56}} P(3, 2^{60}, 120, (7 \cdot 2^{59}, -37 \cdot 2^{60}, -63 \cdot 2^{58}, 85 \cdot 2^{59}, 3861 \cdot 2^{56},$$
$$-3357 \cdot 2^{55}, 0, -655 \cdot 2^{58}, 347 \cdot 2^{54}, 79 \cdot 2^{53}, 0, 4703 \cdot 2^{52}, -7 \cdot 2^{53}, 0,$$
$$-1687 \cdot 2^{52}, -655 \cdot 2^{54}, 7 \cdot 2^{51}, -4067 \cdot 2^{49}, 0, -6695 \cdot 2^{48}, -347 \cdot 2^{48},$$
$$0, 0, -7375 \cdot 2^{46}, -3861 \cdot 2^{46}, -37 \cdot 2^{48}, -63 \cdot 2^{46}, 85 \cdot 2^{47}, -7 \cdot 2^{45},$$
$$-933 \cdot 2^{45}, 0, -655 \cdot 2^{46}, 347 \cdot 2^{42}, -37 \cdot 2^{44}, 875 \cdot 2^{43}, 4703 \cdot 2^{40},$$
$$-7 \cdot 2^{41}, 0, 63 \cdot 2^{40}, -3105 \cdot 2^{38}, 7 \cdot 2^{39}, -4067 \cdot 2^{37}, 0, 85 \cdot 2^{39}, 441 \cdot 2^{39},$$
$$0, 0, -7375 \cdot 2^{34}, 7 \cdot 2^{35}, 79 \cdot 2^{33}, -63 \cdot 2^{34}, 85 \cdot 2^{35}, -7 \cdot 2^{33},$$
$$-3357 \cdot 2^{31}, -875 \cdot 2^{33}, -655 \cdot 2^{34}, 347 \cdot 2^{30}, -37 \cdot 2^{32}, 0, -167 \cdot 2^{32},$$
$$-7 \cdot 2^{29}, 0, 63 \cdot 2^{28}, -655 \cdot 2^{30}, -3861 \cdot 2^{26}, -4067 \cdot 2^{25}, 0, 85 \cdot 2^{27},$$
$$-347 \cdot 2^{24}, -375 \cdot 2^{23}, 0, -7375 \cdot 2^{22}, 7 \cdot 2^{23}, -37 \cdot 2^{24}, 1687 \cdot 2^{22},$$
$$85 \cdot 2^{23}, -7 \cdot 2^{21}, -3357 \cdot 2^{19}, 0, -3105 \cdot 2^{18}, 347 \cdot 2^{18}, -37 \cdot 2^{20}, 0,$$
$$4703 \cdot 2^{16}, 3861 \cdot 2^{16}, 0, 63 \cdot 2^{16}, -655 \cdot 2^{18}, 7 \cdot 2^{15}, -923 \cdot 2^{15}, 0,$$
$$85 \cdot 2^{15}, -347 \cdot 2^{12}, 0, -875 \cdot 2^{13}, -7375 \cdot 2^{10}, 7 \cdot 2^{11}, -37 \cdot 2^{12},$$
$$-63 \cdot 2^{10}, -6695 \cdot 2^8, -7 \cdot 2^9, -3357 \cdot 2^7, 0, -655 \cdot 2^{10}, -441 \cdot 2^9,$$
$$-37 \cdot 2^8, 0, 4703 \cdot 2^4, -224, -375 \cdot 2^3, 63 \cdot 2^4, -655 \cdot 2^6, 56, -8134,$$
$$875 \cdot 2^3, 85 \cdot 2^3, -347, 0, 0, 0)) \tag{35}$$

$$\pi^3 \;=\; \frac{1}{2^{54}}P(3, 2^{60}, 120, (5\cdot 2^{59}, -15\cdot 2^{60}, -225\cdot 2^{58}, 95\cdot 2^{59}, 4115\cdot 2^{56},$$
$$-3735\cdot 2^{55}, 0, -685\cdot 2^{58}, 505\cdot 2^{54}, 5\cdot 2^{53}, 0, 5485\cdot 2^{52}, -5\cdot 2^{53}, 0,$$
$$-1775\cdot 2^{52}, -685\cdot 2^{54}, 5\cdot 2^{51}, -3945\cdot 2^{49}, 0, -7365\cdot 2^{48}, -505\cdot 2^{48},$$
$$0, 0, -8125\cdot 2^{46}, -4115\cdot 2^{46}, -15\cdot 2^{48}, -225\cdot 2^{46}, 95\cdot 2^{47}, -5\cdot 2^{45},$$
$$-965\cdot 2^{45}, 0, -685\cdot 2^{46}, 505\cdot 2^{42}, -15\cdot 2^{44}, 125\cdot 2^{46}, 5485\cdot 2^{40},$$
$$-5\cdot 2^{41}, 0, 225\cdot 2^{40}, -2835\cdot 2^{38}, 5\cdot 2^{39}, -3945\cdot 2^{37}, 0, 95\cdot 2^{39},$$
$$905\cdot 2^{38}, 0, 0, -8125\cdot 2^{34}, 5\cdot 2^{35}, 5\cdot 2^{33}, -225\cdot 2^{34}, 95\cdot 2^{35},$$
$$-5\cdot 2^{33}, -3735\cdot 2^{31}, -125\cdot 2^{36}, -685\cdot 2^{34}, 505\cdot 2^{30}, -15\cdot 2^{32}, 0,$$
$$-165\cdot 2^{32}, -5\cdot 2^{29}, 0, 225\cdot 2^{28}, -685\cdot 2^{30}, -4115\cdot 2^{26}, -3945\cdot 2^{25},$$
$$0, 95\cdot 2^{27}, -505\cdot 2^{24}, -125\cdot 2^{23}, 0, -8125\cdot 2^{22}, 5\cdot 2^{23}, -15\cdot 2^{24},$$
$$1775\cdot 2^{22}, 95\cdot 2^{23}, -5\cdot 2^{21}, -3735\cdot 2^{19}, 0, -2835\cdot 2^{18}, 505\cdot 2^{18},$$
$$-15\cdot 2^{20}, 0, 5485\cdot 2^{16}, 4115\cdot 2^{16}, 0, 225\cdot 2^{16}, -685\cdot 2^{18}, 5\cdot 2^{15},$$
$$-955\cdot 2^{15}, 0, 95\cdot 2^{15}, -505\cdot 2^{12}, 0, -125\cdot 2^{16}, -8125\cdot 2^{10}, 5\cdot 2^{11},$$
$$-15\cdot 2^{12}, -225\cdot 2^{10}, -7365\cdot 2^{8}, -5\cdot 2^{9}, -3735\cdot 2^{7}, 0, -685\cdot 2^{10},$$
$$-905\cdot 2^{8}, -15\cdot 2^{8}, 0, 5485\cdot 2^{4}, -160, -125\cdot 2^{3}, 225\cdot 2^{4}, -685\cdot 2^{6}, 40,$$
$$-7890, 125\cdot 2^{6}, 95\cdot 2^{3}, -505, 0, 0, 0)) \tag{36}$$

$$35\zeta(3) - 2\pi^2\log 2 \;=\; \frac{9}{4}P(3, 64, 6, (16, -24, -8, -6, 1, 0)) \tag{37}$$

$$\log^3 2 \;=\; \frac{1}{3\cdot 2^8}P(3, 2^{12}, 24, (2^{13}, -5\cdot 2^{14}, -2^{12}, 17\cdot 2^{13}, -2^{11},$$
$$5\cdot 19\cdot 2^9, 2^{10}, 9\cdot 2^{12}, 2^9, -5\cdot 2^{10}, -2^8, 2^6, -2^7, -5\cdot 2^8, 2^6,$$
$$9\cdot 2^8, 2^5, 5\cdot 19\cdot 2^3, -2^4, 17\cdot 2^5, -2^3, -5\cdot 2^4, 2^2, 9)) \tag{38}$$

$$\pi^2\log 2 \;=\; \frac{1}{3\cdot 2^6}P(3, 2^{12}, 24, (5\cdot 2^{11}, -41\cdot 2^{11}, -5\cdot 2^{10}, 49\cdot 2^{11},$$
$$-5\cdot 2^9, 67\cdot 2^9, 5\cdot 2^8, 27\cdot 2^{10}, 5\cdot 2^7, -41\cdot 2^7, -5\cdot 2^6, -5\cdot 2^7,$$
$$-5\cdot 2^5, -41\cdot 2^5, 5\cdot 2^4, 27\cdot 2^6, 5\cdot 2^3, 67\cdot 2^3, -5\cdot 2^2, 49\cdot 2^3,$$
$$-5\cdot 2^1, -41\cdot 2^1, 5, 0)) \tag{39}$$

$$\zeta(3) \;=\; \frac{1}{21\cdot 2^5}P(3, 2^{12}, 24, (2^{11}, -11\cdot 2^{10}, -2^{10}, 23\cdot 2^9, -2^9, 2^{12},$$
$$2^8, 27\cdot 2^7, 2^7, -11\cdot 2^6, -2^6, -2^7, -2^5, -11\cdot 2^4, 2^4, 27\cdot 2^3, 2^3,$$
$$2^6, -2^2, 23\cdot 2^1, -2^1, -11, 1, 0)) \tag{40}$$

$$\pi^3 \;=\; \frac{5}{2^{55}}P(3, 2^{60}, 120, (2^{59}, -3\cdot 2^{60}, 11\cdot 2^{57}, 0, 23\cdot 2^{56}, 3\cdot 7\cdot 2^{56}, -2^{56}, 0,$$

$$11\cdot 2^{54}, 13\cdot 2^{54}, 2^{54}, 0, -2^{53}, 3\cdot 2^{54}, 7\cdot 2^{52}, 0, 2^{51}, -3\cdot 7\cdot 2^{50}, 2^{50}, 0,$$

$$-11\cdot 2^{48}, 3\cdot 2^{50}, -2^{48}, 0, -23\cdot 2^{46}, -3\cdot 2^{48}, 11\cdot 2^{45}, 0, -2^{45}, -2^{46}, -2^{44},$$

$$0, 11\cdot 2^{42}, -3\cdot 2^{44}, -23\cdot 2^{41}, 0, -2^{41}, 3\cdot 2^{42}, -11\cdot 2^{39}, 0, 2^{39}, -3\cdot 7\cdot 2^{38},$$

$$2^{38}, 0, 7\cdot 2^{37}, 3\cdot 2^{38}, -2^{36}, 0, 2^{35}, 13\cdot 2^{34}, 11\cdot 2^{33}, 0, -2^{33}, 3\cdot 7\cdot 2^{32},$$

$$23\cdot 2^{31}, 0, 11\cdot 2^{30}, -3\cdot 2^{32}, 2^{30}, 0, -2^{29}, 3\cdot 2^{30}, -11\cdot 2^{27}, 0, -23\cdot 2^{26},$$

$$-3\cdot 7\cdot 2^{26}, 2^{26}, 0, -11\cdot 2^{24}, -13\cdot 2^{24}, -2^{24}, 0, 2^{23}, -3\cdot 2^{24}, -7\cdot 2^{22}, 0,$$

$$-2^{21}, 3\cdot 7\cdot 2^{20}, -2^{20}, 0, 11\cdot 2^{18}, -3\cdot 2^{20}, 2^{18}, 0, 23\cdot 2^{16}, 3\cdot 2^{18}, -11\cdot 2^{15},$$

$$0, 2^{15}, 2^{16}, 2^{14}, 0, -11\cdot 2^{12}, 3\cdot 2^{14}, 23\cdot 2^{11}, 0, 2^{11}, -3\cdot 2^{12}, 11\cdot 2^{9}, 0, -2^{9},$$

$$3\cdot 7\cdot 2^{8}, -2^{8}, 0, -7\cdot 2^{7}, -3\cdot 2^{8}, 2^{6}, 0, -2^{5}, -13\cdot 2^{4}, -11\cdot 2^{3}, 0, 2^{3},$$

$$-3\cdot 7\cdot 2^{2}, -23\cdot 2, 0, -11, 3\cdot 2^{2}, -1, 0)) \tag{41}$$

$$\pi\log^2 2 \;=\; \frac{1}{2^{57}}P(3, 2^{60}, 120, (7\cdot 2^{59}, -37\cdot 2^{60}, 13\cdot 17\cdot 2^{57}, 0, 19^2\cdot 2^{56},$$

$$5\cdot 71\cdot 2^{56}, -7\cdot 2^{56}, 0, 13\cdot 17\cdot 2^{54}, 227\cdot 2^{54}, 7\cdot 2^{54}, 0, -7\cdot 2^{53},$$

$$37\cdot 2^{54}, 7\cdot 11\cdot 2^{52}, 0, 7\cdot 2^{51}, -5\cdot 71\cdot 2^{50}, 7\cdot 2^{50}, 0, -13\cdot 17\cdot 2^{48},$$

$$37\cdot 2^{50}, -7\cdot 2^{48}, 0, -19^2\cdot 2^{46}, -37\cdot 2^{48}, 13\cdot 17\cdot 2^{45}, 0, -7\cdot 2^{45},$$

$$-5\cdot 2^{46}, -7\cdot 2^{44}, 0, 13\cdot 17\cdot 2^{42}, -37\cdot 2^{44}, -19^2\cdot 2^{41}, 0, -7\cdot 2^{41},$$

$$37\cdot 2^{42}, -13\cdot 17\cdot 2^{39}, 0, 7\cdot 2^{39}, -5\cdot 71\cdot 2^{38}, 7\cdot 2^{38}, 0, 7\cdot 11\cdot 2^{37},$$

$$37\cdot 2^{38}, -7\cdot 2^{36}, 0, 7\cdot 2^{35}, 227\cdot 2^{34}, 13\cdot 17\cdot 2^{33}, 0, -7\cdot 2^{33},$$

$$5\cdot 71\cdot 2^{32}, 19^2\cdot 2^{31}, 0, 13\cdot 17\cdot 2^{30}, -37\cdot 2^{32}, 7\cdot 2^{30}, 0, -7\cdot 2^{29},$$

$$37\cdot 2^{30}, -13\cdot 17\cdot 2^{27}, 0, -19^2\cdot 2^{26}, -5\cdot 71\cdot 2^{26}, 7\cdot 2^{26}, 0, -13\cdot 17\cdot 2^{24},$$

$$-227\cdot 2^{24}, -7\cdot 2^{24}, 0, 7\cdot 2^{23}, -37\cdot 2^{24}, -7\cdot 11\cdot 2^{22}, 0, -7\cdot 2^{21},$$

$$5\cdot 71\cdot 2^{20}, -7\cdot 2^{20}, 0, 13\cdot 17\cdot 2^{18}, -37\cdot 2^{20}, 7\cdot 2^{18}, 0, 19^2\cdot 2^{16},$$

$$37\cdot 2^{18}, -13\cdot 17\cdot 2^{15}, 0, 7\cdot 2^{15}, 5\cdot 2^{16}, 7\cdot 2^{14}, 0, -13\cdot 17\cdot 2^{12},$$

$$37\cdot 2^{14}, 19^2\cdot 2^{11}, 0, 7\cdot 2^{11}, -37\cdot 2^{12}, 13\cdot 17\cdot 2^{9}, 0, -7\cdot 2^{9}, 5\cdot 71\cdot 2^{8},$$

$$-7\cdot 2^{8}, 0, -7\cdot 11\cdot 2^{7}, -37\cdot 2^{8}, 7\cdot 2^{6}, 0, -7\cdot 2^{5}, -227\cdot 2^{4}, -13\cdot 17\cdot 2^{3},$$

$$0, 7\cdot 2^{3}, -5\cdot 71\cdot 2^{2}, -19^2\cdot 2, 0, -17\cdot 13, 37\cdot 2^{2}, -7, 0)) \tag{42}$$

$$\zeta(3) \;=\; \frac{1}{7 \cdot 2^{55}} P(3, 2^{60}, 120, (2^{59}, 0, -83 \cdot 2^{57}, 11 \cdot 2^{59}, 3 \cdot 41 \cdot 2^{56}, 0, 2^{56}, -11 \cdot 2^{57},$$
$$83 \cdot 2^{54}, 0, -2^{54}, 5^3 \cdot 2^{53}, -2^{53}, 0, -3 \cdot 7 \cdot 2^{52}, -11 \cdot 2^{53}, 2^{51}, 0, -2^{50}, -3^4 \cdot 2^{49},$$
$$-83 \cdot 2^{48}, 0, 2^{48}, -5^3 \cdot 2^{47}, -3 \cdot 41 \cdot 2^{46}, 0, -83 \cdot 2^{45}, 11 \cdot 2^{47}, -2^{45}, 0, 2^{44},$$
$$-11 \cdot 2^{45}, 83 \cdot 2^{42}, 0, 3 \cdot 41 \cdot 2^{41}, 5^3 \cdot 2^{41}, -2^{41}, 0, 83 \cdot 2^{39}, 3^4 \cdot 2^{39}, 2^{39}, 0, -2^{38},$$
$$11 \cdot 2^{39}, 3 \cdot 7 \cdot 2^{37}, 0, 2^{36}, -5^3 \cdot 2^{35}, 2^{35}, 0, -83 \cdot 2^{33}, 11 \cdot 2^{35}, -2^{33}, 0, -3 \cdot 41 \cdot 2^{31},$$
$$-11 \cdot 2^{33}, 83 \cdot 2^{30}, 0, -2^{30}, 0, -2^{29}, 0, 83 \cdot 2^{27}, -11 \cdot 2^{29}, -3 \cdot 41 \cdot 2^{26}, 0, -2^{26},$$
$$11 \cdot 2^{27}, -83 \cdot 2^{24}, 0, 2^{24}, -5^3 \cdot 2^{23}, 2^{23}, 0, 3 \cdot 7 \cdot 2^{22}, 11 \cdot 2^{23}, -2^{21}, 0, 2^{20}, 3^4 \cdot 2^{19},$$
$$83 \cdot 2^{18}, 0, -2^{18}, 5^3 \cdot 2^{17}, 3 \cdot 41 \cdot 2^{16}, 0, 83 \cdot 2^{15}, -11 \cdot 2^{17}, 2^{15}, 0, -2^{14}, 11 \cdot 2^{15},$$
$$-83 \cdot 2^{12}, 0, -3 \cdot 41 \cdot 2^{11}, -5^3 \cdot 2^{11}, 2^{11}, 0, -83 \cdot 2^9, -3^4 \cdot 2^9, -2^9, 0, 2^8, -11 \cdot 2^9,$$
$$-3 \cdot 7 \cdot 2^7, 0, -2^6, 5^3 \cdot 2^5, -2^5, 0, 83 \cdot 2^3, -11 \cdot 2^5, 2^3, 0, 3 \cdot 41 \cdot 2, 11 \cdot 2^3,$$
$$-83, 0, 1, 0)) \tag{43}$$

$$\pi^2 \log 2 \;=\; \frac{3}{5 \cdot 2^{56}} P(3, 2^{60}, 120, (7 \cdot 2^{59}, 0, -1031 \cdot 2^{57}, 19 \cdot 2^{62}, 3^2 \cdot 179 \cdot 2^{56}, 0, 7 \cdot 2^{56},$$
$$-19 \cdot 2^{60}, 1031 \cdot 2^{54}, 0, -7 \cdot 2^{54}, 5^3 \cdot 13 \cdot 2^{53}, -7 \cdot 2^{53}, 0, -3^3 \cdot 11 \cdot 2^{52},$$
$$-19 \cdot 2^{56}, 7 \cdot 2^{51}, 0, -7 \cdot 2^{50}, -3^2 \cdot 113 \cdot 2^{49}, -1031 \cdot 2^{48}, 0, 7 \cdot 2^{48},$$
$$-5^3 \cdot 13 \cdot 2^{47}, -3^2 \cdot 179 \cdot 2^{46}, 0, -1031 \cdot 2^{45}, 19 \cdot 2^{50}, -7 \cdot 2^{45}, 0, 7 \cdot 2^{44},$$
$$-19 \cdot 2^{48}, 1031 \cdot 2^{42}, 0, 3^2 \cdot 179 \cdot 2^{41}, 5^3 \cdot 13 \cdot 2^{41}, -7 \cdot 2^{41}, 0, 1031 \cdot 2^{39},$$
$$3^2 \cdot 113 \cdot 2^{39}, 7 \cdot 2^{39}, 0, -7 \cdot 2^{38}, 19 \cdot 2^{42}, 3^3 \cdot 11 \cdot 2^{37}, 0, 7 \cdot 2^{36}, -5^3 \cdot 13 \cdot 2^{35},$$
$$7 \cdot 2^{35}, 0, -1031 \cdot 2^{33}, 19 \cdot 2^{38}, -7 \cdot 2^{33}, 0, -3^2 \cdot 179 \cdot 2^{31}, -19 \cdot 2^{36}, 1031 \cdot 2^{30},$$
$$0, -7 \cdot 2^{30}, 0, -7 \cdot 2^{29}, 0, 1031 \cdot 2^{27}, -19 \cdot 2^{32}, -3^2 \cdot 179 \cdot 2^{26}, 0, -7 \cdot 2^{26},$$
$$19 \cdot 2^{30}, -1031 \cdot 2^{24}, 0, 7 \cdot 2^{24}, -5^3 \cdot 13 \cdot 2^{23}, 7 \cdot 2^{23}, 0, 3^3 \cdot 11 \cdot 2^{22}, 19 \cdot 2^{26},$$
$$-7 \cdot 2^{21}, 0, 7 \cdot 2^{20}, 3^2 \cdot 113 \cdot 2^{19}, 1031 \cdot 2^{18}, 0, -7 \cdot 2^{18}, 5^3 \cdot 13 \cdot 2^{17}, 3^2 \cdot 179 \cdot 2^{16},$$
$$0, 1031 \cdot 2^{15}, -19 \cdot 2^{20}, 7 \cdot 2^{15}, 0, -7 \cdot 2^{14}, 19 \cdot 2^{18}, -1031 \cdot 2^{12}, 0, -3^2 \cdot 179 \cdot 2^{11},$$
$$-5^3 \cdot 13 \cdot 2^{11}, 7 \cdot 2^{11}, 0, -1031 \cdot 2^9, -3^2 \cdot 113 \cdot 2^9, -7 \cdot 2^9, 0, 7 \cdot 2^8, -19 \cdot 2^{12},$$
$$-3^3 \cdot 11 \cdot 2^7, 0, -7 \cdot 2^6, 5^3 \cdot 13 \cdot 2^5, -7 \cdot 2^5, 0, 1031 \cdot 2^3, -19 \cdot 2^8, 7 \cdot 2^3, 0,$$
$$3^2 \cdot 179 \cdot 2, 19 \cdot 2^6, -1031, 0, 7, 0)) \tag{44}$$

13

$$\log^3 2 \ = \ \frac{3}{2^{58}}P(3, 2^{60}, 120, (2^{59}, 0, -11 \cdot 19 \cdot 2^{57}, 5 \cdot 2^{62}, 373 \cdot 2^{56}, 0, 2^{56}, -5 \cdot 2^{60},$$
$$11 \cdot 19 \cdot 2^{54}, 0, -2^{54}, 367 \cdot 2^{53}, -2^{53}, 0, -83 \cdot 2^{52}, -5 \cdot 2^{56}, 2^{51}, 0, -2^{50},$$
$$-5 \cdot 43 \cdot 2^{49}, -11 \cdot 19 \cdot 2^{48}, 0, 2^{48}, -367 \cdot 2^{47}, -373 \cdot 2^{46}, 0, -11 \cdot 19 \cdot 2^{45}, 5 \cdot 2^{50},$$
$$-2^{45}, 0, 2^{44}, -5 \cdot 2^{48}, 11 \cdot 19 \cdot 2^{42}, 0, 373 \cdot 2^{41}, 367 \cdot 2^{41}, -2^{41}, 0, 11 \cdot 19 \cdot 2^{39},$$
$$5 \cdot 43 \cdot 2^{39}, 2^{39}, 0, -2^{38}, 5 \cdot 2^{42}, 83 \cdot 2^{37}, 0, 2^{36}, -367 \cdot 2^{35}, 2^{35}, 0, -11 \cdot 19 \cdot 2^{33},$$
$$5 \cdot 2^{38}, -2^{33}, 0, -373 \cdot 2^{31}, -5 \cdot 2^{36}, 11 \cdot 19 \cdot 2^{30}, 0, -2^{30}, -2^{32}, -2^{29}, 0,$$
$$11 \cdot 19 \cdot 2^{27}, -5 \cdot 2^{32}, -373 \cdot 2^{26}, 0, -2^{26}, 5 \cdot 2^{30}, -11 \cdot 19 \cdot 2^{24}, 0, 2^{24}, -367 \cdot 2^{23},$$
$$2^{23}, 0, 83 \cdot 2^{22}, 5 \cdot 2^{26}, -2^{21}, 0, 2^{20}, 5 \cdot 43 \cdot 2^{19}, 11 \cdot 19 \cdot 2^{18}, 0, -2^{18}, 367 \cdot 2^{17},$$
$$373 \cdot 2^{16}, 0, 11 \cdot 19 \cdot 2^{15}, -5 \cdot 2^{20}, 2^{15}, 0, -2^{14}, 5 \cdot 2^{18}, -11 \cdot 19 \cdot 2^{12}, 0, -373 \cdot 2^{11},$$
$$-367 \cdot 2^{11}, 2^{11}, 0, -11 \cdot 19 \cdot 2^{9}, -5 \cdot 43 \cdot 2^{9}, -2^{9}, 0, 2^{8}, -5 \cdot 2^{12}, -83 \cdot 2^{7}, 0, -2^{6},$$
$$367 \cdot 2^{5}, -2^{5}, 0, 11 \cdot 19 \cdot 2^{3}, -5 \cdot 2^{8}, 2^{3}, 0, 373 \cdot 2, 5 \cdot 2^{6}, -19 \cdot 11, 0, 1, 2^{2})) \quad (45)$$

$$\frac{1}{3}\log^3 2 - \frac{5}{12}\pi^2 \log 2 + \frac{35}{4}\zeta(3) \ = \ P(3, 16, 8, (8, 0, -4, -4, -2, 0, 1, 1)) \quad (46)$$

The existence of BBP formulas for these constants was originally established by Broadhurst [18]. However, except for 32, which appeared in [14], the specific explicit formulas listed here were produced by the author's PSLQ program. Formula 34 is proved by subtracting $5/192\times$Relation 118 from Formula 39. Formula 35=Formula 42+37/75/$2^{57}\times$ Relation 119+133/75/$2^{56}\times$ Relation 120. Formula 36=Formula 41+1/5/$2^{55}\times$ Relation 119+ 9/5/$2^{54}\times$ Relation 120. The results for $\pi \log^2 2$ and π^3 were produced by a special parallel version of this program, running on the IBM SP parallel computer system in the NERSC supercomputer facility at the Lawrence Berkeley National Laboratory. Formulas 37 through 46 were recently proven by Kunle Adegoke [6, 2].

7 Degree 4 binary formulas

$$\pi^4 = \frac{1}{164}P(4, 2^{12}, 24, (27 \cdot 2^{11}, -513 \cdot 2^{11}, 135 \cdot 2^{14}, -27 \cdot 2^{11}, -27 \cdot 2^9,$$
$$-621 \cdot 2^{10}, 27 \cdot 2^8, -729 \cdot 2^{10}, -135 \cdot 2^{11}, -513 \cdot 2^7, -27 \cdot 2^6,$$
$$-189 \cdot 2^9, -27 \cdot 2^5, -513 \cdot 2^5, -135 \cdot 2^8, -729 \cdot 2^6, 216, -621 \cdot 2^4,$$
$$-108, -216, 135 \cdot 2^5, -1026, 27, 0)) \tag{47}$$

$$\log^4 2 = \frac{1}{205 \cdot 2^5}P(4, 2^{12}, 24, (73 \cdot 2^{12}, -2617 \cdot 2^{12}, 8455 \cdot 2^{12}, -2533 \cdot 2^{12},$$
$$-73 \cdot 2^{10}, -25781 \cdot 2^9, 73 \cdot 2^9, -6891 \cdot 2^{11}, -8455 \cdot 2^9, -2617 \cdot 2^8,$$
$$-73 \cdot 2^7, -23551 \cdot 2^6, -73 \cdot 2^6, -2617 \cdot 2^6, -8455 \cdot 2^6, -6891 \cdot 2^7,$$
$$73 \cdot 2^4, -25781 \cdot 2^3, -73 \cdot 2^3, -2533 \cdot 2^4, 8455 \cdot 2^3, -10468,$$
$$146, -615)) \tag{48}$$

$$\pi^2 \log^2 2 = \frac{1}{41 \cdot 2^5}P(4, 2^{12}, 24, (121 \cdot 2^{11}, -3775 \cdot 2^{11}, 10375 \cdot 2^{11}, -1597 \cdot 2^{11},$$
$$-121 \cdot 2^9, -3421 \cdot 2^{11}, 121 \cdot 2^8, -7695 \cdot 2^{10}, -10375 \cdot 2^8, -3775 \cdot 2^7,$$
$$-121 \cdot 2^6, -3539 \cdot 2^8, -121 \cdot 2^5, -3775 \cdot 2^5, -10375 \cdot 2^5, -7695 \cdot 2^6,$$
$$121 \cdot 2^3, -3421 \cdot 2^5, -484, -1597 \cdot 2^3, 41500, -7550, 121, 0)) \tag{49}$$

$$\pi^4 = \frac{675}{7 \cdot 71 \cdot 2^{51}}P(4, 2^{60}, 120, (2^{59}, -5 \cdot 2^{61}, 11 \cdot 17 \cdot 2^{57}, -2^{61}, -127 \cdot 2^{56},$$
$$-5 \cdot 2^{59}, 2^{56}, -2^{61}, -11 \cdot 17 \cdot 2^{54}, -5 \cdot 2^{57}, -2^{54}, -5 \cdot 41 \cdot 2^{53},$$
$$-2^{53}, -5 \cdot 2^{55}, -31 \cdot 2^{52}, -2^{57}, 2^{51}, -5 \cdot 2^{53}, -2^{50}, 109 \cdot 2^{49},$$
$$11 \cdot 17 \cdot 2^{48}, -5 \cdot 2^{51}, 2^{48}, 5^3 \cdot 2^{47}, 127 \cdot 2^{46}, -5 \cdot 2^{49}, 11 \cdot 17 \cdot 2^{45},$$
$$-2^{49}, -2^{45}, -5 \cdot 2^{47}, 2^{44}, -2^{49}, -11 \cdot 17 \cdot 2^{42}, -5 \cdot 2^{45}, -127 \cdot 2^{41},$$
$$-5 \cdot 41 \cdot 2^{41}, -2^{41}, -5 \cdot 2^{43}, -11 \cdot 17 \cdot 2^{39}, -3^3 \cdot 7 \cdot 2^{39}, 2^{39}, -5 \cdot 2^{41},$$
$$-2^{38}, -2^{41}, 31 \cdot 2^{37}, -5 \cdot 2^{39}, 2^{36}, 5^3 \cdot 2^{35}, 2^{35}, -5 \cdot 2^{37},$$
$$11 \cdot 17 \cdot 2^{33}, -2^{37}, -2^{33}, -5 \cdot 2^{35}, 127 \cdot 2^{31}, -2^{37}, -11 \cdot 17 \cdot 2^{30},$$
$$-5 \cdot 2^{33}, -2^{30}, -5 \cdot 2^{33}, -2^{29}, -5 \cdot 2^{31}, -11 \cdot 17 \cdot 2^{27}, -2^{33},$$
$$127 \cdot 2^{26}, -5 \cdot 2^{29}, -2^{26}, -2^{29}, 11 \cdot 17 \cdot 2^{24}, -5 \cdot 2^{27}, 2^{24},$$
$$5^3 \cdot 2^{23}, 2^{23}, -5 \cdot 2^{25}, 31 \cdot 2^{22}, -2^{25}, -2^{21}, -5 \cdot 2^{23}, 2^{20},$$
$$-3^3 \cdot 7 \cdot 2^{19}, -11 \cdot 17 \cdot 2^{18}, -5 \cdot 2^{21}, -2^{18}, -5 \cdot 41 \cdot 2^{17}, -127 \cdot 2^{16},$$
$$-5 \cdot 2^{19}, -11 \cdot 17 \cdot 2^{15}, -2^{21}, 2^{15}, -5 \cdot 2^{17}, -2^{14}, -2^{17}, 11 \cdot 17 \cdot 2^{12},$$
$$-5 \cdot 2^{15}, 127 \cdot 2^{11}, 5^3 \cdot 2^{11}, 2^{11}, -5 \cdot 2^{13}, 11 \cdot 17 \cdot 2^9, 109 \cdot 2^9, -2^9,$$
$$-5 \cdot 2^{11}, 2^8, -2^{13}, -31 \cdot 2^7, -5 \cdot 2^9, -2^6, -5 \cdot 41 \cdot 2^5, -2^5, -5 \cdot 2^7,$$
$$-11 \cdot 17 \cdot 2^3, -2^9, 2^3, -5 \cdot 2^5, -127 \cdot 2, -2^5, 17 \cdot 11, -5 \cdot 2^3, 1, 0)) \tag{50}$$

$$\log^4 2 = \frac{1}{7 \cdot 71 \cdot 2^{54}} P(4, 2^{60}, 120, (823 \cdot 2^{59}, -5 \cdot 3137 \cdot 2^{60}, 11 \cdot 40829 \cdot 2^{57}, -18047 \cdot 2^{60},$$
$$-277 \cdot 1723 \cdot 2^{56}, -5 \cdot 3137 \cdot 2^{58}, 823 \cdot 2^{56}, 1181 \cdot 2^{59}, -11 \cdot 40829 \cdot 2^{54},$$
$$-5 \cdot 3137 \cdot 2^{56}, -823 \cdot 2^{54}, -595141 \cdot 2^{53}, -823 \cdot 2^{53}, -5 \cdot 3137 \cdot 2^{54},$$
$$29 \cdot 457 \cdot 2^{52}, 1181 \cdot 2^{55}, 823 \cdot 2^{51}, -5 \cdot 3137 \cdot 2^{52}, -823 \cdot 2^{50},$$
$$331249 \cdot 2^{49}, 11 \cdot 40829 \cdot 2^{48}, -5 \cdot 3137 \cdot 2^{50}, 823 \cdot 2^{48}, 19^2 \cdot 1301 \cdot 2^{47},$$
$$277 \cdot 1723 \cdot 2^{46}, -5 \cdot 3137 \cdot 2^{48}, 11 \cdot 40829 \cdot 2^{45}, -18047 \cdot 2^{48}, -823 \cdot 2^{45},$$
$$-5 \cdot 3137 \cdot 2^{46}, 823 \cdot 2^{44}, 1181 \cdot 2^{47}, -11 \cdot 40829 \cdot 2^{42}, -5 \cdot 3137 \cdot 2^{44},$$
$$-277 \cdot 1723 \cdot 2^{41}, -595141 \cdot 2^{41}, -823 \cdot 2^{41}, -5 \cdot 3137 \cdot 2^{42}, -11 \cdot 40829 \cdot 2^{39},$$
$$-3 \cdot 7^2 \cdot 13 \cdot 239 \cdot 2^{39}, 823 \cdot 2^{39}, -5 \cdot 3137 \cdot 2^{40}, -823 \cdot 2^{38}, -18047 \cdot 2^{40},$$
$$-29 \cdot 457 \cdot 2^{37}, -5 \cdot 3137 \cdot 2^{38}, 823 \cdot 2^{36}, 19^2 \cdot 1301 \cdot 2^{35}, 823 \cdot 2^{35},$$
$$-5 \cdot 3137 \cdot 2^{36}, 11 \cdot 40829 \cdot 2^{33}, -18047 \cdot 2^{36}, -823 \cdot 2^{33}, -5 \cdot 3137 \cdot 2^{34},$$
$$277 \cdot 1723 \cdot 2^{31}, 1181 \cdot 2^{35}, -11 \cdot 40829 \cdot 2^{30}, -5 \cdot 3137 \cdot 2^{32}, -823 \cdot 2^{30},$$
$$-29879 \cdot 2^{31}, -823 \cdot 2^{29}, -5 \cdot 3137 \cdot 2^{30}, -11 \cdot 40829 \cdot 2^{27}, 1181 \cdot 2^{31},$$
$$277 \cdot 1723 \cdot 2^{26}, -5 \cdot 3137 \cdot 2^{28}, -823 \cdot 2^{26}, -18047 \cdot 2^{28}, 11 \cdot 40829 \cdot 2^{24},$$
$$-5 \cdot 3137 \cdot 2^{26}, 823 \cdot 2^{24}, 19^2 \cdot 1301 \cdot 2^{23}, 823 \cdot 2^{23}, -5 \cdot 3137 \cdot 2^{24},$$
$$-29 \cdot 457 \cdot 2^{22}, -18047 \cdot 2^{24}, -823 \cdot 2^{21}, -5 \cdot 3137 \cdot 2^{22}, 823 \cdot 2^{20},$$
$$-3 \cdot 7^2 \cdot 13 \cdot 239 \cdot 2^{19}, -11 \cdot 40829 \cdot 2^{18}, -5 \cdot 3137 \cdot 2^{20}, -823 \cdot 2^{18},$$
$$-595141 \cdot 2^{17}, -277 \cdot 1723 \cdot 2^{16}, -5 \cdot 3137 \cdot 2^{18}, -11 \cdot 40829 \cdot 2^{15},$$
$$1181 \cdot 2^{19}, 823 \cdot 2^{15}, -5 \cdot 3137 \cdot 2^{16}, -823 \cdot 2^{14}, -18047 \cdot 2^{16},$$
$$11 \cdot 40829 \cdot 2^{12}, -5 \cdot 3137 \cdot 2^{14}, 277 \cdot 1723 \cdot 2^{11}, 19^2 \cdot 1301 \cdot 2^{11},$$
$$823 \cdot 2^{11}, -5 \cdot 3137 \cdot 2^{12}, 11 \cdot 40829 \cdot 2^9, 331249 \cdot 2^9, -823 \cdot 2^9,$$
$$-5 \cdot 3137 \cdot 2^{10}, 823 \cdot 2^8, 1181 \cdot 2^{11}, 29 \cdot 457 \cdot 2^7, -5 \cdot 3137 \cdot 2^8,$$
$$-823 \cdot 2^6, -595141 \cdot 2^5, -823 \cdot 2^5, -5 \cdot 3137 \cdot 2^6, -11 \cdot 40829 \cdot 2^3,$$
$$1181 \cdot 2^7, 823 \cdot 2^3, -5 \cdot 3137 \cdot 2^4, -277 \cdot 1723 \cdot 2, -18047 \cdot 2^4,$$
$$40829 \cdot 11, -5 \cdot 3137 \cdot 2^2, 823, -3 \cdot 7 \cdot 71 \cdot 2)) \tag{51}$$

$$\pi^2 \log^2 2 = \frac{1}{7 \cdot 71 \cdot 2^{54}} P(4, 2^{60}, 120, (13^2 \cdot 19 \cdot 2^{59}, -37 \cdot 179 \cdot 2^{63}, 5 \cdot 13 \cdot 19973 \cdot 2^{57},$$
$$-5 \cdot 1663 \cdot 2^{62}, -17 \cdot 179 \cdot 379 \cdot 2^{56}, -37 \cdot 179 \cdot 2^{61}, 13^2 \cdot 19 \cdot 2^{56}, -4931 \cdot 2^{60},$$
$$-5 \cdot 13 \cdot 19973 \cdot 2^{54}, -37 \cdot 179 \cdot 2^{59}, -13^2 \cdot 19 \cdot 2^{54}, -43 \cdot 36529 \cdot 2^{53}, -13^2 \cdot 19 \cdot 2^{53},$$
$$-37 \cdot 179 \cdot 2^{57}, -5 \cdot 15137 \cdot 2^{52}, -4931 \cdot 2^{56}, 13^2 \cdot 19 \cdot 2^{51}, -37 \cdot 179 \cdot 2^{55},$$
$$-13^2 \cdot 19 \cdot 2^{50}, 5 \cdot 176159 \cdot 2^{49}, 5 \cdot 13 \cdot 19973 \cdot 2^{48}, -37 \cdot 179 \cdot 2^{53}, 13^2 \cdot 19 \cdot 2^{48},$$
$$5^5 \cdot 367 \cdot 2^{47}, 17 \cdot 179 \cdot 379 \cdot 2^{46}, -37 \cdot 179 \cdot 2^{51}, 5 \cdot 13 \cdot 19973 \cdot 2^{45}, -5 \cdot 1663 \cdot 2^{50},$$
$$-13^2 \cdot 19 \cdot 2^{45}, -37 \cdot 179 \cdot 2^{49}, 13^2 \cdot 19 \cdot 2^{44}, -4931 \cdot 2^{48}, -5 \cdot 13 \cdot 19973 \cdot 2^{42},$$
$$-37 \cdot 179 \cdot 2^{47}, -17 \cdot 179 \cdot 379 \cdot 2^{41}, -43 \cdot 36529 \cdot 2^{41}, -13^2 \cdot 19 \cdot 2^{41}, -37 \cdot 179 \cdot 2^{45},$$
$$-5 \cdot 13 \cdot 19973 \cdot 2^{39}, -3^5 \cdot 7 \cdot 13 \cdot 59 \cdot 2^{39}, 13^2 \cdot 19 \cdot 2^{39}, -37 \cdot 179 \cdot 2^{43}, -13^2 \cdot 19 \cdot 2^{38},$$
$$-5 \cdot 1663 \cdot 2^{42}, 5 \cdot 15137 \cdot 2^{37}, -37 \cdot 179 \cdot 2^{41}, 13^2 \cdot 19 \cdot 2^{36}, 5^5 \cdot 367 \cdot 2^{35},$$
$$13^2 \cdot 19 \cdot 2^{35}, -37 \cdot 179 \cdot 2^{39}, 5 \cdot 13 \cdot 19973 \cdot 2^{33}, -5 \cdot 1663 \cdot 2^{38}, -13^2 \cdot 19 \cdot 2^{33},$$
$$-37 \cdot 179 \cdot 2^{37}, 17 \cdot 179 \cdot 379 \cdot 2^{31}, -4931 \cdot 2^{36}, -5 \cdot 13 \cdot 19973 \cdot 2^{30}, -37 \cdot 179 \cdot 2^{35},$$
$$-13^2 \cdot 19 \cdot 2^{30}, -37 \cdot 179 \cdot 2^{35}, -13^2 \cdot 19 \cdot 2^{29}, -37 \cdot 179 \cdot 2^{33}, -5 \cdot 13 \cdot 19973 \cdot 2^{27},$$
$$-4931 \cdot 2^{32}, 17 \cdot 179 \cdot 379 \cdot 2^{26}, -37 \cdot 179 \cdot 2^{31}, -13^2 \cdot 19 \cdot 2^{26}, -5 \cdot 1663 \cdot 2^{30},$$
$$5 \cdot 13 \cdot 19973 \cdot 2^{24}, -37 \cdot 179 \cdot 2^{29}, 13^2 \cdot 19 \cdot 2^{24}, 5^5 \cdot 367 \cdot 2^{23}, 13^2 \cdot 19 \cdot 2^{23},$$
$$-37 \cdot 179 \cdot 2^{27}, 5 \cdot 15137 \cdot 2^{22}, -5 \cdot 1663 \cdot 2^{26}, -13^2 \cdot 19 \cdot 2^{21}, -37 \cdot 179 \cdot 2^{25},$$
$$13^2 \cdot 19 \cdot 2^{20}, -3^5 \cdot 7 \cdot 13 \cdot 59 \cdot 2^{19}, -5 \cdot 13 \cdot 19973 \cdot 2^{18}, -37 \cdot 179 \cdot 2^{23},$$
$$-13^2 \cdot 19 \cdot 2^{18}, -43 \cdot 36529 \cdot 2^{17}, -17 \cdot 179 \cdot 379 \cdot 2^{16}, -37 \cdot 179 \cdot 2^{21}, -5 \cdot 13 \cdot 19973 \cdot 2^{15},$$
$$-4931 \cdot 2^{20}, 13^2 \cdot 19 \cdot 2^{15}, -37 \cdot 179 \cdot 2^{19}, -13^2 \cdot 19 \cdot 2^{14}, -5 \cdot 1663 \cdot 2^{18},$$
$$5 \cdot 13 \cdot 19973 \cdot 2^{12}, -37 \cdot 179 \cdot 2^{17}, 17 \cdot 179 \cdot 379 \cdot 2^{11}, 5^5 \cdot 367 \cdot 2^{11}, 13^2 \cdot 19 \cdot 2^{11},$$
$$-37 \cdot 179 \cdot 2^{15}, 5 \cdot 13 \cdot 19973 \cdot 2^9, 5 \cdot 176159 \cdot 2^9, -13^2 \cdot 19 \cdot 2^9, -37 \cdot 179 \cdot 2^{13},$$
$$13^2 \cdot 19 \cdot 2^8, -4931 \cdot 2^{12}, -5 \cdot 15137 \cdot 2^7, -37 \cdot 179 \cdot 2^{11}, -13^2 \cdot 19 \cdot 2^6,$$
$$-43 \cdot 36529 \cdot 2^5, -13^2 \cdot 19 \cdot 2^5, -37 \cdot 179 \cdot 2^9, -5 \cdot 13 \cdot 19973 \cdot 2^3, -4931 \cdot 2^8,$$
$$13^2 \cdot 19 \cdot 2^3, -37 \cdot 179 \cdot 2^7, -17 \cdot 179 \cdot 379 \cdot 2, -5 \cdot 1663 \cdot 2^6, 13 \cdot 19973 \cdot 5,$$
$$-37 \cdot 179 \cdot 2^5, 19 \cdot 13^2, 0)) \tag{52}$$

$$20 \operatorname{Cl}_4\left(\frac{\pi}{2}\right) + \frac{3\pi \log^3 2}{32} - \frac{27\pi^3 \log 2}{128} = \frac{9}{2^{10}} P(4, 2^{12}, 24, (2^{11}, -2^{12}, -7 \cdot 2^9, 0, -2^9,$$
$$-5 \cdot 2^8, -2^8, 0, -7 \cdot 2^6, -2^8, 2^6, 0, -2^5, 2^6, 7 \cdot 2^3, 0,$$
$$2^3, 5 \cdot 2^2, 2^2, 0, 7, 2^2, -1, 0)) \tag{53}$$

$$12\,\mathrm{Cl}_4\left(\frac{\pi}{2}\right) + \frac{29\pi \log^3 2}{192} - \frac{47\pi^3 \log 2}{256}$$
$$= \frac{1}{2^{58}}P(4, 2^{60}, 120, (2^{60}, 23 \cdot 2^{60},$$
$$-239 \cdot 2^{57}, 0, -3 \cdot 211 \cdot 2^{55}, -5 \cdot 67 \cdot 2^{56}, -2^{57}, 0, -239 \cdot 2^{54},$$
$$-3^2 \cdot 7^2 \cdot 2^{53}, 2^{55}, 0, -2^{54}, -23 \cdot 2^{54}, -3 \cdot 7^2 \cdot 2^{50}, 0,$$
$$2^{52}, 5 \cdot 67 \cdot 2^{50}, 2^{51}, 0, 239 \cdot 2^{48}, -23 \cdot 2^{50}, -2^{49}, 0,$$
$$3 \cdot 211 \cdot 2^{45}, 23 \cdot 2^{48}, -239 \cdot 2^{45}, 0, -2^{46}, -3^2 \cdot 5 \cdot 2^{43},$$
$$-2^{45}, 0, -239 \cdot 2^{42}, 23 \cdot 2^{44}, 3 \cdot 211 \cdot 2^{40}, 0, -2^{42}, -23 \cdot 2^{42}, 239 \cdot 2^{39},$$
$$0, 2^{40}, 5 \cdot 67 \cdot 2^{38}, 2^{39}, 0, -3 \cdot 7^2 \cdot 2^{35}, -23 \cdot 2^{38}, -2^{37}, 0, 2^{36},$$
$$-3^2 \cdot 7^2 \cdot 2^{33}, -239 \cdot 2^{33}, 0, -2^{34}, -5 \cdot 67 \cdot 2^{32}, -3 \cdot 211 \cdot 2^{30}, 0,$$
$$-239 \cdot 2^{30}, 23 \cdot 2^{32}, 2^{31}, 0, -2^{30}, -23 \cdot 2^{30}, 239 \cdot 2^{27}, 0, 3 \cdot 211 \cdot 2^{25},$$
$$5 \cdot 67 \cdot 2^{26}, 2^{27}, 0, 239 \cdot 2^{24}, 3^2 \cdot 7^2 \cdot 2^{23}, -2^{25}, 0, 2^{24}, 23 \cdot 2^{24},$$
$$3 \cdot 7^2 \cdot 2^{20}, 0, -2^{22}, -5 \cdot 67 \cdot 2^{20}, -2^{21}, 0, -239 \cdot 2^{18}, 23 \cdot 2^{20}, 2^{19}, 0,$$
$$-3 \cdot 211 \cdot 2^{15}, -23 \cdot 2^{18}, 239 \cdot 2^{15}, 0, 2^{16}, 3^2 \cdot 5 \cdot 2^{13}, 2^{15}, 0, 239 \cdot 2^{12},$$
$$-23 \cdot 2^{14}, -3 \cdot 211 \cdot 2^{10}, 0, 2^{12}, 23 \cdot 2^{12}, -239 \cdot 2^9, 0, -2^{10},$$
$$-5 \cdot 67 \cdot 2^8, -2^9, 0, 3 \cdot 7^2 \cdot 2^5, 23 \cdot 2^8, 2^7, 0, -2^6, 3^2 \cdot 7^2 \cdot 2^3,$$
$$239 \cdot 2^3, 0, 2^4, 5 \cdot 67 \cdot 2^2, 211 \cdot 3, 0, 239, -23 \cdot 2^2, -2, 0)) \tag{54}$$

The existence of BBP-type formulas for these constants was originally established by Broadhurst [18], although the explicit formulas given here were found by the author's PSLQ program. Formulas 47 to 49 were subsequently proved by Kunle Adegoke [8] who also found formulas 50 to 54. Here Cl_4 is a Clausen function (see section 9).

8 Degree 5 binary formulas

$$
\begin{aligned}
\zeta(5) \;=\; & \frac{1}{62651 \cdot 2^{49}} P(5, 2^{60}, 120, (279 \cdot 2^{59}, -7263 \cdot 2^{60}, 293715 \cdot 2^{57}, \\
& -13977 \cdot 2^{60}, -1153683 \cdot 2^{56}, 28377 \cdot 2^{60}, 279 \cdot 2^{56}, 83871 \cdot 2^{59}, \\
& -293715 \cdot 2^{54}, -7263 \cdot 2^{56}, -279 \cdot 2^{54}, -889173 \cdot 2^{53}, -279 \cdot 2^{53}, \\
& -7263 \cdot 2^{54}, 429705 \cdot 2^{52}, 83871 \cdot 2^{55}, 279 \cdot 2^{51}, 28377 \cdot 2^{54}, \\
& -279 \cdot 2^{50}, 1041309 \cdot 2^{49}, 293715 \cdot 2^{48}, -7263 \cdot 2^{50}, 279 \cdot 2^{48}, \\
& 1153125 \cdot 2^{47}, 1153683 \cdot 2^{46}, -7263 \cdot 2^{48}, 293715 \cdot 2^{45}, -13977 \cdot 2^{48}, \\
& -279 \cdot 2^{45}, 28377 \cdot 2^{48}, 279 \cdot 2^{44}, 83871 \cdot 2^{47}, -293715 \cdot 2^{42}, \\
& -7263 \cdot 2^{44}, -1153683 \cdot 2^{41}, -889173 \cdot 2^{41}, -279 \cdot 2^{41}, -7263 \cdot 2^{42}, \\
& -293715 \cdot 2^{39}, 188811 \cdot 2^{39}, 279 \cdot 2^{39}, 28377 \cdot 2^{42}, -279 \cdot 2^{38}, \\
& -13977 \cdot 2^{40}, -429705 \cdot 2^{37}, -7263 \cdot 2^{38}, 279 \cdot 2^{36}, 1153125 \cdot 2^{35}, \\
& 279 \cdot 2^{35}, -7263 \cdot 2^{36}, 293715 \cdot 2^{33}, -13977 \cdot 2^{36}, -279 \cdot 2^{33}, \\
& 28377 \cdot 2^{36}, 1153683 \cdot 2^{31}, 83871 \cdot 2^{35}, -293715 \cdot 2^{30}, -7263 \cdot 2^{32}, \\
& -279 \cdot 2^{30}, 16497 \cdot 2^{33}, -279 \cdot 2^{29}, -7263 \cdot 2^{30}, -293715 \cdot 2^{27}, \\
& 83871 \cdot 2^{31}, 1153683 \cdot 2^{26}, 28377 \cdot 2^{30}, -279 \cdot 2^{26}, -13977 \cdot 2^{28}, \\
& 293715 \cdot 2^{24}, -7263 \cdot 2^{26}, 279 \cdot 2^{24}, 1153125 \cdot 2^{23}, 279 \cdot 2^{23}, \\
& -7263 \cdot 2^{24}, -429705 \cdot 2^{22}, -13977 \cdot 2^{24}, -279 \cdot 2^{21}, 28377 \cdot 2^{24}, \\
& 279 \cdot 2^{20}, 188811 \cdot 2^{19}, -293715 \cdot 2^{18}, -7263 \cdot 2^{20}, -279 \cdot 2^{18}, \\
& -889173 \cdot 2^{17}, -1153683 \cdot 2^{16}, -7263 \cdot 2^{18}, -293715 \cdot 2^{15}, 83871 \cdot 2^{19}, \\
& 279 \cdot 2^{15}, 28377 \cdot 2^{18}, -279 \cdot 2^{14}, -13977 \cdot 2^{16}, 293715 \cdot 2^{12}, \\
& -7263 \cdot 2^{14}, 1153683 \cdot 2^{11}, 1153125 \cdot 2^{11}, 279 \cdot 2^{11}, -7263 \cdot 2^{12}, \\
& 293715 \cdot 2^{9}, 1041309 \cdot 2^{9}, -279 \cdot 2^{9}, 28377 \cdot 2^{12}, 279 \cdot 2^{8}, \\
& 83871 \cdot 2^{11}, 429705 \cdot 2^{7}, -7263 \cdot 2^{8}, -279 \cdot 2^{6}, -889173 \cdot 2^{5}, \\
& -279 \cdot 2^{5}, -7263 \cdot 2^{6}, -293715 \cdot 2^{3}, 83871 \cdot 2^{7}, 279 \cdot 2^{3}, \\
& 28377 \cdot 2^{6}, -2307366, -13977 \cdot 2^{4}, 293715, -29052, 279, 0))
\end{aligned}
$$

(55)

$$\log^5 2 \;=\; \frac{1}{2021 \cdot 2^{52}} P(5, 2^{60}, 120, (2783 \cdot 2^{59}, -32699 \cdot 2^{62}, 7171925 \cdot 2^{57},$$
$$-187547 \cdot 2^{61}, -41252441 \cdot 2^{56}, 9391097 \cdot 2^{57}, 2783 \cdot 2^{56},$$
$$52183 \cdot 2^{65}, -7171925 \cdot 2^{54}, -32699 \cdot 2^{58}, -2783 \cdot 2^{54},$$
$$-29483621 \cdot 2^{53}, -2783 \cdot 2^{53}, -32699 \cdot 2^{56}, 17037475 \cdot 2^{52},$$
$$52183 \cdot 2^{61}, 2783 \cdot 2^{51}, 9391097 \cdot 2^{51}, -2783 \cdot 2^{50},$$
$$38246123 \cdot 2^{49}, 7171925 \cdot 2^{48}, -32699 \cdot 2^{52}, 2783 \cdot 2^{48},$$
$$41307505 \cdot 2^{47}, 41252441 \cdot 2^{46}, -32699 \cdot 2^{50}, 7171925 \cdot 2^{45},$$
$$-187547 \cdot 2^{49}, -2783 \cdot 2^{45}, 9391097 \cdot 2^{45}, 2783 \cdot 2^{44},$$
$$52183 \cdot 2^{53}, -7171925 \cdot 2^{42}, -32699 \cdot 2^{46}, -41252441 \cdot 2^{41},$$
$$-29483621 \cdot 2^{41}, -2783 \cdot 2^{41}, -32699 \cdot 2^{44}, -7171925 \cdot 2^{39},$$
$$12188517 \cdot 2^{39}, 2783 \cdot 2^{39}, 9391097 \cdot 2^{39}, -2783 \cdot 2^{38},$$
$$-187547 \cdot 2^{41}, -17037475 \cdot 2^{37}, -32699 \cdot 2^{40}, 2783 \cdot 2^{36},$$
$$41307505 \cdot 2^{35}, 2783 \cdot 2^{35}, -32699 \cdot 2^{38}, 7171925 \cdot 2^{33},$$
$$-187547 \cdot 2^{37}, -2783 \cdot 2^{33}, 9391097 \cdot 2^{33}, 41252441 \cdot 2^{31},$$
$$52183 \cdot 2^{41}, -7171925 \cdot 2^{30}, -32699 \cdot 2^{34}, -2783 \cdot 2^{30},$$
$$5881627 \cdot 2^{30}, -2783 \cdot 2^{29}, -32699 \cdot 2^{32}, -7171925 \cdot 2^{27},$$
$$52183 \cdot 2^{37}, 41252441 \cdot 2^{26}, 9391097 \cdot 2^{27}, -2783 \cdot 2^{26},$$
$$-187547 \cdot 2^{29}, 7171925 \cdot 2^{24}, -32699 \cdot 2^{28}, 2783 \cdot 2^{24},$$
$$41307505 \cdot 2^{23}, 2783 \cdot 2^{23}, -32699 \cdot 2^{26}, -17037475 \cdot 2^{22},$$
$$-187547 \cdot 2^{25}, -2783 \cdot 2^{21}, 9391097 \cdot 2^{21}, 2783 \cdot 2^{20},$$
$$12188517 \cdot 2^{19}, -7171925 \cdot 2^{18}, -32699 \cdot 2^{22}, -2783 \cdot 2^{18},$$
$$-29483621 \cdot 2^{17}, -41252441 \cdot 2^{16}, -32699 \cdot 2^{20}, -7171925 \cdot 2^{15},$$
$$52183 \cdot 2^{25}, 2783 \cdot 2^{15}, 9391097 \cdot 2^{15}, -2783 \cdot 2^{14}, -187547 \cdot 2^{17},$$
$$7171925 \cdot 2^{12}, -32699 \cdot 2^{16}, 41252441 \cdot 2^{11}, 41307505 \cdot 2^{11}, 2783 \cdot 2^{11},$$
$$-32699 \cdot 2^{14}, 7171925 \cdot 2^{9}, 38246123 \cdot 2^{9}, -2783 \cdot 2^{9}, 9391097 \cdot 2^{9},$$
$$2783 \cdot 2^{8}, 52183 \cdot 2^{17}, 17037475 \cdot 2^{7}, -32699 \cdot 2^{10}, -2783 \cdot 2^{6},$$
$$-29483621 \cdot 2^{5}, -2783 \cdot 2^{5}, -32699 \cdot 2^{8}, -7171925 \cdot 2^{3}, 52183 \cdot 2^{13},$$
$$2783 \cdot 2^{3}, 9391097 \cdot 2^{3}, -82504882, -187547 \cdot 2^{5}, 7171925,$$
$$-32699 \cdot 2^{4}, 2783, 30315)) \tag{56}$$

$$\pi^2 \log^3 2 \;=\; \frac{1}{2021 \cdot 2^{53}} P(5, 2^{60}, 120, (21345 \cdot 2^{59}, -464511 \cdot 2^{61}, 47870835 \cdot 2^{57},$$
$$-1312971 \cdot 2^{61}, -236170815 \cdot 2^{56}, 1579179 \cdot 2^{62}, 21345 \cdot 2^{56},$$
$$286131 \cdot 2^{65}, -47870835 \cdot 2^{54}, -464511 \cdot 2^{57}, -21345 \cdot 2^{54},$$
$$-173704605 \cdot 2^{53}, -21345 \cdot 2^{53}, -464511 \cdot 2^{55}, 94128645 \cdot 2^{52},$$
$$286131 \cdot 2^{61}, 21345 \cdot 2^{51}, 1579179 \cdot 2^{56}, -21345 \cdot 2^{50},$$
$$215120589 \cdot 2^{49}, 47870835 \cdot 2^{48}, -464511 \cdot 2^{51}, 21345 \cdot 2^{48},$$
$$236128125 \cdot 2^{47}, 236170815 \cdot 2^{46}, -464511 \cdot 2^{49}, 47870835 \cdot 2^{45},$$
$$-1312971 \cdot 2^{49}, -21345 \cdot 2^{45}, 1579179 \cdot 2^{50}, 21345 \cdot 2^{44},$$
$$286131 \cdot 2^{53}, -47870835 \cdot 2^{42}, -464511 \cdot 2^{45}, -236170815 \cdot 2^{41},$$
$$-173704605 \cdot 2^{41}, -21345 \cdot 2^{41}, -464511 \cdot 2^{43}, -47870835 \cdot 2^{39},$$
$$56870019 \cdot 2^{39}, 21345 \cdot 2^{39}, 1579179 \cdot 2^{44}, -21345 \cdot 2^{38},$$
$$-1312971 \cdot 2^{41}, -94128645 \cdot 2^{37}, -464511 \cdot 2^{39}, 21345 \cdot 2^{36},$$
$$236128125 \cdot 2^{35}, 21345 \cdot 2^{35}, -464511 \cdot 2^{37}, 47870835 \cdot 2^{33},$$
$$-1312971 \cdot 2^{37}, -21345 \cdot 2^{33}, 1579179 \cdot 2^{38}, 236170815 \cdot 2^{31},$$
$$286131 \cdot 2^{41}, -47870835 \cdot 2^{30}, -464511 \cdot 2^{33}, -21345 \cdot 2^{30},$$
$$1950735 \cdot 2^{34}, -21345 \cdot 2^{29}, -464511 \cdot 2^{31}, -47870835 \cdot 2^{27},$$
$$286131 \cdot 2^{37}, 236170815 \cdot 2^{26}, 1579179 \cdot 2^{32}, -21345 \cdot 2^{26},$$
$$-1312971 \cdot 2^{29}, 47870835 \cdot 2^{24}, -464511 \cdot 2^{27}, 21345 \cdot 2^{24},$$
$$236128125 \cdot 2^{23}, 21345 \cdot 2^{23}, -464511 \cdot 2^{25}, -94128645 \cdot 2^{22},$$
$$-1312971 \cdot 2^{25}, -21345 \cdot 2^{21}, 1579179 \cdot 2^{26}, 21345 \cdot 2^{20},$$
$$56870019 \cdot 2^{19}, -47870835 \cdot 2^{18}, -464511 \cdot 2^{21}, -21345 \cdot 2^{18},$$
$$-173704605 \cdot 2^{17}, -236170815 \cdot 2^{16}, -464511 \cdot 2^{19}, -47870835 \cdot 2^{15},$$
$$286131 \cdot 2^{25}, 21345 \cdot 2^{15}, 1579179 \cdot 2^{20}, -21345 \cdot 2^{14}, -1312971 \cdot 2^{17},$$
$$47870835 \cdot 2^{12}, -464511 \cdot 2^{15}, 236170815 \cdot 2^{11}, 236128125 \cdot 2^{11},$$
$$21345 \cdot 2^{11}, -464511 \cdot 2^{13}, 47870835 \cdot 2^{9}, 215120589 \cdot 2^{9}, -21345 \cdot 2^{9},$$
$$1579179 \cdot 2^{14}, 21345 \cdot 2^{8}, 286131 \cdot 2^{17}, 94128645 \cdot 2^{7}, -464511 \cdot 2^{9},$$
$$-21345 \cdot 2^{6}, -173704605 \cdot 2^{5}, -21345 \cdot 2^{5}, -464511 \cdot 2^{7},$$
$$-47870835 \cdot 2^{3}, 286131 \cdot 2^{13}, 21345 \cdot 2^{3}, 1579179 \cdot 2^{8}, -472341630,$$
$$-1312971 \cdot 2^{5}, 47870835, -464511 \cdot 2^{3}, 21345, 0)) \tag{57}$$

$$\pi^4 \log 2 = \frac{1}{2021 \cdot 2^{50}} P(5, 2^{60}, 120, (5157 \cdot 2^{59}, -89127 \cdot 2^{61}, 7805295 \cdot 2^{57},$$
$$-195183 \cdot 2^{61}, -32325939 \cdot 2^{56}, 1621107 \cdot 2^{59}, 5157 \cdot 2^{56},$$
$$37287 \cdot 2^{65}, -7805295 \cdot 2^{54}, -89127 \cdot 2^{57}, -5157 \cdot 2^{54},$$
$$-24620409 \cdot 2^{53}, -5157 \cdot 2^{53}, -89127 \cdot 2^{55}, 12255165 \cdot 2^{52},$$
$$37287 \cdot 2^{61}, 5157 \cdot 2^{51}, 1621107 \cdot 2^{53}, -5157 \cdot 2^{50},$$
$$29192697 \cdot 2^{49}, 7805295 \cdot 2^{48}, -89127 \cdot 2^{51}, 5157 \cdot 2^{48},$$
$$32315625 \cdot 2^{47}, 32325939 \cdot 2^{46}, -89127 \cdot 2^{49}, 7805295 \cdot 2^{45},$$
$$-195183 \cdot 2^{49}, -5157 \cdot 2^{45}, 1621107 \cdot 2^{47}, 5157 \cdot 2^{44},$$
$$37287 \cdot 2^{53}, -7805295 \cdot 2^{42}, -89127 \cdot 2^{45}, -32325939 \cdot 2^{41},$$
$$-24620409 \cdot 2^{41}, -5157 \cdot 2^{41}, -89127 \cdot 2^{43}, -7805295 \cdot 2^{39},$$
$$5866263 \cdot 2^{39}, 5157 \cdot 2^{39}, 1621107 \cdot 2^{41}, -5157 \cdot 2^{38},$$
$$-195183 \cdot 2^{41}, -12255165 \cdot 2^{37}, -89127 \cdot 2^{39}, 5157 \cdot 2^{36},$$
$$32315625 \cdot 2^{35}, 5157 \cdot 2^{35}, -89127 \cdot 2^{37}, 7805295 \cdot 2^{33},$$
$$-195183 \cdot 2^{37}, -5157 \cdot 2^{33}, 1621107 \cdot 2^{35}, 32325939 \cdot 2^{31},$$
$$37287 \cdot 2^{41}, -7805295 \cdot 2^{30}, -89127 \cdot 2^{33}, -5157 \cdot 2^{30},$$
$$480951 \cdot 2^{33}, -5157 \cdot 2^{29}, -89127 \cdot 2^{31}, -7805295 \cdot 2^{27},$$
$$37287 \cdot 2^{37}, 32325939 \cdot 2^{26}, 1621107 \cdot 2^{29}, -5157 \cdot 2^{26},$$
$$-195183 \cdot 2^{29}, 7805295 \cdot 2^{24}, -89127 \cdot 2^{27}, 5157 \cdot 2^{24},$$
$$32315625 \cdot 2^{23}, 5157 \cdot 2^{23}, -89127 \cdot 2^{25}, -12255165 \cdot 2^{22},$$
$$-195183 \cdot 2^{25}, -5157 \cdot 2^{21}, 1621107 \cdot 2^{23}, 5157 \cdot 2^{20},$$
$$5866263 \cdot 2^{19}, -7805295 \cdot 2^{18}, -89127 \cdot 2^{21}, -5157 \cdot 2^{18},$$
$$-24620409 \cdot 2^{17}, -32325939 \cdot 2^{16}, -89127 \cdot 2^{19}, -7805295 \cdot 2^{15},$$
$$37287 \cdot 2^{25}, 5157 \cdot 2^{15}, 1621107 \cdot 2^{17}, -5157 \cdot 2^{14}, -195183 \cdot 2^{17},$$
$$7805295 \cdot 2^{12}, -89127 \cdot 2^{15}, 32325939 \cdot 2^{11}, 32315625 \cdot 2^{11}, 5157 \cdot 2^{11},$$
$$-89127 \cdot 2^{13}, 7805295 \cdot 2^9, 29192697 \cdot 2^9, -5157 \cdot 2^9, 1621107 \cdot 2^{11},$$
$$5157 \cdot 2^8, 37287 \cdot 2^{17}, 12255165 \cdot 2^7, -89127 \cdot 2^9, -5157 \cdot 2^6,$$
$$-24620409 \cdot 2^5, -5157 \cdot 2^5, -89127 \cdot 2^7, -7805295 \cdot 2^3, 37287 \cdot 2^{13},$$
$$5157 \cdot 2^3, 1621107 \cdot 2^5, -64651878, -195183 \cdot 2^5, 7805295,$$
$$-89127 \cdot 2^3, 5157, 0)) \tag{58}$$

As before, the existence of BBP-type formulas for these constants was originally established by Broadhurst [18], although the explicit formulas given here were found by the author's PSLQ program. Proofs for these formulas were subsequently found by Kunle Adegoke [7].

9 Ternary (base-3) formulas

No ternary BBP formulas (i.e. formulas with $b = 3^m$ for some integer $m > 0$) were presented in [12], but several have subsequently been discovered. Here are some that are now known:

$$\log 2 = \frac{2}{3} P(1, 9, 2, (1, 0)) \tag{59}$$

$$\sqrt{3} \tan^{-1}\left(\frac{\sqrt{3}}{7}\right) = \frac{1}{6} P(1, 27, 3, (3, -1, 0)) \tag{60}$$

$$\pi\sqrt{3} = \frac{1}{9} P(1, 3^6, 12, (81, -54, 0, -9, 0, -12, -3, -2, 0, -1, 0, 0)) \tag{61}$$

$$\log 3 = \frac{1}{9} P(1, 9, 2, (9, 1)) \tag{62}$$

$$\log 3 = \frac{1}{3^6} P(1, 3^6, 6, (729, 81, 81, 9, 9, 1)) \tag{63}$$

$$\log 5 = \frac{4}{27} P(1, 3^4, 4, (9, 3, 1, 0)) \tag{64}$$

$$\log 7 = \frac{1}{3^5} P(1, 3^6, 6, (405, 81, 72, 9, 5, 0)) \tag{65}$$

$$\log 11 = \frac{1}{2 \cdot 3^9} P(1, 3^{10}, 10, (85293, 10935, 9477, 1215, 648, 135, 117, \\ 15, 13, 0)) \tag{66}$$

$$\log 13 = \frac{1}{3^5} P(1, 3^6, 6, (567, 81, 36, 9, 7, 0)) \tag{67}$$

$$\pi^2 = \frac{2}{27} P(2, 3^6, 12, (243, -405, 0, -81, -27, -72, -9, -9, 0, \\ -5, 1, 0)) \tag{68}$$

$$\log^2 3 = \frac{1}{3^6} P(2, 3^6, 12, (4374, -13122, 0, -2106, -486, -1944, \\ -162, -234, 0, -162, 18, -8)) \tag{69}$$

$$\pi\sqrt{3}\log 3 = \frac{2}{27} P(2, 3^6, 12, (243, -405, -486, -135, 27, 0, -9, 15, 18, \\ 5, -1, 0)) \tag{70}$$

$$13\zeta(3) - \pi^2 \log 3 + \log^3 3 \ = \ \frac{2}{3}P(3,9,2,(9,1)) \tag{71}$$

$$\frac{1}{\sqrt{3}}\left(\frac{29\pi^3}{1296} - \frac{\pi \log^2 3}{48}\right) \ = \ \frac{1}{2\cdot 3^6}P(3,3^6,12,(3^6,-5\cdot 3^5,-2\cdot 3^6,$$
$$-5\cdot 3^4, 3^4, 0, -3^3, 5\cdot 3^2, 2\cdot 3^3, 5\cdot 3, -3, 0))$$

$$\frac{13}{3}\zeta(3) + \frac{\log^3 3}{8} - \frac{5\pi^2 \log 3}{24} \ = \ \frac{1}{3^5}P(3,3^6,12,(3^6,3^5,0,-3^4,-3^4,-2\cdot 3^3,$$
$$-3^3, -3^2, 0, 3, 3, 2)) \tag{72}$$

$$\frac{127\pi^4}{5184} - \frac{\pi^2 \log^2 3}{32} + \frac{5\log^4 3}{192} \ = \ \frac{1}{3^6}P(4,3^6,12,(3^7,-5\cdot 3^7,0,-19\cdot 3^4,-3^5,$$
$$-2\cdot 3^6, -3^4, -19\cdot 3^2, 0, -5\cdot 3^3, 3^2,$$
$$-10)) \tag{73}$$

$$\frac{1}{\sqrt{3}}\left(11\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{29\pi^3 \log 3}{288} + \frac{\pi \log^3 3}{32}\right) \ = \ \frac{1}{2}P(4,-27,6,(9,-15,-18,-5,1,0)) \tag{74}$$

$$\frac{1573}{144}\zeta(5) - \frac{1}{128}\log^5 3 + \frac{1}{64}\pi^2 \log^3 3 - \frac{127}{3456}\pi^4 \log 3$$
$$= \ \frac{1}{3^5}P(5,3^6,12,(3^7,-5\cdot 3^7,0,-19\cdot 3^4,-3^5,-2\cdot 3^6,-3^4,$$
$$-19\cdot 3^2, 0, -5\cdot 3^3, 3^2, -10)) \tag{75}$$

In Formula 74 above, Cl denotes the Clausen function: $\mathrm{Cl}_n(t) = \sum_{k\geq 1} \cos(kt)/k^n$ if n is odd, otherwise $\mathrm{Cl}_n(t) = \sum_{k\geq 1} \sin(kt)/k^n$.

Formulas 59 and 60 appeared in [14]. Alexander Povolotsky discovered the formula $\log 3 = 1/4 + 1/4\sum_{k\geq 0} 1/9^{k+1}(27/(2k+1)+4/(2k+2)+1/(2k+3))$. Subsequently Jaume Oliver i Lafont simplified this to $\log 3 = \sum_{k\geq 0} 1/9^{k+1}(9/(2k+1)+1/(2k+2))$, which after minor modification yields Formula 62 [25]. Formulas 64 through 67 are due to Oliver i Lafont. Formulas 61 and 68 through 70 are due to Broadhurst [17]. Formulas 71 through 75 are due to Kunle Adegoke [3]. Formula 74 was first found by Broadhurst [18], using PSLQ.

10 Other specific bases

Here are several interesting results in other bases, together with two formulas for an arbitrary base b. Here $\tau = (1 + \sqrt{5})/2$ is the golden mean.

$$\log\left(\frac{9}{10}\right) = \frac{-1}{10}P(1,10,1,(1)) \tag{76}$$

$$\log\left(\frac{3}{2}\right) = \frac{2}{5}P(1,25,2,(1,0)) \tag{77}$$

$$\sqrt{5}\log\phi = P(1,5,2,(1,0)) \tag{78}$$

$$\frac{25}{2}\log\left(\frac{781}{256}\left(\frac{57-5\sqrt{5}}{57+5\sqrt{5}}\right)^{\sqrt{5}}\right) = P(1,5^5,5,(0,5,1,0,0)) \tag{79}$$

$$\frac{1}{\sqrt{\tau}}\tan^{-1}\left(\frac{5^{1/4}}{\sqrt{\tau}}\frac{233-329\sqrt{5}}{5938}\right) + \sqrt{\tau}\tan^{-1}\left(\frac{5^{1/4}}{\sqrt{\tau}}\frac{939+281\sqrt{5}}{5938}\right)$$
$$= \frac{1}{2\cdot 5^{13/4}}P(1,5^5,5,(125,-25,5,-1,0)) \tag{80}$$

$$\log\left(\frac{1111111111}{387420489}\right) = \frac{1}{10^8}P(1,10^{10},10,(10^8,10^7,10^6,10^5,10^4,10^3,$$
$$10^2,10^1,1,0)) \tag{81}$$

Formula 76 appeared in [12] (although it is an elementary observation). Formulas 77 and 78 are due to Jaume Oliver i Lafont. Formulas 79 through 81 appeared in [14].

11 General bases

$$b^2\log\left(\frac{b^2+b+1}{b^2-2b+1}\right) = 3P(1,b^3,3,(b,1,0)) \tag{82}$$

$$b^{b-2}\log\left(\frac{b^b-1}{(b-1)^b}\right) = P(1,b^b,b,(b^{b-2},b^{b-3},\cdots,b^2,b,1)) \tag{83}$$

$$\sqrt{b}\arctan(\frac{1}{\sqrt{b}}) = \frac{1}{b}P(1, b^2, 4, (b, 0, -1, 0)) \tag{84}$$

$$b^2\arctan\left(\frac{\sqrt{3}}{2b-1}\right) = \frac{\sqrt{3}}{2}P(1, -b^3, 3, (b, 1, 0)) \tag{85}$$

$$b^2\arctan\left(\frac{\sqrt{3}}{2b+1}\right) = \frac{\sqrt{3}}{2}P(1, b^3, 3, (b, -1, 0)) \tag{86}$$

$$b^7\arctan\left(\frac{1}{2b-1}\right) = \frac{1}{16}P(1, 16b^8, 8, (8b^6, 8b^5, 4b^4, 0, -2b^2, -2b,$$
$$-1, 0)) \tag{87}$$

$$b^7\arctan\left(\frac{1}{2b+1}\right) = \frac{1}{16}P(1, 16b^8, 8, (8b^6, -8b^5, 4b^4, 0, -2b^2, 2b,$$
$$-1, 0)) \tag{88}$$

$$b^3\sqrt{b}\arctan\left(\frac{\sqrt{b}}{b-1}\sqrt{2}\right) = \sqrt{2}P(1, b^4, 8, (b^3, 0, b^2, 0, -b, 0, -1, 0)) \tag{89}$$

$$27b^5\arctan\left(\frac{1}{\sqrt{3}}\frac{1}{2b-1}\right) = \frac{\sqrt{3}}{2}P(1, -27b^6, 6, (9b^4, 9b^3, 6b^2, 3b, 1, 0)) \tag{90}$$

$$27b^5\arctan\left(\frac{1}{\sqrt{3}}\frac{1}{2b+1}\right) = \frac{\sqrt{3}}{2}P(1, -27b^6, 6, (9b^4, -9b^3, 6b^2, -3b, 1, 0)) \tag{91}$$

$$b^2\sqrt{b}\arctan\left(\frac{\sqrt{b}}{b-1}\right) = P(1, -b^3, 6, (b^2, 0, 2b, 0, 1, 0) \tag{92}$$

$$\log\left(\frac{b+1}{b}\right) = \frac{1}{b^2}P(1, b^2, 2, (b, -1)) \tag{93}$$

$$\log\left(\frac{b-1}{b}\right) = -\frac{1}{b^2}P(1, b^2, 2, (b, 1)) \tag{94}$$

$$\log\left(\frac{\sqrt{b}+1}{\sqrt{b}-1}\right) = \frac{2}{\sqrt{b}}P(1, b, 2, (1, 0)) \tag{95}$$

$$\log\left(\frac{b^2+b+1}{b^2}\right) = \frac{1}{b^3}P(1,b^3,3,(b^2,b,-2)) \tag{96}$$

$$\log\left(\frac{b^2-b+1}{b^2}\right) = -\frac{1}{b^3}P(1,-b^3,3,(b^2,-b,-2)) \tag{97}$$

$$\log\left(\frac{2b^2-2b+1}{2b^2}\right) = -\frac{1}{2b^4}P(1,-4b^4,4,(2b^3,0,-b,-1)) \tag{98}$$

$$\log\left(\frac{2b^2+2b+1}{2b^2}\right) = \frac{1}{2b^4}P(1,-4b^4,4,(2b^3,0,-b,1)) \tag{99}$$

$$\frac{b\sqrt{b}}{\sqrt{2}}\log\left(\frac{b+\sqrt{2}\sqrt{b}+1}{b-\sqrt{2}\sqrt{b}+1}\right) = 2P(1,-b^2,4,(b,0,-1,0)) \tag{100}$$

$$\log\left(\frac{3b^2\pm 3b+1}{3b^2}\right) = \pm\frac{1}{27b^6}P(1,-27b^6,6,(27b^5,\mp 9b^4,0,\pm 3b^2,$$
$$-3b,\pm 2)) \tag{101}$$

$$\frac{b^2\sqrt{b}}{\sqrt{3}}\log\left(\frac{b+\sqrt{3}\sqrt{b}+1}{b-\sqrt{3}\sqrt{b}+1}\right) = 2P(1,-b^3,6,(b^2,0,0,0,-1,0)) \tag{102}$$

Formulas 82 and 83 appeared in [14]. Formulas 84 through 102 are due to Kunle Adegoke [5]. Formula 95 was first obtained by Jaume Oliver Lafont [28].

12 Zero relations

Below are some of the known BBP zero relations, or in other words BBP-type formulas that evaluate to zero. These have been discovered using the author's PSLQ program, and most are new with this compilation. For brevity, not all of the zero relations that have been found are listed here — some of the larger ones are omitted — although the author has a complete set. Further, zero relations that are merely a rewriting of another on the list, such as by expanding a relation with base b and length n to one with base b^r and length rn, are not included in these listings. For convenience, however, the total number of linearly independent zero relations for various choices of s, b and n, including rewritings and unlisted relations, are tabulated in Table 1.

Knowledge of these zero relations is essential for finding formulas such as those above using integer relation programs (such as PSLQ). This is because unless these zero relations are excluded from the search for a conjectured BBP-type formula, the search may only recover a zero relation. A zero relation may be excluded from a integer relation search by setting the input vector element whose position corresponds to the zero relation's smallest nonzero element to some value that is not linearly related to the other entries of the input vector.

For example, note in Table 1 below that there are five zero relations with $s = 1$, $b = 2^{12}$ and $n = 24$. These relations are given below as formulas 105 through 110. If one is

searching for a conjectured formula with these parameters using PSLQ, then these five zero relations must be excluded. This can be done by setting entries 19 through 23 of the PSLQ input vector to e, e^2, e^3, e^4 and e^5, respectively, where e is the base of natural logarithms. Positions 19 through 23 are specified here because in relations 105 through 110 below, the smallest nonzero entries appear in positions 23, 22, 21, 20 and 19, respectively. Powers of e are specified here because, as far as anyone can tell (although this has not been rigorously proven), e is not a polylogarithmic constant in the sense of this paper, and thus it and its powers are not expected to satisfy BBP-type linear relations (this assumption is confirmed by extensive e xperience using the author's PSLQ programs). In any event, it is clear that many other sets of transcendental constants could be used here.

Note that by simply adding a rational multiple of one of these zero relations to one the formulas above (with matching arguments s, b and n), one can produce a valid variant of that formula. Clearly infinitely many variants can be produced in this manner.

Aside from the discussion in [19], these zero relations are somewhat mysterious — it is not understood why zero relations occur for certain s, b and n, but not others. It should also be noted that in most but not all cases where a zero relation has been found, nontrivial BBP-type formulas have been found with the same parameters. This suggests that significant BBP-type results may remain to be discovered. In any event, it is hoped that this compilation will spur some additional insight into these questions.

Note that all of these formulas except for the last two are binary formulas (i.e. $b = 2^m$ for some integer $m > 0$).

s	b	n	No. zero relations	s	b	n	No. zero relations
1	16	8	1	1	2^{48}	48	1
1	64	6	1	1	2^{48}	96	5
1	2^8	16	1	1	2^{52}	104	1
1	2^{12}	12	1	1	2^{54}	54	1
1	2^{12}	24	5	1	2^{56}	112	1
1	2^{16}	32	1	1	2^{60}	60	1
1	2^{18}	18	1	1	2^{60}	120	7
1	2^{20}	40	3	2	2^{12}	24	2
1	2^{24}	24	1	2	2^{20}	40	1
1	2^{24}	48	5	2	2^{24}	48	2
1	2^{28}	56	1	2	2^{36}	72	2
1	2^{30}	30	1	2	2^{40}	80	1
1	2^{30}	60	1	2	2^{48}	96	2
1	2^{32}	64	1	2	2^{60}	120	4
1	2^{36}	36	1	3	2^{12}	24	1
1	2^{36}	72	5	3	2^{24}	48	1
1	2^{40}	80	3	3	2^{36}	72	1
1	2^{42}	42	1	3	2^{48}	96	1
1	2^{42}	84	1	3	2^{60}	120	2
1	2^{44}	88	1	4	2^{60}	120	1
1	3^6	12	2				

Table 1: Zero relation counts for various parameters

$$0 = P(1, 16, 8, (-8, 8, 4, 8, 2, 2, -1, 0)) \tag{103}$$

$$0 = P(1, 64, 6, (16, -24, -8, -6, 1, 0)) \tag{104}$$

$$0 = P(1, 2^{12}, 24, (0, 0, 2^{11}, -2^{11}, 0, -2^9, 256, -3 \cdot 2^8, 0, 0, -64, -128, 0, -32, -32,$$
$$-48, 0, -24, -4, -8, 0, -2, 1, 0)) \tag{105}$$

$$0 = P(1, 2^{12}, 24, (2^{11}, -2^{11}, -2^{11}, 0, -2^9, -2^{10}, -2^8, 0,$$
$$-2^8, -2^7, 2^6, 0, -32, 32, 32, 0, 8, 16, 4, 0, 4, 2, -1, 0)) \tag{106}$$

$$0 = P(1, 2^{12}, 24, (-2^9, -2^{10}, 2^{10}, 7 \cdot 2^8, 256, 3 \cdot 2^8, 64, 3 \cdot 2^7, 0, 0, 0, 0, 8, -32, -16,$$
$$12, -4, 4, -1, 8, 0, -1, 0, 0)) \tag{107}$$

$$0 = P(1, 2^{12}, 24, (2^9, -2^{10}, -2^9, 256, 0, 256, 64, 3 \cdot 2^7, 64, 0, 0, 0, -8, -16, 8, 12, 0,$$
$$4, -1, 2, -1, 0, 0, 0)) \tag{108}$$

$$0 = P(1, 2^{12}, 24, (3 \cdot 2^9, -3 \cdot 2^{10}, 0, -256, 0, 0, 192, 3 \cdot 2^7, 0, 0, 0, -64, -24, -48,$$
$$0, -12, 0, 0, -3, 2, 0, 0, 0, 0)) \tag{109}$$

$$0 = P(1, 2^{12}, 24, (-2^{10}, 3 \cdot 2^9, 2^9, 256, 128, 128, -64, -192, 0, 32, 0, 32, 16, 16,$$
$$-8, 0, -2, -2, 1, 0, 0, 0, 0, 0)) \tag{110}$$

$$0 = P(1, 2^{20}, 40, (0, 2^{18}, -2^{18}, 2^{17}, 0, -5 \cdot 2^{16}, 2^{16}, -5 \cdot 2^{15}, 0, -2^{16}, -2^{14}, 2^{13},$$
$$0, -5 \cdot 2^{12}, -2^{14}, -5 \cdot 2^{11}, 0, 2^{10}, -2^{10}, -2^{11}, 0, -5 \cdot 2^8, 256, -5 \cdot 2^7, 0, 64, -64,$$
$$32, 0, 0, 16, -40, 0, 4, 16, 2, 0, -5, 1, 0)) \tag{111}$$

$$0 = P(1, 2^{20}, 40, (2^{18}, -2^{19}, 0, -2^{17}, 3 \cdot 2^{15}, 2^{16}, 0, 0, 2^{14}, 2^{13}, 0, -2^{13}, -2^{12}, 2^{12},$$
$$5 \cdot 2^{10}, 0, 2^{10}, -2^{11}, 0, -2^9, -256, 256, 0, 0, -96, -128, 0, -32, -16, -24,$$
$$0, 0, 4, -8, -5, -2, -1, 1, 0, 0)) \tag{112}$$

$$0 = P(1, 2^{20}, 40, (-2^{18}, 3 \cdot 2^{18}, 0, -2^{18}, -13 \cdot 2^{15}, 0, 0, 5 \cdot 2^{15}, -2^{14}, 2^{13}, 0, -2^{14},$$
$$2^{12}, 0, 5 \cdot 2^{10}, 5 \cdot 2^{11}, -2^{10}, 3 \cdot 2^{10}, 0, 3 \cdot 2^9, 256, 0, 0, 5 \cdot 2^7, 13 \cdot 2^5,$$
$$192, 0, -64, 16, 40, 0, 40, -4, 12, -5, -4, 1, 0, 0, 0)) \tag{113}$$

$$0 = P(1, 2^{20}, 40, (2^{19}, -3 \cdot 2^{19}, 2^{18}, 0, 2^{19}, 3 \cdot 2^{17},$$
$$-2^{16}, 0, 2^{15}, 2^{16}, 2^{14}, 0, -2^{13}, 3 \cdot 2^{13}, 2^{14}, 0,$$
$$2^{11}, -3 \cdot 2^{11}, 2^{10}, 0, -2^9, 3 \cdot 2^9, -2^8, 0, -2^9, -3 \cdot 2^7,$$
$$2^6, 0, -2^5, -2^6, -2^4, 0, 2^3, -3 \cdot 2^3, -2^4, 0, -2, 6, -1, 0)) \tag{114}$$

$$0 = P(2, 2^{12}, 24, (0, 2^{10}, -3 \cdot 2^{10}, 2^9, 0, 2^{10}, 0, 9 \cdot 2^7, 3 \cdot 2^7, 64, 0, 128, 0, 16, 48, 72,$$
$$0, 16, 0, 2, -6, 1, 0, 0)) \tag{115}$$

$$\begin{aligned}
0 &= P(2, 2^{12}, 24, (-2^{11}, 0, 17 \cdot 2^{11}, -17 \cdot 2^{10}, 2^9, -15 \cdot 2^{10}, -256, -63 \cdot 2^8, \\
&\quad -17 \cdot 2^8, 0, 64, -5 \cdot 2^8, 32, 0, -17 \cdot 2^5, -63 \cdot 2^4, -8, -240, 4, -68, 68, \\
&\quad 0, -1, 0)) \tag{116} \\
0 &= P(2, 2^{20}, 40, (2^{19}, -3 \cdot 2^{20}, -2^{18}, 13 \cdot 2^{18}, 3 \cdot 2^{20}, -3 \cdot 2^{18}, 2^{16}, \\
&\quad -25 \cdot 2^{16}, 2^{15}, -3 \cdot 2^{16}, -2^{14}, 13 \cdot 2^{14}, -2^{13}, -3 \cdot 2^{14}, -3 \cdot 2^{15}, -25 \cdot 2^{12}, \\
&\quad 2^{11}, -3 \cdot 2^{12}, -2^{10}, -3 \cdot 2^{12}, -2^9, -3 \cdot 2^{10}, 256, -25 \cdot 2^8, -3 \cdot 2^{10}, \\
&\quad -3 \cdot 2^8, -64, 13 \cdot 2^6, -32, -192, 16, -25 \cdot 2^4, 8, -48, 96, 52, -2, -12, 1, 0)) \\
&\tag{117} \\
0 &= P(3, 2^{12}, 24, (2^{11}, -19 \cdot 2^{11}, 5 \cdot 2^{14}, -2^{11}, -2^9, -23 \cdot 2^{10}, 256, -27 \cdot 2^{10}, \\
&\quad -5 \cdot 2^{11}, -19 \cdot 2^7, -64, -7 \cdot 2^9, -32, -19 \cdot 2^5, -5 \cdot 2^8, -27 \cdot 2^6, 8, \\
&\quad -23 \cdot 2^4, -4, -8, 160, -38, 1, 0)) \tag{118}
\end{aligned}$$

$$\begin{aligned}
0 &= P(3, 2^{60}, 120, (7 \cdot 2^{59}, -3 \cdot 5^2 \cdot 2^{60}, 1579 \cdot 2^{57}, -29 \cdot 2^{60}, -3 \cdot 23 \cdot 31 \cdot 2^{56}, \\
&\quad 3 \cdot 5 \cdot 2^{60}, 7 \cdot 2^{56}, 67 \cdot 2^{59}, -1579 \cdot 2^{54}, -3 \cdot 5^2 \cdot 2^{56}, -7 \cdot 2^{54}, -5 \cdot 11 \cdot 43 \cdot 2^{53}, \\
&\quad -7 \cdot 2^{53}, -3 \cdot 5^2 \cdot 2^{54}, 3 \cdot 7 \cdot 13 \cdot 2^{52}, 67 \cdot 2^{55}, 7 \cdot 2^{51}, 3 \cdot 5 \cdot 2^{54}, -7 \cdot 2^{50}, \\
&\quad 3 \cdot 631 \cdot 2^{49}, 1579 \cdot 2^{48}, -3 \cdot 5^2 \cdot 2^{50}, 7 \cdot 2^{48}, 5^3 \cdot 17 \cdot 2^{47}, 3 \cdot 23 \cdot 31 \cdot 2^{46}, \\
&\quad -3 \cdot 5^2 \cdot 2^{48}, 1579 \cdot 2^{45}, -29 \cdot 2^{48}, -7 \cdot 2^{45}, 3 \cdot 5 \cdot 2^{48}, 7 \cdot 2^{44}, 67 \cdot 2^{47}, \\
&\quad -1579 \cdot 2^{42}, -3 \cdot 5^2 \cdot 2^{44}, -3 \cdot 23 \cdot 31 \cdot 2^{41}, -5 \cdot 11 \cdot 43 \cdot 2^{41}, -7 \cdot 2^{41}, \\
&\quad -3 \cdot 5^2 \cdot 2^{42}, -1579 \cdot 2^{39}, -3^4 \cdot 13 \cdot 2^{39}, 7 \cdot 2^{39}, 3 \cdot 5 \cdot 2^{42}, -7 \cdot 2^{38}, \\
&\quad -29 \cdot 2^{40}, -3 \cdot 7 \cdot 13 \cdot 2^{37}, -3 \cdot 5^2 \cdot 2^{38}, 7 \cdot 2^{36}, 5^3 \cdot 17 \cdot 2^{35}, 7 \cdot 2^{35}, \\
&\quad -3 \cdot 5^2 \cdot 2^{36}, 1579 \cdot 2^{33}, -29 \cdot 2^{36}, -7 \cdot 2^{33}, 3 \cdot 5 \cdot 2^{36}, 3 \cdot 23 \cdot 31 \cdot 2^{31}, \\
&\quad 67 \cdot 2^{35}, -1579 \cdot 2^{30}, -3 \cdot 5^2 \cdot 2^{32}, -7 \cdot 2^{30}, -3 \cdot 5 \cdot 2^{33}, -7 \cdot 2^{29}, \\
&\quad -3 \cdot 5^2 \cdot 2^{30}, -1579 \cdot 2^{27}, 67 \cdot 2^{31}, 3 \cdot 23 \cdot 31 \cdot 2^{26}, 3 \cdot 5 \cdot 2^{30}, -7 \cdot 2^{26}, \\
&\quad -29 \cdot 2^{28}, 1579 \cdot 2^{24}, -3 \cdot 5^2 \cdot 2^{26}, 7 \cdot 2^{24}, 5^3 \cdot 17 \cdot 2^{23}, 7 \cdot 2^{23}, \\
&\quad -3 \cdot 5^2 \cdot 2^{24}, -3 \cdot 7 \cdot 13 \cdot 2^{22}, -29 \cdot 2^{24}, -7 \cdot 2^{21}, 3 \cdot 5 \cdot 2^{24}, 7 \cdot 2^{20}, \\
&\quad -3^4 \cdot 13 \cdot 2^{19}, -1579 \cdot 2^{18}, -3 \cdot 5^2 \cdot 2^{20}, -7 \cdot 2^{18}, -5 \cdot 11 \cdot 43 \cdot 2^{17}, \\
&\quad -3 \cdot 23 \cdot 31 \cdot 2^{16}, -3 \cdot 5^2 \cdot 2^{18}, -1579 \cdot 2^{15}, 67 \cdot 2^{19}, 7 \cdot 2^{15}, 3 \cdot 5 \cdot 2^{18}, \\
&\quad -7 \cdot 2^{14}, -29 \cdot 2^{16}, 1579 \cdot 2^{12}, -3 \cdot 5^2 \cdot 2^{14}, 3 \cdot 23 \cdot 31 \cdot 2^{11}, 5^3 \cdot 17 \cdot 2^{11}, \\
&\quad 7 \cdot 2^{11}, -3 \cdot 5^2 \cdot 2^{12}, 1579 \cdot 2^9, 3 \cdot 631 \cdot 2^9, -7 \cdot 2^9, 3 \cdot 5 \cdot 2^{12}, 7 \cdot 2^8, \\
&\quad 67 \cdot 2^{11}, 3 \cdot 7 \cdot 13 \cdot 2^7, -3 \cdot 5^2 \cdot 2^8, -7 \cdot 2^6, -5 \cdot 11 \cdot 43 \cdot 2^5, -7 \cdot 2^5, \\
&\quad -3 \cdot 5^2 \cdot 2^6, -1579 \cdot 2^3, 67 \cdot 2^7, 7 \cdot 2^3, 3 \cdot 5 \cdot 2^6, -3 \cdot 23 \cdot 31 \cdot 2, \\
&\quad -29 \cdot 2^4, 1579, -3 \cdot 5^2 \cdot 2^2, 7, 0)) \tag{119}
\end{aligned}$$

$$\begin{aligned}
0 \;=\; &P(3, 2^{60}, 120, (2^{59}, 0, -353 \cdot 2^{57}, 7 \cdot 2^{62}, 3 \cdot 7 \cdot 113 \cdot 2^{56}, -3^3 \cdot 5 \cdot 2^{59}, 2^{56}, \\
&-97 \cdot 2^{60}, 353 \cdot 2^{54}, 0, -2^{54}, 5 \cdot 331 \cdot 2^{53}, -2^{53}, 0, -3 \cdot 337 \cdot 2^{52}, -97 \cdot 2^{56}, 2^{51}, \\
&-3^3 \cdot 5 \cdot 2^{53}, -2^{50}, -3^2 \cdot 239 \cdot 2^{49}, -353 \cdot 2^{48}, 0, 2^{48}, -5^3 \cdot 19 \cdot 2^{47}, -3 \cdot 7 \cdot 113 \cdot 2^{46}, \\
&0, -353 \cdot 2^{45}, 7 \cdot 2^{50}, -2^{45}, -3^3 \cdot 5 \cdot 2^{47}, 2^{44}, -97 \cdot 2^{48}, 353 \cdot 2^{42}, 0, 3 \cdot 7 \cdot 113 \cdot 2^{41}, \\
&5 \cdot 331 \cdot 2^{41}, -2^{41}, 0, 353 \cdot 2^{39}, -3^6 \cdot 2^{39}, 2^{39}, -3^3 \cdot 5 \cdot 2^{41}, -2^{38}, 7 \cdot 2^{42}, \\
&3 \cdot 337 \cdot 2^{37}, 0, 2^{36}, -5^3 \cdot 19 \cdot 2^{35}, 2^{35}, 0, -353 \cdot 2^{33}, 7 \cdot 2^{38}, -2^{33}, -3^3 \cdot 5 \cdot 2^{35}, \\
&-3 \cdot 7 \cdot 113 \cdot 2^{31}, -97 \cdot 2^{36}, 353 \cdot 2^{30}, 0, -2^{30}, -3^2 \cdot 5 \cdot 2^{33}, -2^{29}, 0, 353 \cdot 2^{27}, \\
&-97 \cdot 2^{32}, -3 \cdot 7 \cdot 113 \cdot 2^{26}, -3^3 \cdot 5 \cdot 2^{29}, -2^{26}, 7 \cdot 2^{30}, -353 \cdot 2^{24}, 0, 2^{24}, \\
&-5^3 \cdot 19 \cdot 2^{23}, 2^{23}, 0, 3 \cdot 337 \cdot 2^{22}, 7 \cdot 2^{26}, -2^{21}, -3^3 \cdot 5 \cdot 2^{23}, 2^{20}, -3^6 \cdot 2^{19}, \\
&353 \cdot 2^{18}, 0, -2^{18}, 5 \cdot 331 \cdot 2^{17}, 3 \cdot 7 \cdot 113 \cdot 2^{16}, 0, 353 \cdot 2^{15}, -97 \cdot 2^{20}, 2^{15}, \\
&-3^3 \cdot 5 \cdot 2^{17}, -2^{14}, 7 \cdot 2^{18}, -353 \cdot 2^{12}, 0, -3 \cdot 7 \cdot 113 \cdot 2^{11}, -5^3 \cdot 19 \cdot 2^{11}, 2^{11}, 0, \\
&-353 \cdot 2^9, -3^2 \cdot 239 \cdot 2^9, -2^9, -3^3 \cdot 5 \cdot 2^{11}, 2^8, -97 \cdot 2^{12}, -3 \cdot 337 \cdot 2^7, 0, -2^6, \\
&5 \cdot 331 \cdot 2^5, -2^5, 0, 353 \cdot 2^3, -97 \cdot 2^8, 2^3, -3^3 \cdot 5 \cdot 2^5, \\
&3 \cdot 7 \cdot 113 \cdot 2, 7 \cdot 2^6, -353, 0, 1, 0))
\end{aligned}$$

<div style="text-align:right">(120)</div>

$$
\begin{aligned}
0 \;=\; &P(4, 2^{60}, 120, (-31 \cdot 2^{59}, 3 \cdot 269 \cdot 2^{60}, -5 \cdot 61 \cdot 107 \cdot 2^{57}, \\
&1553 \cdot 2^{60}, 3^2 \cdot 14243 \cdot 2^{56}, -3 \cdot 1051 \cdot 2^{60}, -31 \cdot 2^{56}, \\
&-9319 \cdot 2^{59}, 5 \cdot 61 \cdot 107 \cdot 2^{54}, 3 \cdot 269 \cdot 2^{56}, 31 \cdot 2^{54}, \\
&31 \cdot 3187 \cdot 2^{53}, 31 \cdot 2^{53}, 3 \cdot 269 \cdot 2^{54}, -3^2 \cdot 5 \cdot 1061 \cdot 2^{52}, \\
&-9319 \cdot 2^{55}, -31 \cdot 2^{51}, -3 \cdot 1051 \cdot 2^{54}, 31 \cdot 2^{50}, -3 \cdot 38567 \cdot 2^{49}, \\
&-5 \cdot 61 \cdot 107 \cdot 2^{48}, 3 \cdot 269 \cdot 2^{50}, -31 \cdot 2^{48}, -5^5 \cdot 41 \cdot 2^{47}, \\
&-3^2 \cdot 14243 \cdot 2^{46}, 3 \cdot 269 \cdot 2^{48}, -5 \cdot 61 \cdot 107 \cdot 2^{45}, 1553 \cdot 2^{48}, \\
&31 \cdot 2^{45}, -3 \cdot 1051 \cdot 2^{48}, -31 \cdot 2^{44}, -9319 \cdot 2^{47}, 5 \cdot 61 \cdot 107 \cdot 2^{42}, \\
&3 \cdot 269 \cdot 2^{44}, 3^2 \cdot 14243 \cdot 2^{41}, 31 \cdot 3187 \cdot 2^{41}, 31 \cdot 2^{41}, \\
&3 \cdot 269 \cdot 2^{42}, 5 \cdot 61 \cdot 107 \cdot 2^{39}, -3^4 \cdot 7 \cdot 37 \cdot 2^{39}, -31 \cdot 2^{39}, \\
&-3 \cdot 1051 \cdot 2^{42}, 31 \cdot 2^{38}, 1553 \cdot 2^{40}, 3^2 \cdot 5 \cdot 1061 \cdot 2^{37}, \\
&3 \cdot 269 \cdot 2^{38}, -31 \cdot 2^{36}, -5^5 \cdot 41 \cdot 2^{35}, -31 \cdot 2^{35}, \\
&3 \cdot 269 \cdot 2^{36}, -5 \cdot 61 \cdot 107 \cdot 2^{33}, 1553 \cdot 2^{36}, 31 \cdot 2^{33}, \\
&-3 \cdot 1051 \cdot 2^{36}, -3^2 \cdot 14243 \cdot 2^{31}, -9319 \cdot 2^{35}, \\
&5 \cdot 61 \cdot 107 \cdot 2^{30}, 3 \cdot 269 \cdot 2^{32}, 31 \cdot 2^{30}, -3 \cdot 13 \cdot 47 \cdot 2^{33}, \\
&31 \cdot 2^{29}, 3 \cdot 269 \cdot 2^{30}, 5 \cdot 61 \cdot 107 \cdot 2^{27}, -9319 \cdot 2^{31}, \\
&-3^2 \cdot 14243 \cdot 2^{26}, -3 \cdot 1051 \cdot 2^{30}, 31 \cdot 2^{26}, 1553 \cdot 2^{28}, \\
&-5 \cdot 61 \cdot 107 \cdot 2^{24}, 3 \cdot 269 \cdot 2^{26}, -31 \cdot 2^{24}, -5^5 \cdot 41 \cdot 2^{23}, \\
&-31 \cdot 2^{23}, 3 \cdot 269 \cdot 2^{24}, 3^2 \cdot 5 \cdot 1061 \cdot 2^{22}, 1553 \cdot 2^{24}, \\
&31 \cdot 2^{21}, -3 \cdot 1051 \cdot 2^{24}, -31 \cdot 2^{20}, -3^4 \cdot 7 \cdot 37 \cdot 2^{19}, \\
&5 \cdot 61 \cdot 107 \cdot 2^{18}, 3 \cdot 269 \cdot 2^{20}, 31 \cdot 2^{18}, 31 \cdot 3187 \cdot 2^{17}, \\
&3^2 \cdot 14243 \cdot 2^{16}, 3 \cdot 269 \cdot 2^{18}, 5 \cdot 61 \cdot 107 \cdot 2^{15}, -9319 \cdot 2^{19}, \\
&-31 \cdot 2^{15}, -3 \cdot 1051 \cdot 2^{18}, 31 \cdot 2^{14}, 1553 \cdot 2^{16}, \\
&-5 \cdot 61 \cdot 107 \cdot 2^{12}, 3 \cdot 269 \cdot 2^{14}, -3^2 \cdot 14243 \cdot 2^{11}, \\
&-5^5 \cdot 41 \cdot 2^{11}, -31 \cdot 2^{11}, 3 \cdot 269 \cdot 2^{12}, \\
&-5 \cdot 61 \cdot 107 \cdot 2^9, -3 \cdot 38567 \cdot 2^9, 31 \cdot 2^9, -3 \cdot 1051 \cdot 2^{12}, \\
&-31 \cdot 2^8, -9319 \cdot 2^{11}, -3^2 \cdot 5 \cdot 1061 \cdot 2^7, 3 \cdot 269 \cdot 2^8, \\
&31 \cdot 2^6, 31 \cdot 3187 \cdot 2^5, 31 \cdot 2^5, 3 \cdot 269 \cdot 2^6, \\
&5 \cdot 61 \cdot 107 \cdot 2^3, -9319 \cdot 2^7, -31 \cdot 2^3, -3 \cdot 1051 \cdot 2^6, \\
&3^2 \cdot 14243 \cdot 2, 1553 \cdot 2^4, -61 \cdot 107 \cdot 5, 3 \cdot 269 \cdot 2^2, -31, 0)) \qquad (121)
\end{aligned}
$$

$$
\begin{aligned}
0 \;&=\; P(1, 729, 12, (0, 81, -162, 0, 27, 36, 0, 9, 6, 4, -1, 0)) &&(122) \\
0 \;&=\; P(1, 729, 12, (243, -324, -162, -81, 0, -36, -9, 0, 6, -1, 0, 0)) &&(123)
\end{aligned}
$$

Relation 103 appeared in [12]. Relation 104 and 105 were given in [14]. Relations 106 and 114 are due to Jaume Oliver Lafont [26]. Lafont subsequently proved 105 through 113 [27]. Relations 107 through 113 and 115 through 118 and 122 and 123 were found experimentally by the author using his PSLQ program, but in the wake of the other relations in this list are now proven. Relation 118 has been proved by Kunle Adegoke as the difference of Formulas 32 and 40. Relations 119 through 121 are due to Kunle Adegoke [2, 3], who also proved 122 and 123 [8].

13 Curiosities

There are two other formulas worth mentioning, although neither, technically speaking, is a BBP-type formula. The first formula employs the irrational base $b = 2/\tau = 2\tau - 2$, where τ is the golden mean (see Section 9):

$$\frac{3\pi\sqrt{\tau}}{5^{5/4}} = \frac{1}{2^9}P(1, 2/\tau, 10, (256\tau, 128\tau^3, 64\tau^4, 32\tau^4, 0, -8\tau^6, -4\tau^8, -2\tau^9, 0)). \quad (124)$$

The second example of this class is the formula

$$\frac{1}{\sqrt{19}}\cos^{-1}\left(\frac{9}{10}\right) = \frac{1}{10}\sum_{k=0}^{\infty}\frac{D_k}{10^k}\left(\frac{1}{k+1}\right), \quad (125)$$

where the D coefficients satisfy the recurrence $D_0 = D_1 = 1$, and $D_{k+1} = D_k - 5D_{k-1}$ for $k \geq 2$. It is possible that a variant of the original BBP algorithm can be fashioned for this case, on the idea that the D_k comprise a Lucas sequence, and as is known, evaluations of sequence elements mod n can be effected via exponential-ladder methods. These two formulas appeared in [14].

14 Acknowledgements

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