A (Leibnizian) Theory of Concepts*

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> If we had it [a *characteristica universalis*], we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis. ... If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other ... : Let us calculate.[1] *G. W. Leibniz*

> I have therefore, in what follows, ... endeavored as far as possible to exhibit the theory of monads as a rigid deduction from a small number of premisses.[2] *B. Russell*

In the eyes of many philosophers, Leibniz established his credentials as a clear and logically precise thinker by having invented the differential and integral calculus. However, his philosophical and metaphysical views were never expressed as precisely as the mathematics he developed. His ideas about concept summation, concept inclusion, complete individual concepts, possible worlds, and the containment theory of truth were never developed within a single, overarching system as precise as the mathematical calculus. In this paper, we describe a system that may rectify this omission.

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[1]This is the translation in Russell [1900], pp. 169-170. The source is G.vii 21 and G.vii 200. ('G' refers to C. Gerhardt (ed.), *Die philosophischen Schriften von Gottfried Wilhelm Leibniz*; see the entry for Leibniz in the Bibliography.)

[2]See Russell [1900], p. viii.

In what follows, we articulate a theory of concepts using Leibniz's work on the logic and metaphysics of concepts as a guide. The development of the theory consists primarily of proving facts about concepts that are often just stipulated to be true. The derivations of these facts take place within the context of a precise metaphysical theory that has been developed in previous work, namely, the axiomatic theory of abstract objects.[3] The truth of the axioms of this theory will be preserved in the theorems of the theory of concepts, but putting aside the question of truth, we put forward the following system as one way of reconstructing many of Leibniz's ideas about concepts.

Leibniz's views on concepts are developed in two strands of his work, which we shall refer to as his 'logic of concepts' and his '(modal) metaphysics of individual concepts', respectively. The first strand, the logic of concepts, is outlined in the series of unpublished sketches in which he formulated and reformulated algebras of concepts.[4] We shall focus on what appears to be his most mature logic, namely, the one developed in the 1690 fragment G.vii 236-247. We shall derive both the axioms and theorems of this fragment in what follows.

The second strand of Leibniz's work on concepts is the metaphysics he develops in connection with the notion of an 'individual concept'. This notion played a significant role in the *Discourse on Metaphysics*, the *Correspondence with Arnauld*, the *Theodicy*, and the *Monadology*. In these works, it seems clear that Leibniz thought that the properties of an individual could be derived from its individual concept. To account for contingent truths, he theorized about individual concepts in the context of his metaphysics of possibility and possible worlds.

[3]See Zalta [1983], [1988a], [1993], and [1999]. Certain Leibnizian features of this metaphysics were first sketched in Zalta [1983], but this work is redeveloped and enhanced here (Section 8) in numerous ways. The present effort represents a more considered view.

[4]See the fragments in G. W. Leibniz, *Logical Papers*, edited and translated by G. H. R. Parkinson. Hereafter, we refer to this work as 'LP'. (The full reference is in the Bibliography.) Leibniz referred to these algebras as 'calculi' and Rescher [1954] separates these calculi into 3 basic systems. He documents Leibniz's intensional and extensional interpretations of these systems, noting that the second system has a propositional interpretation.

Although Leibniz didn't integrate his modal metaphysics of individual concepts and his logic of concepts, it seems clear that this is something that should be done, since he applied the concept containment theory of truth in both of these strands to his work. In what follows, we integrate the logic and metaphysics of concepts and prove many of the claims that Leibniz asserted in connection with individual concepts, including what appears to be the fundamental theorem of his metaphysics of concepts, namely, that if an object x has F but might not have had F, then (i) the individual concept of x contains the concept F and (ii) there is a individual concept y (a 'counterpart' of the concept of x) which doesn't contain the concept F and which appears at some other possible world. I think that the formal derivations that we construct in developing the logic and metaphysics of concepts goes some way towards understanding Leibniz's idea of a *characteristica universalis* and *calculus ratiocinator*.

Our principal goal in what follows, then, is to develop a correct theory of concepts that links the two strands in Leibniz's work and to do so by proving metaphysical claims that are often stipulated. In working towards this goal, we shall use *neither* 'possible world semantics' nor set theory, nor shall we build a semantical system which models the truth of logical and metaphysical claims; indeed we hope to show that *no set theory or other mathematics* is needed to develop a Leibnizian theory of concepts with rigor, clarity, and logical precision. Concepts will be systematized by precisely identifying them within an axiomatized realm of abstract objects which have well-defined, and antecedently-specified, existence and identity conditions. The interesting web of logical and metaphysical theorems that result should establish that the existence and identity conditions for abstract objects correctly apply to (Leibnizian) concepts. Moreover, the containment theory of truth will be defined so that: (a) it applies both to the logic and modal metaphysics of concepts, and (b) it is consistent with the existence of contingent facts.

These accomplishments should justify the present contribution both to the literature on concepts and to the secondary literature on Leibniz. Of course, some theories of concepts in the literature do not look back to Leibniz for inspiration.[5] Of the ones that do, some only treat the logic of concepts and not the modal metaphysics of individual concepts.[6]

[5]See, for example, Peacocke [1991] and Bealer [1998].

[6]See, for example, Rescher [1954], Kauppi [1960], [1967], Castañeda [1976], [1990], and Swoyer [1994], [1995].

Others treat only the modal metaphysics of individual concepts and not the logic.[7] Although Lenzen [1990] attempts to treat both, it is not made clear which of the Leibnizian laws that he formalizes are axioms and which are theorems.[8] Most of these works in the secondary literature assume certain basic axioms (often stated in set-theoretic terms) which are then used in the reconstruction of Leibniz's ideas. In what follows, we plan to derive these axioms as theorems.

§1: Concepts, Properties, and Concepts of Properties

Before we turn to the definitions and theorems that articulate the theory of concepts, it will serve well if we spend some time discussing the distinction between concepts and properties in what follows. We shall not identify concepts and properties. There are good reasons to distinguish these two kinds of entity. Two of these reasons are 'negative' ones; they tell us why we shouldn't identify concepts and properties. There is also a 'positive' reason for distinguishing concepts and properties, which we will get to in just a moment.

The first reason not to identify concepts and properties is that it would get the Leibnizian *logic* of concepts wrong. The most important theorem in Leibniz's of logic concepts is:

(A) The concept F contains the concept G iff the concept F is identical with the sum of the concept F and the concept G.[9]

Swoyer [1995] and [1994] designates this principle 'Leibniz's Equivalence'. We may represent this formally as follows:

(A) $F \succeq G \;\equiv\; F = F \oplus G$

Principle (A) becomes false when analyzed in terms of one of the property

[7]See Mates [1968], Mondadori [1973], and Fitch [1979].

[8]See the various formalizations that Lenzen presents throughout [1990] and which are listed on pp. 217-223.

[9]Principle (A) is the 'containment' version of the conjunction of Principles 13 and 14 in LP 135:

> Principle 13: If $F \oplus G$ is identical to F, then G is in included in F.
>
> Principle 14: If G is included in F, then $F \oplus G$ is identical to F.

The source is G.vii 239.

theories that are now available.[10] Just consider the following property-theoretic analysis of the notions involved:

$$\textit{The concept } F \ =_{df} \ F$$

$$F \textit{ contains } G \ (`F \succeq G\text{'}) \ =_{df} \ F \Rightarrow G \ \ (\text{i.e., } \Box\forall x(Fx \rightarrow Gx))$$

$$\textit{The sum of concepts } F \textit{ and } G \ (`F \oplus G\text{'}) \ =_{df} \ [\lambda x\, Fx \,\&\, Gx]^{11}$$

These definitions do have the virtue of preserving the Leibnizian intuition that the concept person contains the concept rational.[12] But it is now easy to see that (A) is false from the point of view of any property theory that treats properties in a fine-grained way. For on the property-theoretic analysis of containment and summation just proposed, Principle (A) would amount to the following:

$$F \Rightarrow G \ \equiv \ F = [\lambda x\, Fx \,\&\, Gx]$$

This is clearly false as a principle governing intensionally conceived properties.[13]

[10]See Cocchiarella [1978], Bealer [1982], Zalta [1983], Chierchia and Turner [1985], Menzel [1986], and Swoyer [1998] and [1996]. These theories allow for the possibility of distinct, though necessarily equivalent properties.

[11]The λ-predicate used in the definition of concept summation denotes a complex property. We read the λ-predicate as follows: being an x such that both x exemplifies F and x exemplifies G. Intuitively, the λ-predicate denotes the complex conjunctive property: *being F and G*.

[12]To see why, note that Leibniz regarded the concept person (or 'man') as the sum of the concepts rational and animal. On the above analysis, once the concepts rational and animal are identified with the properties of being rational and being an animal, respectively, the sum of the concepts rational and animal is identified as the conjunctive property *being rational and an animal* ($[\lambda x\, Rx \,\&\, Ax]$). It therefore follows that the concept person (i.e., the sum of the concepts rational and animal) contains the concept rational, since the property of being rational is necessarily implied by the conjunctive property *being rational and an animal*.

[13]From the fact that a property F implies a property G, it does *not* follow that F just is identical to the conjunctive property *being F and G*. Intuitively, F may be distinct from *being F and G* even if F implies G. Indeed, on a fine-grained theory of properties, the right hand side of this biconditional is always false—it is always false that F is identical with the property of $[\lambda x\, Fx \& \, Gx]$, no matter what G you pick.

Of course, one could try to reinterpret the identity sign in Principle (A) in terms of some weaker notion, but what notion that might be remains a mystery. Castañeda [1976] and [1990] suggests that Leibniz's relation of congruence is a weaker congruence relation on concepts, but Ishiguro [1990] argues against this idea, in Chapter 2. Among other things, it conflicts with Leibniz's reading of the symbol '=' (which, in his logical papers is usually symbolized as '∞') as 'identity' or 'sameness'.

The second reason not to identify concepts and properties is that it would get the Leibnizian *metaphysics* of concepts wrong. It is central to Leibniz's view of individual concepts that each individual x has a unique individual concept. So suppose that Adam's complete individual concept is the concept (i.e., property) P. Then pick your favorite proposition, say q, and consider the distinct property $[\lambda y\, Py \,\&\, (q \vee \neg q)]$. Call this property '$Q$'. P and Q are exactly the kind of necessarily equivalent but distinct properties that are the subject of property theory. But if P is a complete individual concept of Adam, so is Q. If Adam is the unique thing exemplifying P, he is the unique thing exemplifying Q.[14] So we have distinct complete individual concepts that are both complete individual concepts of Adam, contrary to the assumption that there is a unique individual concept of Adam.

Although, in what follows, we shall employ a fine-grained, intensional property theory as a part of the theory of abstract objects, we shall not identify (Leibnizian) concepts with such properties, but instead identify them with abstract objects that encode such properties. But before we turn to the development of this idea, it is important to discuss the positive reason and textual support for distinguishing properties and concepts.

The positive reason for distinguishing properties and concepts is that it allows us to separate individuals and properties, on the one hand, from the concepts of individuals and the concepts of properties, on the other. In what follows, we shall not simply distinguish the individual x from the concept of that individual c_x, but also distinguish the property F from the concept of that property c_F. This exploits a distinction that seems to be latent in Leibniz's views on truth and predication. Here is how.

Leibniz agrees with Aristotle that a substance is something of which attributes can be predicated but which itself cannot be predicated of

[14]In the formal terms of property theory, one defines a property F to be *complete* iff for every property G, either F implies G or F implies the negation of G. Where the negation of G ('$\bar{G}$') is defined as $[\lambda x\, \neg Gx]$, this definition becomes:

$$\textit{Complete}(F) \ =_{df} \ \forall G(F \Rightarrow G \ \vee \ F \Rightarrow \bar{G})$$

A property F is an *individual* property iff necessarily, at most one object exemplifies F:

$$\textit{Individual}(F) \ =_{df} \ \Box\forall x\forall y(Fx \,\&\, Fy \rightarrow x = y)$$

(This is a weak notion of an individual property, and although stronger notions are definable, the problem we are discussing applies already to the weaker notion of individual property.) It is now straightforward to show that if P is a complete, individual concept of Adam, then so is Q.

anything.[15] But Leibniz offers a theory about what it is for an attribute to be truly predicated of a substance, namely, when the concept of the former is contained in the concept of the latter. This is his concept containment theory of truth. Leibniz frequently states this theory in what appear to be linguistic terms: a 'proposition' with subject-predicate form is true iff the concept of the predicate is contained in the concept of the subject. Thus, in the *Discourse on Metaphysics* (Article 8), we find:

> It is necessary, therefore, to consider what it is to be truly attributed to a subject. ... The subject-term must always include the predicate-term, in such a way that a man who understood the notion of the subject perfectly would also judge that the predicate belongs to it.[16]

In this passage, Leibniz is talking about a subject term that refers to an individual substance. He says something similar about 'universal propositions' in a piece entitled 'Elements of a Calculus', written in April 1679:

> ...every true universal affirmative categorical proposition simply shows some connection between predicate and subject (a *direct* connection, which is what is always meant here). This connection is, that the predicate is said to be in the subject, or to be contained in the subject; either absolutely and regarded in itself, or at any rate, in some instance; i.e., that the subject is said to contain the predicate in a stated fashion. This is to say that the concept of the subject, either in itself or with some addition, involves the concept of the predicate....[17]

Notice that in the last line of this passage, Leibniz talks about 'the concept of the predicate'. This seems to distinguish the predicate as a linguistic entity from the ordinary concept it expresses. But consider what happens if we interpret Leibniz in the material mode, by understanding his use of 'predicate' as referring to the property (or attribute) expressed.

[15]In the *Discourse on Metaphysics*, Article 8, Leibniz says:

> It is very true that when several predicates are attributed to one and the same subject and this subject is not attributed to any other, one calls the subject an individual substance.

This is the translation in PW 18. The source is G.iv 432.

[16]This is the translation in PW 18. The source is G.iv 433.

[17]This is the translation in LP 18-19. The source is C 51. Parkinson entitles the piece from which this quote is taken 'Elements of a Calculus'.

Then 'the concept of the predicate' would be some further entity, namely, something like the concept of the property. But this is the distinction we are suggesting is latent in Leibniz's work, namely, between the property F and the concept c_F of the property.[18]

To exploit this distinction further, let us return to the previous passage, in which Leibniz talks not only of the 'subject' but also of the 'notion of the subject'. Let us interpret this talk also in the material mode, so that 'subject' refers to the individual substance being discussed in that passage from the *Discourse* and 'notion of the subject' refers to the concept of the subject. Putting these suggestions together, we could interpret Leibniz as having invoked a distinction between the individual substance x and the concept of the individual substance c_x, on the one hand, and the property or attribute F of the substance and the concept of that property c_F, on the other. In somewhat more formal terms, we might say that the Leibnizian analysis of the ordinary singular statement 'x is F' amounts to: c_x contains c_F. Whereas most philosophers distinguish the individual x from its concept c_x, it is rare to find the property F distinguished from its concept c_F. But it seems natural to group ordinary individuals and properties together and to suppose that there is a kind of concept appropriate to each.

Further evidence for this view occurs later on in Article 8 of the *Discourse on Metaphysics*, where Leibniz talks about the 'notion' of an accident:

> ...it is in the nature of an individual substance, or complete being, to have a notion so complete that it is sufficient to contain and render deducible from itself, all the predicates of the subject to which this notion is attributed. On the other hand, an accident is a being whose notion does not include all that can be attributed to

[18]The reader should note that our particular distinction between properties and the concepts of properties is drawn here for the specific purpose of developing a Leibnizian theory of concepts. In the context of Frege's work, however, the distinction between a property and the concept of that property would be drawn rather differently. Elsewhere we have suggested that a predicate denotes an ordinary property and has an *abstract* property as its sense. Abstract properties encode properties of properties and they are axiomatized in the *type-theoretic* development of the theory of abstract objects. As such, these abstract properties would serve as the 'concept' a person x might have of the property F, for such a concept would encode the properties of the property F that x takes to be characteristic of F. But this is a matter that does not affect the present development. Interested readers should consult Zalta [1983] (Chapters V and VI), and [1988] (Chapters 9 – 12).

> the subject to which this notion is attributed. Take, for example, the quality of being a king, which belongs to Alexander the Great. This quality, when abstracted from its subject, is not sufficiently determinate for an individual and does not contain the other qualities of the same subject, nor everything that the notion of this prince contains.[19]

In this passage, Leibniz seems to distinguish an accident from the notion of that accident ("an accident is a being whose notion ..."). But then, in the sentences that follow, he reverts back to talk about the quality of being a king, as if that were the entity that is contained in the concept of Alexander. In what follows, however, we continue to distinguish between properties and their concepts. We think that it provides the key to a thorough reconstruction of Leibniz's logic and metaphysics of concepts.

§2: Concepts As Abstract Objects

In what follows, we shall try, whenever possible, to match our results to the propositions asserted or proved in the 1690 fragment.[20] Often, however, the propositions that Leibniz considers in the 1690 fragment appear in somewhat different notation in some of his earlier logical sketches.

We propose that Leibnizian concepts (both the concepts of individuals and the concepts of properties) are *abstract* objects that *encode* properties. Encoding is a mode of predication that has been formalized, axiomatized, and applied in Zalta [1983], [1988a], [1993], [1999], and elsewhere. Those readers unfamiliar with this system should consult Appendix I, which contains a reasonably thorough sketch of the theory. In what follows, we shall identify the principles of the theory according to the scheme in Appendix I.

Here is a one paragraph, intuitive description of the idea that we shall formalize in what follows. The Leibnizian concept (of) F will be identified as the abstract object that *encodes* all and only the properties necessarily implied by F. Thus, *the concept person* will encode the property of being rational, assuming that the property of being rational is necessarily implied by the property of being a person. Moreover, we shall define: x *contains* y iff x encodes every property y encodes. So, assuming that

[19] This is the translation in PW 18-19. The source is G.iv 433.

[20] Parkinson entitles this paper 'A Study in the Calculus of Real Addition'. See LP 131-144 (= G.vii 236-247).

the property of being a person necessarily implies the property of being rational, it will follow that the concept person contains the concept rational.[21] By way of contrast, the *individual concept* of Alexander will be identified as the abstract object that encodes all and only the properties the ordinary object Alexander exemplifies. It will then follow that the concept Alexander contains the concept person if given the assumption that Alexander exemplifies the property of being a person.

Given this sketch, we begin our formal analysis by identifying Leibnizian concepts in terms of the abstract objects of our background metaphysics. *Principle 3* (Appendix I) is our comprehension principle for abstract objects and it asserts that for any condition ϕ (in which x isn't free), there is an abstract object x that encodes all and only the properties satisfying the condition. Since our system uses the predicate '$A!x$' to denote the property of being abstract, we can turn our theory of abstract objects into a theory of concepts by employing the following definition:

$$Concept(x) \ =_{df} \ A!x$$

Instances of the comprehension principle for abstract objects now assert the existence of Leibnizian concepts, and so we may think of the comprehension schema as providing the existence conditions for concepts. Similarly, *Principle 4* (Appendix I) offers well-defined identity conditions for abstract objects and these conditions now tell us that concepts x and y are identical whenever they necessarily encode the same properties. Note that the identity conditions for concepts are defined in terms what we take to be their distinctive feature, namely, their encoded properties.

It would serve well to look at a particular example of a concept, say the concept rational. It should now be clear from our discussion that *the concept rational* ('c_R') is to be identified with the abstract object that encodes all and only the properties necessarily implied by the property of *being rational* ('R'):

$$c_R \ =_{df} \ \imath x(Concept(x) \,\&\, \forall F(xF \equiv R \Rightarrow F))$$

[21] For a quick proof sketch, assume that the property of being a person necessarily implies the property of being rational. To show that the concept person contains the concept rational, assume that the concept rational encodes property P (to show that the concept person encodes P). If the concept rational encodes P, then by definition, the property of being rational necessarily implies P. But since being a person necessarily implies being rational, and being rational necessarily implies P, being a person necessarily implies P. So the concept person encodes P, since it encodes just the properties necessarily implied by the property of being a person.

In this definition, 'the concept rational' is defined in terms of a definite description. *Principle 3′* and its Corollary (Appendix I) establishes that every such description denotes, so our new term 'c_R' is well-defined. To take another example, we may identify the concept animal ('c_A') as the (abstract) object that encodes just the properties implied by the property of being an animal ('A').

In general, we define *the concept G* ('c_G'), for any property G, as follows:

$$c_G \;=_{df}\; \iota x(Concept(x)\;\&\;\forall F(xF \equiv G \Rightarrow F))$$

That there is a unique such object for each property G follows from *Principle 3′*.[22] Clearly, then, we are distinguishing the property G from its concept c_G.

§3: Concept Identity

Now let x, y, and z be any Leibnizian concepts. Then the following three facts concerning concept identity are immediate consequences of *Principles 4* and *5* (Appendix I):[23]

Theorem 1: $x = x$

Theorem 2: $x = y \rightarrow y = x$

Theorem 3: $x = y \;\&\; y = z \rightarrow x = z$

So we should represent Leibniz's claim:[24]

If $A = B$, then $B = A$

as the claim:

$$c_A = c_B \rightarrow c_B = c_A$$

This is an instance of Theorem 2. Leibniz proves other corollaries and theorems with respect to concept identity.[25] However, these are all simple consequences of the fact that concept identity is an equivalence condition, and we shall omit mention and derivation of these simple consequences here.[26]

[22]The following is derivable from *Principle 3′* by generalizing on the variable G: $\forall G \exists ! x(A!x \;\&\; \forall F(xF \equiv G \Rightarrow F))$.

[23]See LP 131 (= G.vii 236), Propositions 1 and 3, where Leibniz proves the symmetricality and transitivity of concept identity, respectively.

[24]See LP 131 (= G.vii 236), Proposition 1.

[25]See LP 131-132 (= G.vii 236-237), Proposition 2, Corollary to Proposition 3, and Proposition 4.

§4: Concept Addition

If we continue to let x, y, z range over Leibnizian concepts, then we may define the real sum of the concepts x and y ('$x \oplus y$') as follows:

$$x \oplus y \;=_{df}\; \iota z(Concept(z)\;\&\;\forall F(zF \equiv xF \vee yF))$$

In other words, the real sum concept $x \oplus y$ is the concept that encodes a property F iff either x encodes F or y encodes F. *Principle 3′* (Appendix I) guarantees that there is a unique concept that meets the definition of $x \oplus y$, for any x and y.[27] To take a particular example, the real sum of the concept rational and the concept animal ('$c_R \oplus c_A$') is now well-defined. It is a simple consequence of these definitions that the sum of the concept G and the concept H is identical to the (abstract) object that encodes just the properties implied by G or implied by H:

Theorem 4: $c_G \oplus c_H = \iota x(Concept(x)\;\&\;\forall F(xF \equiv G \Rightarrow F \vee H \Rightarrow F))$

The proof of Theorem 4 is given in Appendix II. Note that we must prove that x and y necessarily encode the same properties to show that they are identical. Since the logic of encoding (Appendix I) guarantees that encoded properties are rigidly encoded, it suffices to show that x and y encode the same properties to prove them identical. So, in the left-to-right direction of the proof, we assume that $c_G \oplus c_H$ encodes an arbitrary property, say P, and then show that the object described on the right hand side of the identity sign encodes P, and similarly for the right-to-left direction.[28]

[26]Note also that Theorems 1 – 3 remain true when the range of the variables are extended to include ordinary individuals. This is a consequence of *Principles 2* and *5* (Appendix I). Since the domain of objects contains *only* ordinary and abstract objects, our theorems of identity can be thought of as completely general, covering any objects whatsoever.

[27]The following is a consequence of *Principle 3′*, by generalizing on the variables x and y in the relevant instance: $\forall x \forall y \exists ! z(A!z \;\&\; \forall F(zF \equiv xF \vee yF))$.

[28]In the proof, we use the notation '$c_G \oplus c_H P$' to assert that the sum concept $c_G \oplus c_H$ encodes property P. This is an encoding formula of the form 'xG' in which the variable 'x' has been replaced by a complex object term.

Now to confirm that $\oplus$ behaves in the manner that Leibniz prescribed, note that it follows immediately from these definitions that the operation $\oplus$ is idempotent and commutative:

Theorem 5: $x \oplus x = x$

Theorem 6: $x \oplus y = y \oplus x$

Leibniz takes these two principles as *axioms* of his calculus, whereas we derive them as theorems.[29] Leibniz omits associativity from his list of axioms for $\oplus$, but as Swoyer ([1995], [1994]) points out, it must be included for the proofs of certain theorems to go through. In our system, the associativity of $\oplus$ is almost as immediate as idempotency and commutativity:

Theorem 7: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

The proof of *Theorem 7* is also in Appendix II. Recall that to show that $(x \oplus y) \oplus z$ and $x \oplus (y \oplus z)$ are identical, we take an arbitrary property P and show that $(x \oplus y) \oplus z$ encodes P if and only if $x \oplus (y \oplus z)$ encodes P.[30]

So $\oplus$ is associative. In virtue of this fact, we may leave off the parentheses in '$(x \oplus y) \oplus z$' and '$x \oplus (y \oplus z)$'. Indeed, this suggests that we may generalize the real sum operation. We may define the real sum of concepts $x_1, \ldots, x_n$ as follows:

$$x_1 \oplus \ldots \oplus x_n =_{df} \imath z(\mathit{Concept}(z) \,\&\, \forall F(zF \equiv x_1F \vee \ldots \vee x_nF))$$

That there is a unique such object, for any concepts $x_1, \ldots, x_n$, is guaranteed by *Principle 3′*.[31] Since $c_{G_1} \oplus \ldots \oplus c_{G_n}$ is an instance of this definition, 'the sum of the concept G_1 and ... and the concept G_n' is well-defined. We can therefore generalize an earlier theorem:

Theorem 8: $c_{G_1} \oplus \ldots \oplus c_{G_n} =$
$\imath x(\mathit{Concept}(x) \,\&\, \forall F(xF \equiv G_1 \Rightarrow F \vee \ldots \vee G_n \Rightarrow F))$

[29] See LP 132 (= G.vii 237), Axioms 2 and 1, respectively. Other idempotency assertions appear in LP 40 (= G.vii 222), LP 56 (= C 366), LP 85 (= C 396), LP 90 (= C 235), LP 93 (= C 421) and LP 124 (= G.vii 230). Swoyer [1995], in footnote 5, also cites C 260 and C 262. Lenzen [1990] also cites GI 171 for idempotency. Other commutativity assertions appear in LP 40 (= G.vii 222), LP 90 (= C 235), and LP 93 (= C 421).

[30] The proof uses the notation '$(x \oplus y) \oplus zP$' to express the claim that $(x \oplus y) \oplus z$ encodes P. Again, this is an encoding formula of the form 'xP' in which the variable 'x' has been replaced by a complex term.

[31] By the following consequence of the comprehension and identity principles governing A-objects: $\forall x_1 \ldots \forall x_n \exists! z(A!z \,\&\, \forall F(zF \equiv x_1F \vee \ldots \vee x_nF))$.

The following two theorems also prove to be of interest:[32]

Theorem 9: $x = y \rightarrow x \oplus z = y \oplus z$

Theorem 10: $x = y \,\&\, z = u \rightarrow x \oplus z = y \oplus u$

Leibniz is careful to note that the *converses* of these two simple theorems are *not* true, for there are counterexamples.[33] We can produce our own counterexample to the converse of Theorem 9: let x be $c_F \oplus c_G \oplus c_H$, let y be $c_F \oplus c_G$, and let z be c_H. Then $x \oplus z = y \oplus z$, but $x \neq y$. Similarly, as a counterexample to the converse of Theorem 10, let x be $c_F \oplus c_G$, let z be $c_H \oplus c_I$, let y be c_F, and let u be $c_G \oplus c_H \oplus c_I$. Then, $x \oplus z = y \oplus u$, but neither $x = y$ nor $z = u$. Finally, Leibniz describes other theorems concerning concept addition and identity, but these are less interesting (since easily derived by substitution of identicals), and will be omitted here.[34]

§5: Concept Inclusion and Containment

Leibniz defined the notions of concept inclusion and concept containment in terms of 'sameness' or 'coincidence'. He says:

> Definition 1. Those terms are 'the same' or 'coincident' of which either can be substituted for the other wherever we please without loss of truth—for example, 'triangle' and 'trilateral'. ... 'A = B' means that A and B are the same.[35]
>
> Definition 3. That A 'is in' L, or, that L 'contains' A, is the same as that L is assumed to be coincident with several terms taken together, among which is A.[36]

In other words, Leibniz defines: x is included in y just in case there is a concept z such that the sum of x and z is identical to y.

However, in our reconstruction of Leibniz, we shall define concept inclusion and containment in terms of our notion of encoding and then prove Leibniz's definition as a theorem (in Section 7). We define the notions of concept inclusion ('$x \preceq y$') and concept containment ('$y \succeq x$') in our system as follows:

[32] See LP 133-4 (= G.vii 238), Propositions 9 and 10.

[33] See the Notes (Scholia) to Propositions 9 and 10 in LP 133-134 (= G.vii 238).

[34] See LP 133 (= G.vii 237-238), Propositions 5, 6, and 8.

[35] This is the translation in LP 131. The source is G.vii 236.

[36] This is the translation in LP 132. The source is G.vii 237.

$$x \preceq y \ =_{df} \ \forall F(xF \rightarrow yF)$$

$$y \succeq x \ =_{df} \ x \preceq y$$

To show that these definitions are good ones, we establish that the notions defined behave the way Leibniz says they are supposed to behave. In what follows, we identify the relevant theorems in pairs, a theorem governing concept inclusion and the counterpart theorem governing concept containment. We prove the theorem only as it pertains to concept inclusion. First, we note that concept inclusion and containment are reflexive, anti-symmetric, and transitive:[37]

Theorem 11i: $x \preceq x$
Theorem 11c: $x \succeq x$

Theorem 12i: $x \preceq y \ \rightarrow \ (x \neq y \rightarrow y \not\preceq x)$
Theorem 12c: $x \succeq y \ \rightarrow \ (x \neq y \rightarrow y \not\succeq x)$

Theorem 13i: $x \preceq y \,\&\, y \preceq z \ \rightarrow \ x \preceq z$
Theorem 13c: $x \succeq y \,\&\, y \succeq z \ \rightarrow \ x \succeq z$

The proof of Theorem 12i is in Appendix II. Leibniz goes on to prove that when concepts x and y are included or contained in each other, they are identical:[38]

Theorem 14i: $x \preceq y \,\&\, y \preceq x \ \rightarrow \ x = y$
Theorem 14c: $x \succeq y \,\&\, y \succeq x \ \rightarrow \ x = y$

Two other interesting consequences of concept inclusion and identity that Leibniz doesn't appear to have considered are that x and y are identical concepts if (a) concept z is included in x iff z is included in y, or if (b) x is included in z iff y is included in z:

Theorem 15i: $\forall z(z \preceq x \equiv z \preceq y) \ \rightarrow \ x = y$
Theorem 15c: $\forall z(x \succeq z \equiv y \succeq z) \rightarrow x = y$

Theorem 16i: $\forall z(x \preceq z \equiv y \preceq z) \ \rightarrow \ x = y$
Theorem 16c: $\forall z(z \succeq x \equiv z \succeq y) \rightarrow x = y$

[37]See LP 133 (= G.vii 238), Proposition 7, where Leibniz proves the reflexivity of inclusion. See LP 135 (= G.vii 240), Proposition 15, where Leibniz proves the transitivity of inclusion. See also LP 33 (= G.vii 218) for the reflexivity of containment.

[38]See LP 136 (= G.vii 240), Proposition 17.

I conclude this initial discussion of concept inclusion and containment with two other consequences of the foregoing:

Theorem 17i: $c_{G_1} \preceq c_{G_1} \oplus c_{G_2} \preceq \ldots \preceq c_{G_1} \oplus \ldots \oplus c_{G_n}$

Theorem 18i: $c_{G_{i_1}} \oplus \ldots \oplus c_{G_{i_j}} \preceq c_{G_1} \oplus \ldots \oplus c_{G_n}$, where $1 \leq i_1 \leq \ldots \leq i_j \leq n$

Similar claims apply to concept containment.

§6: Concept Inclusion, Containment, and Addition

Leibniz's theorems that govern just concept inclusion (containment) and addition are also derivable. The concept x is included in the sum $x \oplus y$, and so is the concept y:[39]

Theorem 19i: $x \preceq x \oplus y$
Theorem 19c: $x \oplus y \succeq x$

Theorem 20i: $y \preceq x \oplus y$
Theorem 20c: $x \oplus y \succeq y$

Moreover, if y is included in z, then $x \oplus y$ is included in $x \oplus z$:[40]

Theorem 21i: $y \preceq z \ \rightarrow \ x \oplus y \preceq x \oplus z$
Theorem 21c: $y \succeq z \ \rightarrow \ x \oplus y \succeq x \oplus z$

It also follows that if $x \oplus y$ is included in z, then both x and y are included in z:[41]

Theorem 22i: $x \oplus y \preceq z \ \rightarrow \ x \preceq z \,\&\, y \preceq z$
Theorem 22c: $z \succeq x \oplus y \ \rightarrow \ z \succeq x \,\&\, z \succeq y$

And it follows that if both x and y are included in z, then $x \oplus y$ is included in z:[42]

Theorem 23i: $x \preceq z \,\&\, y \preceq z \ \rightarrow x \oplus y \preceq z$
Theorem 23c: $z \succeq x \,\&\, z \succeq y \ \rightarrow z \succeq x \oplus y$

[39]This fact appears not to have been mentioned in the fragment of 1690, but see LP 33 (= G.vii 218) for the corresponding fact about containment.

[40]See LP 134 (= G.vii 239), Proposition 12. See also LP 41 (= G.vii 223), for the version governing containment.

[41]See LP 136 (= G.vii 240), Corollary to Proposition 15.

[42]See LP 137 (= G.vii 241), Proposition 18.

Finally, we may prove that if x is included in y and z is included in u, then $x \oplus z$ is included in $y \oplus u$:[43]

Theorem 24i: $x \preceq y \;\&\; z \preceq u \;\rightarrow x \oplus z \preceq y \oplus u$
Theorem 24c: $x \succeq y \;\&\; z \succeq u \;\rightarrow x \oplus z \succeq y \oplus u$

§7: Concept Inclusion, Addition, and Identity

The most important theorems of our Leibnizian calculus are the ones that relate the notions of concept inclusion (containment), addition, and identity. Recall that Leibniz defines x *is included in* y iff there is a concept z such that the sum of x and z is identical to y.[44] This definition, and the corresponding definition of x *contains* y, fall out as theorems:[45]

Theorem 25i: $x \preceq y \;\equiv\; \exists z(x \oplus z = y)$

Theorem 25c: $x \succeq y \;\equiv\; \exists z(x = y \oplus z)$

(The proof of Theorem 25i is in Appendix II.) Second, our definition of $\preceq$ also validates both forms of the principal theorem governing Leibniz's calculus: (i) x is included in y iff the sum of x and y is identical with y and (c) x contains y iff x is identical with the sum of x and y. In formal terms:[46]

Theorem 26i: $x \preceq y \;\equiv\; x \oplus y = y$

Theorem 26c: $x \succeq y \;\equiv\; x = x \oplus y$

(The proof is in Appendix II.) Though Leibniz proves Theorem 26i using our Theorem 25i as a definition, on the present theory, no appeal to Theorem 25i needs to be made.

We can now prove Leibniz's Principle (A) discussed at the outset of the paper as a simple instantiation of Theorem 26c. Recall that Principle (A) is:

(A) The concept F contains the concept G iff the concept F is identical to the sum of the concept F and the concept G.

[43] See LP 137 (= G.vii 241), Proposition 20.
[44] See LP 132 (= G.vii 237), Definition 3.
[45] Compare Swoyer [1995], p. 99, who builds this condition into Leibnizian Relational Structures.
[46] See LP 135 (= G.vii 239), Propositions 13 and 14.

The formal representation of Principle (A) is a simple corollary to Theorem 26c by instantiation:

Corollary: $c_F \succeq c_G \;\equiv\; c_F = c_F \oplus c_G$

So Leibniz's main principles governing the relationship between concept inclusion ($\preceq$), concept containment ($\succeq$), concept identity, and concept addition ($\oplus$) are derivable. Moreover, since Theorem 26i establishes the equivalence of $x \oplus y = y$ and $x \preceq y$, and Theorem 25i establishes the equivalence of $x \preceq y$ and $\exists z(x \oplus z = y)$, it follows that:

Theorem 27i: $x \oplus y = y \;\equiv\; \exists z(x \oplus z = y)$

Theorem 27c: $x = x \oplus y \;\equiv\; \exists z(x = y \oplus z)$

These results show that our definition of $\preceq$ and $\succeq$ preserve Leibniz's logic of concepts.

We conclude this derivation of the basic logic of Leibnizian concepts with the following extended observation. Since we have derived Principle (A), the following instance of Principle (A) is therefore provable: the concept person contains the concept rational iff the concept person is identical with the sum of the concept person and the concept rational. In formal terms:

Instance of Principle (A): $c_P \succeq c_R \;\equiv\; c_P = c_P \oplus c_R$

The reason this is true (in the left-right direction) is that if every property encoded in the concept rational is encoded in the concept person, then the concept person encodes the same properties as the concept that encodes all the properties encoded in either the concept person or the concept rational (since c_R contributes to $c_P \oplus c_R$ no properties not already encoded in c_P).[47] However, recall that Leibniz believed that the concept person is itself the sum of the concept rational and the concept animal. In formal terms, Leibniz's suggestion amounts to:

$$c_P = c_R \oplus c_A$$

If this is an accurate representation of Leibniz's view, then if we add this as an hypothesis to our system, we may derive from it the following fact:[48]

[47] Of course, we have assumed here that the property of being a person implies the property of being rational; i.e., $P \Rightarrow R$.
[48] Given the identity of c_P with $c_R \oplus c_A$, the fact that $c_P \succeq c_R$ is a simple consequence of Theorem 19c: $x \oplus y \succeq x$.

$$c_P \succeq c_R$$

So from Leibniz's analysis of the concept person, we may derive the left side of the above *Instance* of Principle (A). Such a derivation, together with the instance itself, yields the consequence that the concept person is identical to the sum of the concept person and the concept rational.

However, it seems reasonable to argue that, strictly speaking, c_P is not the same concept as $c_R \oplus c_A$. By definition, $c_R \oplus c_A$ encodes all and only the properties either implied by the property of *being rational* or implied by the property of *being an animal*. It therefore fails to encode the property $[\lambda x\, Rx \,\&\, Ax]$, since this conjunctive property is neither implied by the property of being rational nor implied by the property of being an animal. But one might argue that *the concept person* does encode the conjunctive property. This suggests that it might be preferable to distinguish c_P from $c_R \oplus c_A$ by identifying the former with the concept that encodes just the properties implied by the conjunctive property of being a rational animal:[49]

$$c_P = c_{[\lambda x\, Rx \,\&\, Ax]}$$

This makes good sense. For one might argue that, strictly speaking, it is the property of being a person that is identical with the property of being a rational animal (just as the property of being a brother is identical with the property of being a male sibling, and the property of being a circle is identical with the property of being a closed, plane figure every point of which lies equidistant from some given point, etc.). So simply by adding the hypothesis that $P = [\lambda x\, Rx \,\&\, Ax]$ to our system, we could derive that c_P is identical with the concept $c_{[\lambda x\, Rx \,\&\, Ax]}$ (since abstract objects that encode the same properties are identical).

Notice that even if we were to represent the concept person in this way, it still follows that the concept person contains the concept rational, since the conjunctive property of being a rational animal implies the property of being rational. But, strictly speaking, this is a departure from the letter of the Leibnizian corpus, in which a complex concept such as c_P is analyzed in terms of the sum of its simpler concepts. Our theory of concepts suggests that there is subtle and important difference between $c_R \oplus c_A$ and $c_{[\lambda x\, Rx \,\&\, Ax]}$. This difference may not have been observed in Leibniz's own theory.

[49]Recall that $c_{[\lambda x\, Rx \,\&\, Ax]}$ is defined as the abstract object that encodes all the properties necessarily implied by the conjunctive property $[\lambda x\, Rx \,\&\, Ax]$.

§8: The Modal Metaphysics of Individual Concepts

In this section, we define individual concepts and derive the basic modal facts that govern them.[50] In the next section, we show how to combine these facts with the containment theory of truth to establish the fundamental theorem of Leibniz's metaphysics of concepts. To define the notion of an individual concept, we employ the notion of a possible world. However, our use of possible worlds is based on an explicit definition of worlds that can be given in object theory. This definition and the derivation of the theorems of world theory can be found in previous work (Zalta [1993]). Readers unfamiliar with this work may wish to consult Appendix III, where one will find a review of the theorems of world theory. Of course, the fact that Leibniz appeals to the notion of a possible world throughout his work is well known.[51]

§8.1: Must We Use Counterpart Theory?

In what follows, we plan to offer an alternative to the widespread view that the best way to reconstruct a Leibnizian modal metaphysics of *concepts* is to use some version of D. Lewis's [1968] counterpart theory. This view traces back to work of Mondadori ([1973] and [1975]), who notes that the natural reading of certain passages in the Leibnizian corpus are suggestive of counterpart theory.[52] Here is an example from the *Theodicy* which Mondadori cites:

> I will now show you some [worlds], wherein shall be found, not absolutely the same Sextus as you have seen (that is not possible, he carries with him always that which he shall be) but several Sextuses resembling him, possessing all that you know already of the true Sextus, but not that is already in him imperceptibly, nor in consequence all that shall yet happen to him. You will find in one

[50]The ideas that follow were initially developed, in somewhat different guise, in Zalta [1983] (pp. 84-90). The reader familiar with the earlier work will find that we have replaced our [1983] definition of 'monad' by a definition of 'individual concept' and replace the [1983] definition of 'concept containment' by the definition developed in the present paper. It should be mentioned that the material in [1983] was indebted to Parsons [1978] (pp. 146-148) and [1980] (pp. 219-224).

[51]Two particularly clearcut examples are in the *Theodicy* (see T 128 = G.vi 107) and in the *Monadology* §53 (see PW 187 = G.vi 615-616).

[52]See also Ishiguro [1972], pp. 123-134), for a possible version of this view.

> world a very happy and noble Sextus, in another a Sextus content with a mediocre state, ...[53]

Mondadori also cites the letter to Landgraf Ernst von Hessen-Rheinfels of April 12, 1686, where Leibniz talks about the different possible Adams, all of which differ from each other:

> For by the individual notion of Adam I undoubtedly mean a perfect representation of a particular Adam, with given individual conditions and distinguished thereby from an infinity of other possible persons very much like him, but yet different from him... There is one possible Adam whose posterity is such and such, and an infinity of others whose posterity would be different; is it not the case that these possible Adams (if I may so speak of them) are different from one another, and that God has chosen only one of them, who is exactly our Adam?[54]

Many commentators have accepted Mondadori's understanding of these passages, though some have had certain reservations and qualifications about understanding Leibniz's work in terms of counterpart theory.[55]

The case for reconstructing Leibniz's work in terms of counterpart theory is not perfectly straightforward, however. When Mondadori introduces the suggestion of using counterpart theory to model Leibniz's views, he notes that whereas for Lewis the counterpart relation is a relation on individuals, "in Leibniz's case, it is best regarded as being a relation between (complete) concepts" ([1973], 248). This is explicitly built into the Leibnizian system described in Fitch [1979]. The idea is that in a Leibnizian modal metaphysics, the possible worlds are not inhabited by Lewis's possibilia, but rather by complete individual concepts. Indeed, these authors just model possible worlds as sets of compossible individual concepts. So their reconstruction of the modal metaphysics of complete concepts involves: (1) equivalence classes of compossible complete individual concepts to define the possible worlds, and (2) a counterpart relation which connects each complete individual concept of a given world to various other complete individual concepts in other worlds. The Leibnizian analysis of the contingent claim 'Alexander might not have been a king' becomes: there is a possible world w (other than the actual world) and an individual concept c such that: (i) c is in w, (ii) c is a counterpart of the concept c_a of Alexander, and (iii) c doesn't contain the concept of being a king.

So when Leibniz talks about the 'many possible Adams' and 'several Sextuses' which are all distinct from one another, these commentators take him to be talking about different individual concepts rather than different possible individuals. This is, strictly speaking, a departure from the texts. The commentators suppose that it is legitimate to 'interpret' Leibniz's talk of 'several Sextuses' and 'the many possible Adams' not as referring to possibilia but rather to individual concepts. Similarly, in what follows, we describe a modal metaphysics in which we interpret Leibniz's talk of 'the many possible Adams' and 'the several Sextuses' as referring to various distinct complete individual concepts. However, we shall not employ counterpart theory to link the counterparts of the concept Adam which appear at various other possible worlds. Instead, we shall utilize the variety of complete individual concepts which become defined when we consider the properties that Adam exemplifies at each possible world.

In the modal metaphysics we present in what follows, it is part of the logic that the single ordinary individual Adam exemplifies properties at *every* possible world. Of course, the properties that Adam exemplifies at one world differ from the properties he exemplifies at other worlds. For example, although Adam is concrete (spatiotemporal) at our world and at certain other possible worlds, there are possible worlds where he fails to be concrete. (Some philosophers have been tempted to call those worlds where Adam isn't concrete worlds where 'Adam doesn't exist', but we will not follow this usage.)[56] There will be numerous possible worlds where Adam is concrete and his 'posterity is different'. (At worlds where Adam is not concrete, he has no posterity.)[57]

[53] This is the translation in T 371. The source is G.vi 363.

[54] This is the translation in PW 51. The source is G.ii 20.

[55] See, for example, Fitch [1979], Wilson [1979], and Vailati [1986]. Lloyd [1978] also accepts that Leibniz 'resorts to counterparts' (p. 379), though she discovers some Leibnizian features in a Kripkean semantics of rigid designators, which supposes that the same individual can appear in other possible worlds.

[56] Given our fixed domain quantified modal logic governed by the Barcan formulas, every object exists necessarily in the sense that $\forall x \Box \exists y (y = x)$. The contingency of an ordinary object such as Adam is preserved by the fact that he is not concrete at every world. While other philosophers would say that these latter worlds are ones where Adam fails to exist, it would be a mistake to say this in the present version of modal logic, which was defended in Linsky and Zalta [1994] and Williamson [1998].

[57] It might help to point out here that, in the case of ordinary objects such as Adam, the claim 'Adam is F essentially' is to be understood as the claim that Adam is F in every world in which he is concrete. For example, the claim 'Adam is essentially a

Now although the same ordinary individual Adam exemplifies properties at other worlds, a Leibnizian metaphysics of 'world-bound' individuals emerges once we consider, for each world, the 'individual concept' (i.e., abstract object) that encodes exactly the properties that Adam exemplifies at that world. At each world, Adam realizes a *different* individual concept, since individual concepts will differ whenever they encode distinct properties. The individual concept that encodes all and only the properties Adam exemplifies at one world is distinct from the individual concept that encodes all and only the properties Adam exemplifies at a different world. Of course, we will define only one of these individual concepts to be 'the concept of Adam', namely, the concept that encodes just what Adam exemplifies at the actual world. Strictly speaking, the various individual 'concepts of Adam' are 'concepts-of-Adam-at-w' (for some w) and so *the* concept of Adam will be identified with the concept of Adam at the actual world. When Leibniz talks about 'possible Adams', we may take him to be talking about different individual concepts, namely different 'Adam-at-w' concepts. We'll explain this in more detail once the definitions and theorems have been presented.

What is more interesting is the fact that these individual concepts have certain other Leibnizian features. By way of contrast to other commentators on Leibniz, we shall not just *stipulate* that individual concepts are partitioned into compossibility (equivalence) classes but rather define compossibility and *prove* that it is a condition which partitions the individual concepts. Moreover, we shall not define possible worlds as sets of compossible individual concepts, but rather prove that there is a one-to-one correlation between each group of compossible concepts and the possible worlds. We shall also prove that any arbitrarily chosen member of a group of compossible individual concepts is a perfect mirror of its corresponding world. And, finally, we shall prove that individual concepts are complete concepts that are realized by at most one object if realized at all.

So, although we do not use counterpart theory in our reconstruction, and we suppose, in departure from the strict Leibnizian text, that there is exactly one Adam who exemplifies properties at different worlds, we nevertheless *recover* a Leibnizian modal metaphysics in which the complete individual concepts are the 'world-bound' individuals, not the ordinary

man' is to be understood as: Adam is a man in every world in which he is concrete. See Linsky and Zalta [1994], §4, for further discussion.

objects themselves. These individual concepts will provide an interpretation for much of what Leibniz says about necessity, contingency, completeness, mirroring, etc., as we shall see.

§8.2: The Definitions and Theorems

We begin by defining what it is for an ordinary object to *realize* a concept *at* a world. We shall say that an object y realizes a concept x at world w just in case y is an ordinary object which exemplifies at w all and only the properties x encodes.[58] This definition can be formalized with the help of notions of object theory defined elsewhere. First, the notion of an ordinary object ('$O!x$') is defined as any object that might have been concrete (Appendix I). Second, the claim 'proposition p is true at world w' ('$\models_w p$') is defined and properly systematized (Appendices I and III) by the claim that w encodes the property *being such that* p (i.e., $w[\lambda y\, p]$). With these notions, we can formalize our definition of 'realizes at' as follows:[59]

$$RealizesAt(y, x, w) =_{df} O!y \;\&\; \forall F(\models_w Fy \equiv xF)$$

Or, letting 'u' be a restricted variable ranging only over ordinary objects, we may define this notion more simply as:

$$RealizesAt(u, x, w) =_{df} \forall F(\models_w Fu \equiv xF)$$

Important notions of Leibniz's modal metaphysics can be defined in terms of realization:

$$Appears(x, w) =_{df} \exists u RealizesAt(u, x, w)$$

$$IndividualConcept(x) =_{df} \exists w Appears(x, w)$$

The first definition tells us that a concept appears at a world just in case some ordinary object realizes that concept at that world. The second tells us that a concept is an individual concept just in case it appears at some world.

[58] This represents another minor change of terminology from our previous work. In Zalta [1983], p. 85, we said that the ordinary object y 'is the correlate of' the abstract object x when this condition obtains.

[59] The symbol '$\models_w$' should be read with the smallest possible scope. For example, $\models_w p \equiv q$ is to be parsed as $(\models_w p) \equiv q$. To claim that the biconditional $p \equiv q$ is true at w, we write: $\models_w (p \equiv q)$.

To get some practice with these notions, let us now introduce concepts of particular individuals and show that they are individual concepts. (This is not a triviality!) In general, whenever u is an ordinary individual (i.e., whenever $O!u$), we define *the concept of* u ('c_u') as the concept that encodes just the properties u exemplifies:

$$c_u =_{df} \imath x(Concept(x) \,\&\, \forall F(xF \equiv Fu))$$

It follows from the Corollary to *Principle 3′* (Appendix I) that for any ordinary object u, the concept of u exists; i.e., $\forall u \exists z(z = c_u)$.

We can now prove that the concept of u is an individual concept. The proof depends on a simple lemma, namely, that c_u encodes a property G iff u exemplifies G:

Lemma: $c_u G \equiv Gu$

This Lemma is a consequence of the logic of definite descriptions.[60] (It is important to remember here that our definite descriptions are rigid designators. The logic of these descriptions is governed by a logical axiom that is a contingent logical truth. See the axiom *Descriptions* in Appendix I. In the next section, we'll see how this fact about *Descriptions* becomes important.)

Now with the help of this Lemma, it is straightforward to show that the concept of u is an individual concept:

Theorem 28: $IndividualConcept(c_u)$

(The proof is in Appendix II.) *A fortiori*, for every ordinary object u, there is exactly one individual concept that serves as c_u.

To see how these theoretical definitions work in a concrete case, let us assume the contingent fact that Alexander ('a') is a concrete object ($E!a$). It then follows by the T schema of modal logic and the definition of ordinary object (Appendix I) that Alexander is an ordinary object ($O!a$) and it follows from the foregoing that the concept of Alexander (c_a) exists and is an individual concept. So, thus far, we have used the contingent premise that Alexander is a concrete object to show that individual concepts exist.

However, it is possible to prove that individual concepts exist without appealing to any contingent premises. Notice that our theory doesn't assert the contingent claim that Alexander is a concrete object as an axiom,

[60] The Lemma follows immediately from *A-Descriptions* (Appendix I), which in turn is provable from the *Descriptions* axiom.

nor does it assert either the existence of any other particular concrete object or the existence of concrete objects in general. However, we can and ought to extend our theory with the *a priori* axiom that it is *possible* that there are concrete objects; i.e., $\Diamond \exists x E!x$. Such an axiom, in the context of an S5 modal logic, is provably a necessary truth. Moreover, by the Barcan formula, it follows that $\exists x \Diamond E!x$. In other words, it follows from *a priori* assumptions alone that there are ordinary objects ($\exists x O!x$). It then immediately follows from Theorem 28 and the Corollary to *Principle 3′*:

Theorem 29: $\exists x[IndividualConcept(x)]$

Thus we have an *a priori* proof that individual concepts exist.

Now one of Leibniz's most interesting metaphysical ideas is that of *mirroring*. We can show that individual concepts mirror any world where they appear. We define the notion of mirroring as follows:

$$Mirrors(x, w) =_{df} \forall p(x[\lambda y\, p] \equiv w[\lambda y\, p])$$

In other words, a concept x will mirror a world w just in case x encodes all and only those propositions true in w. It now follows that:

Theorem 30: $Appears(x, w) \rightarrow Mirrors(x, w)$

(The proof of Theorem 30 is in Appendix II.) From this theorem it now follows that every individual concept appears at a unique world:

Theorem 31: $IndividualConcept(x) \rightarrow \exists! w Appears(x, w)$

(This too is proved in Appendix II.) This last theorem demonstrates that individual concepts are, in an important sense, 'world-bound'. Since the properties these world-bound individual concepts encode are necessarily encoded, we have recovered a kind of Leibnizian 'super-essentialism' at the level of individual concepts.[61] (In the next section, we will see just how this super-essentialism is consistent with the existence of contingent truths.)

Since it is now established that every individual concept appears at a unique world, we may legitimately talk about *the* world w_c where individual concept c appears. Letting 'c' be a restricted variable ranging over individual concepts, we may define:

$$w_c =_{df} \imath w Appears(c, w)$$

[61] See Mondadori [1973].

The previous two theorems, that any concept appearing at a world mirrors that world and that every individual concept appears at a unique world, allow us to assert that every individual concept c mirrors *its* world and indeed, even contains its world:

Corollary: $Mirrors(c, w_c)$ & $c \succeq w_c$

The proof of the left conjunct is immediate from the preceding theorems and the proof of the right conjunct is immediate from the definition of containment and the fact that propositional properties of the form $[\lambda y\, p]$ are the only kind of properties that worlds encode.

Another important notion of Leibnizian metaphysics is *compossibility*. Two individual concepts are compossible just in case they appear at the same world. Compossibility should partition the individual concepts into equivalence classes that correspond with the possible worlds. We define compossibility formally as follows:

$$Compossible(c_1, c_2) =_{df} \exists w(Appears(c_1, w) \;\&\; Appears(c_2, w))$$

Given the previous theorem, it is an immediate consequence of this definition that:

Theorem 32: $Compossible(c_1, c_2) \equiv w_{c_1} = w_{c_2}$

With the help of this theorem, we get the following immediate results:

Theorem 33: $Compossible(c_1, c_1)$

Theorem 34: $Compossible(c_1, c_2) \rightarrow Compossible(c_2, c_1)$

Theorem 35: $Compossible(c_1, c_2)$ & $Compossible(c_2, c_3) \rightarrow Compossible(c_1, c_3)$

Since compossibility is a condition that is reflexive, symmetrical, and transitive, we know the individual concepts are partitioned. For each group of compossible individual concepts, there is a unique world where all of the individual concepts in that group appear.[62] Each member of the group not only mirrors the world corresponding to that group but also contains that world.

We look next at what kind of concepts individual concepts are. Individual concepts are suppose to be 'complete' and 'singular'. Intuitively, a complete concept is any concept x such that, for every property F, either x encodes F or x encodes $\bar{F}$:[63]

$$Complete(x) =_{df} \forall F(xF \vee x\bar{F})$$

The theory now predicts that every individual concept is complete:[64]

Theorem 36: $IndividualConcept(x) \rightarrow Complete(x)$

(The proof of this theorem is in Appendix II.)[65]

Finally, a concept x is singular just in case, for any world w, if ordinary objects u and v realize x at world w, then $u = v$. In formal terms:

[62] Compare Mates [1968], Mondadori [1973], and Fitch [1979], who stipulate most of these claims. Note also the difference between our definition of compossibility and that in Lenzen [1990], p. 186. Lenzen defines compossibility as follows: concepts x and y are compossible just in case it is possible that both x and y contain the concept of existence. On this definition, the transitivity of compossibility does not follow. But if individual concepts mirror their worlds in a genuine sense, transitivity has to be a property of compossibility.

[63] Recall that $\bar{F}$ is defined as: $[\lambda x\, \neg Fx]$.

[64] However, see Mondadori [1973] (p. 239) and [1975] (p. 257), who argues that Leibniz doesn't regard the complete concept of x as involving all of the properties that x exemplifies. Mondadori cites CA 48 (= G.ii 44). However, the reader should consider the passages cited at the end of this section, where Leibniz seems to say that individual concepts 'involve' even extrinsic, relational properties.

Nevertheless, if Mondadori is right, there could be a way to capture this view, if there is some clearcut notion of 'primitive', 'simple', or 'basic' property. Both Leibniz (CA 48 = G.ii 44) and Mates [1968] appeal to some such notion. If such a notion can be well defined, we could stipulate:

The core concept of $u =_{df}$
$\imath x(Concept(x) \;\&\; \forall F(xF \equiv Fu \;\&\; Basic(F)))$

As an example, the core concept of Adam would encode all and only the basic properties exemplified by Adam. However, now we have two candidates for *the* concept of Adam: c_a (as originally defined) and the core concept of Adam. These are distinct concepts. We will leave it to others to determine which one is best matched to Leibniz's entire corpus. Of course, before one can utilize 'the core concept of Adam', one will have to somehow demonstrate that all of Adam's other properties can be derived from this core concept.

[65] It might have been more in the spirit of Leibniz's framework to have defined a complete concept to be one which, for any property F, contains either the concept F or the concept $\bar{F}$. In other words, we might have defined the notion of completeness as follows:

$$Complete(x) =_{df} \forall F(x \succeq c_F \vee x \succeq c_{\bar{F}})$$

This, it turns out, is equivalent to the above definition, at least as far as individual concepts go. For by Theorem 39b (which is proved in the next section), it follows by simple disjunctive syllogism that if individual concepts are complete in the sense proved in Theorem 36, then they are complete in the sense defined in this footnote, and vice versa.

$$Singular(x) =_{df} \forall u, v, w(RealizesAt(u, x, w) \ \& \ RealizesAt(v, x, w) \rightarrow u = v)$$

It now follows that every individual concept is singular:

Theorem 37: $IndividualConcept(x) \rightarrow Singular(x)$

(The proof of this theorem is in Appendix II.) So every individual concept is complete and singular.

§8.3: Some Observations About the Metaphysics

The results just outlined offer a precise picture which seems to underlie such claims as the following (*Discourse on Metaphysics*):

> ...it is in the nature of an individual substance, or complete being, to have a notion so complete that it is sufficient to contain, and render deducible from itself, all the predicates of the subject to which this notion is attributed. (Article 8)[66]
>
> ...every substance is like an entire world and like a mirror of God, or of the whole universe, which each one expresses in its own way. ... Thus the universe is in a way multiplied as many times as there are substances. (Article 9)[67]

This very last claim is made true by the fact that every substance (i.e., non-concept) has an individual concept which mirrors, and indeed, contains, the actual world (by the Corollary to Theorem 31). A world w is 'multiplied' since each individual concept in the corresponding group of compossible concepts has w as a part.

There is no doubt that Leibniz scholars would question these results by pointing out that they are grounded in a logic which accepts that ordinary individuals have properties at every possible world. But there is also no doubt that the fact that these results are provable constitute a reason to take them seriously that Leibniz himself would have found compelling. (The results proved in the next section may be even more compelling.) The *subsystem* of individual concepts and worlds (i.e., the group of theorems that pertain to individual concepts and worlds) still preserves Leibniz's conception of the work that God had to do to 'create'

[66]This is the translation in PW 18. The source is G.iv 433.
[67]This is the translation in PW 19. The source is G.iv 434.

the actual world. If we focus narrowly on that subsystem, it becomes evident that, in order to evaluate all the possible worlds, God simply has to inspect an arbitrarily chosen individual concept from each group of compossible individual concepts. His inspection will reveal to Him the entire corresponding world, since every individual concept of the group mirrors that world. Note that our metaphysics doesn't tell us which world is the actual world other than by describing the actual world as the one that encodes all and only the true propositions. So God could 'create' (i.e., actualize) world w (after deciding it was the best) by making it the case that every proposition p encoded in w is true. Alternatively, when inspecting the possible worlds by examining a representative individual concept from each group of compossible concepts, he could 'create' (i.e., actualize) the individual concept c which was representative of the possible world which turned out to be the best. To do so, He would make it the case that there is in fact an ordinary object which exemplifies all the properties c encodes. In the process of doing this, he would have to create an entire world, since c encodes all the propositions encoded in the world it mirrors.

We now adopt the strategy of the other Leibnizian commentators, who interpret Leibniz's talk of 'the several Sextuses' and 'the many possible Adams' as a reference to the variety of individual concepts 'connected with' (see below) Sextus and Adam. Even though our subsystem of individual concepts was carved out without appeal to counterpart theory, we can, nevertheless, still speak with the counterpart theorists! To do so, we first define 'the concept of Adam-at-w':

$$c^w_a =_{df} \imath x(Concept(x) \ \& \ \forall F(xF \equiv \models_w Fa))$$

So, if w_α is the actual world, the concept of Adam-at-w_α (i.e., $c^{w_\alpha}_a$) is provably identical to *the* concept of Adam (c_a). Moreover, we may now say precisely when two individual concepts c and c' are counterparts:

$$Counterparts(c, c') =_{df} \exists u \exists w_1 \exists w_2 (c = c^{w_1}_u \ \& \ c' = c^{w_2}_u)$$

In other worlds, individual concepts c and c' are counterparts whenever there is an ordinary object u and there are worlds w_1 and w_2 such that c is the concept of u-at-w_1 and c' is the concept of u-at-w_2. Thus, when Leibniz talks of the 'many possible Adams', we take him to be referring to the many different concepts of Adam-at-w (for the various w). These are all counterparts of one another. There is nothing mysterious at all about this notion of counterpart. Of course, our notion of counterpart is an equivalence condition, and so differs from that of Lewis. But, to my knowledge, there is nothing in Leibniz's work that would rule this out. (We will employ our notion of counterpart further in the next section, when we consider the question of how our Leibnizian modal metaphysics of individual concepts accounts for contingency.)

We conclude this section by noting that our reconstruction of Leibniz's modal metaphysics is based on a general theory of relations and the relational properties definable in terms of such a theory. (Readers unfamiliar with the theory of abstract objects will find a very brief discussion of its theory of relations in Appendix I.) The Leibnizian justification for relying on a general theory of relations comes from the July 14, 1686 letter to Arnauld:

> I say that the concept of an individual substance involves all its changes and all its relations, even those which are commonly called extrinsic, that is to say which pertain to it only by virtue of the general interconnection of things, and in so far as it expresses the whole universe in its own way...[68]

The question here is, what did Leibniz mean by 'involves' if not 'contains'? Even if Leibniz takes relations to be abstract, ideal entities, it would seem that they play role in the background of his metaphysics of modality.[69] In any case, the appeal to a general theory of relations facilitates our reconstruction of a Leibnizian metaphysics.

§9: Containment and the Fundamental Theorem

In this section, we continue to use the variable u to range over ordinary objects. In what follows, we analyze the Leibnizian notion of concept con-

[68]This is the translation in Ishiguro [1972], p. 99. One may wish to check the source G.ii 56, for it might be thought that Leibniz didn't intend to mention 'relations' in this passage, given that Morris and Parkinson translate the classical French word 'denominations' as the English 'denominations' rather than as 'relations'. In PW 62, they translate the passage from G.ii 56 as follows:

> ...I say that the individual substance includes all its events and all its denominations, even those which are commonly called extrinsic (that is, they belong to it only by virtue of the general interconnexion of things and because it expresses the whole universe in its own way), ...

However, Morris and Parkinson, in note 'g' placed at the word 'extrinsic' in the above passage, indicate that relations are indeed intended. See PW 248, note g.

[69]For further confirmation, see D'Agostino [1976].

tainment using the notion of containment that we defined at the beginning of Section 5. That is:

The concept x contains the concept z

is analyzed as:

$x \succeq z$

Now Leibniz analyses:

Alexander is a king

in the following terms:

The concept of Alexander contains the concept king

On our theory of concepts, this Leibnizian analysis becomes:

$c_a \succeq c_K$

So truth is a matter of the subject concept containing the predicate concept.

The more modern analysis of 'Alexander is a king', of course, is: Alexander exemplifies (the property of) being king; i.e., Ka. But consider the following fact:

Theorem 38: $Fu \equiv c_u \succeq c_F$

(The proof is in Appendix II.) This establishes that our Leibnizian analysis of "The concept of Alexander contains the concept king" as $c_a \succeq c_K$ is equivalent to the more modern analysis of 'Alexander is a king' as Ka.

These results present us with a defense of a distinction that Leibniz draws in Article 13 of the *Discourse on Metaphysics* and in a letter to Hessen-Rheinfels. Recall that Arnauld charged that Leibniz's analyses turn contingent truths into necessary truths. Arnauld worried that the contingent statement that 'Alexander is a king' is analyzed in terms of the necessary truth 'the concept of Alexander contains the concept king'. Leibniz defends himself against Arnauld's charges by distinguishing absolute necessity from 'hypothetical necessity'.[70] Although Leibniz explains

[70]In Article 13, Leibniz says:

> To give a satisfactory answer to it, I assert that connexion or sequence is of two kinds. The one is absolutely necessary, whose contrary implies a

this distinction in terms of the difference between what God is free to do absolutely and what God is contrained to do given previous choices, there is a distinctive way of explaining 'hypothetical necessities' in the present framework.

The explanation begins by pointing to the fact that any *proof* of $c_a \succeq c_K$ would have to rest on the contingent premise that Alexander is a king (i.e., it rests on Ka). Just consider the fact that if we are to cite Theorem 38 to prove that $c_a \succeq c_K$, we must assume the contingent premise Ka. Moreover, it is important to observe that (it is a theorem that) if $x \succeq y$, then $\Box(x \succeq y)$, for the logic of encoding (Appendix I) guarantees that encoded properties are necessarily encoded. So, as an instance of this result, we know: if $c_a \succeq c_K$, then $\Box(c_a \succeq c_K)$. But since a proof of the antecedent would rely on a contingent fact, it follows that the proof of the consequence would as well. So we have a case where the proof of a necessary truth depends on a contingent assumption. It may be that this is the way to understand 'hypothetical necessities'.[71]

(Some reader might find it important to observe here that Theorem 38 is a logical truth that is not necessary. To see why, reconsider the Lemma preceding Theorem 28 (i.e., $c_uF \equiv Fu$) more carefully. Note that the Lemma itself is a an example of a logical truth that is not necessary. It is a logical truth because it is derivable from our logical axioms and rules alone. But it is not metaphysically necessary. To see why, suppose that Ka is true (at the actual world) and consider a world, say w_1, where Alexander is not a king. At w_1, Ka is false. But note that at such a world, c_aK is true. That's because c_a, the object that (at the actual world) encodes all and only the properties that Alexander actually exemplifies, encodes its properties rigidly and so encodes K even at w_1.[72]

> contradiction; this kind of deduction holds in the case of eternal truths, such as those of geometry. The other is only necessary by hypothesis (*ex hypothesi*), and so to speak by accident; it is contingent in itself, since its contrary does not imply a contradiction.

This is the translation in PW 23-24. The source is G.iv 437. In a letter to Hessen-Rheinfels, Leibniz says:

> These last words contain the proof of the consequence; but it is very clear that they confuse *necessitatem ex hypothesi* with absolute necessity.

This is the translation in PW 49. The source is G.ii 18.

[71] This analysis corrects an error in Zalta [1983], where it was claimed (p. 90) without qualification that c_aK (which was formalized in [1983] as '$\bar{a}K$') is a necessary truth.

[72] The description that serves to define c_a is a rigid designator and so still denotes at w_1 the object x that satisfies '$\forall F(xF \equiv Fa)$' at the actual world.

So w_1 is a world where the left condition (c_uF) of the Lemma is true but the right condition (Fu) is false. Hence the Lemma is not a necessary truth. Similarly, Theorem 38 is a logical truth that is not necessary. Even though it is *logically* required (provable) that Ka is equivalent to $c_a \succeq c_K$, this equivalence is not necessary in the metaphysical sense. For a fuller discussion of such logical truths that are not necessary, see Zalta [1988b].)

We turn next to the proof of the fundamental theorem of Leibniz's metaphysics of individual concepts. This theorem can be stated in one of two equivalent ways: (1) if F is a property that ordinary object u exemplifies but might not have exemplified, then (i) the concept of u contains the concept F and (ii) there is a (complete individual concept which is a) counterpart of the concept of u which doesn't contain the concept F and which appears at some other possible world, or (2) if F is a property that ordinary object u doesn't exemplify but might have exemplified, then (a) the concept of u doesn't contain the concept F and (b) there is a (complete individual concept which is a) counterpart of the concept of u which both contains the concept F and which appears at some other possible world. In order to prove this theorem, we first draw attention to two consequences of our work so far. The first is that it is an immediate consequence of Theorem 38 and the Lemma to Theorem 28 that individual concept c_u encodes a property F iff c_u contains c_F:

Theorem 39a: $c_uF \equiv c_u \succeq c_F$

However, a somewhat more general result is now provable, namely, for any individual concept x, x encodes a property F iff x contains the concept c_F:

Theorem 39b: $\mathit{IndividualConcept}(x) \rightarrow \forall F(xF \equiv x \succeq c_F)$

(The proof is in Appendix II.)

With these two facts in hand, we proceed to establish the fundamental theorem of Leibniz's metaphysics of concepts. Consider a simple claim that expresses a contingent fact, for example, that Alexander is a king but might not have been:

$$Ka \,\&\, \Diamond\neg Ka$$

Now from the first conjunct, we know (by Theorem 38) that the concept of Alexander (c_a) contains the concept king:

$c_a \succeq c_K$

Moreover, it is a theorem of world theory (Appendix III) that the second conjunct $\diamond\neg Ka$ is equivalent to:

$\exists w(\models_w \neg Ka)$

Since there is some world where Alexander is not a king, pick one, say w_1. We can easily establish (from work in the previous section) that the concept of Alexander-at-w_1 ($c_a^{w_1}$) exists and is an individual concept. We also know that $c_a^{w_1}$ is a counterpart of c_a (in the sense defined in the previous section). We furthermore know that $c_a^{w_1}$ appears at w_1, by the definition of appearance and the fact (provable from definitions) that Alexander realizes $c_a^{w_1}$ at w_1. Moreover, we know (as a provable consequence of our world theory) that w_1 is not the actual world w_α, since the proposition that Alexander is a king is true at w_α and not at w_1. We also know that since (by definition) $c_a^{w_1}$ encodes exactly the properties that Alexander exemplifies at w_1, it follows that $c_a^{w_1}$ fails to encode the property of being a king. From this, it follows (by Theorem 39b and the fact that $c_a^{w_1}$ is an individual concept) that $c_a^{w_1}$ fails to contain the concept king (c_K). So, assembling what we know and generalizing on w_1, the following is a consequence of the fact that Alexander is a king but might not have been:

> The (complete, individual) concept of Alexander contains the concept king, but there is a (complete, individual concept which is a) counterpart of the concept of Alexander which doesn't contain the concept king and which appears at some other possible world.

If we generalize on Alexander and the property of being a king, then we have, in essence, just established the following fundamental theorem of Leibniz's metaphysics of individual concepts:

Fundamental Theorem of Leibniz's Metaphysics of Concepts:

Theorem 40a: $(Fu \,\&\, \diamond\neg Fu) \rightarrow [c_u \succeq c_F \,\&$
$\exists x(\mathit{Counterparts}(x, c_u) \,\&\, x \not\succeq c_F \,\&\, \exists w(w \neq w_\alpha \,\&\, \mathit{Appears}(x, w)))]$

Theorem 40b: $(\neg Fu \,\&\, \diamond Fu) \rightarrow [c_u \not\succeq c_F \,\&$
$\exists x(\mathit{Counterparts}(x, c_u) \,\&\, x \succeq c_F \,\&\, \exists w(w \neq w_\alpha \,\&\, \mathit{Appears}(x, w)))]$

Thus, from a simple, pre-theoretic modal statement which asserts that a certain fact about an ordinary object is contingent, we can *prove* the facts

that are supposed to obtain in a Leibnizian metaphysics of individual concepts. Indeed, these are the very facts that are stipulated (in set-theoretic terms) as truth conditions for such statements in the possible-world semantics developed by other commentators on Leibniz.[73] It should now be clear why we have called this the fundamental theorem of Leibniz's metaphysics of individual concepts—the particular systems of set-theoretic, possible world semantics constructed by other commentators to model Leibniz's metaphysics were *designed expressly* to assign the very truth conditions for statements of contingency that are derived in Theorem 40. Taken together, Theorem 40 and the other theorems proved in this section and in the previous section establish that our metaphysics of individual concepts has the distinguishing features of Leibniz's metaphysics of individual concepts. These results also show that our system of (syntactically second-order) quantified modal logic with encoding and comprehension over abstract objects offers a genuine alternative to set theory and possible world semantics, at least insofar as we are interested in formulating an overarching system that unifies Leibniz's logic and metaphysics of concepts.

Now although any discussion following the proof of the fundamental theorem is bound to be anti-climactic, it may still be of interest to indicate briefly that Leibniz may have anticipated not only Montague's [1974] subject-predicate analysis of basic sentences of natural language, but also the idea of a generalized quantifier. Recall that Montague was able to give a uniform subject-predicate analysis of a fundamental class of English sentences by treating such noun phrases as 'Adam' and 'every person' as sets of properties. He supposed that the proper name 'Adam' denotes the set of all and only the properties F that the individual Adam exemplifies and supposed that the noun phrase 'every person' denoted the set of all and only the properties F that every person exemplifies. Then, English sentences such as 'Adam is happy' and 'Every person is happy' could be given a subject-predicate analysis: such sentences are true iff the property denoted by the predicate 'is happy' is a member of the set of properties denoted by the subject term.

[73]See, in particular, Mondadori [1973] (p. 250) and Fitch [1979] (pp. 300-303). Theorem 40 also seems to capture the main idea developed in Mates [1968], though for Mates, you have to drop the clause concerning 'counterparts' (since he doesn't use them). There are other minor differences as well.

With this in mind, it seems worthwhile to point out that quantified sentences such as 'Every person is happy' and 'Some person is happy' can be given an analysis in terms of Leibniz's containment theory of truth. The similarity to the Montagovian analysis should then be obvious. Let us define 'the concept *every* G' ('$c_{\forall G}$') and 'the concept *some* G' ('$c_{\exists G}$') as follows:

$$c_{\forall G} =_{df} \imath z(Concept(z) \,\&\, \forall F(zF \equiv \forall x(Gx \rightarrow Fx)))$$

$$c_{\exists G} =_{df} \imath z(Concept(z) \,\&\, \forall F(zF \equiv \exists x(Gx \,\&\, Fx)))$$

Then, if we let G be the property of being a person ('P'), we may analyze:

Every person is happy

Some person is happy

respectively, as:

$$c_{\forall P} \succeq c_H$$

$$c_{\exists P} \succeq c_H$$

Notice that if there are at least two distinct ordinary things exemplifying G, then neither $c_{\forall G}$ nor $c_{\exists G}$ are individual concepts.

Conclusion

The results described in the previous section are somewhat different from the ones we developed in [1983]. In that earlier work, certain analyses were somewhat simpler. The Leibnizian concept G was identified with the property G and the Leibnizian claim that 'the individual concept of u contains the concept G' was analyzed more simply as 'c_uG'. But under that scheme, concept containment became a connection between two different kinds of things (namely, A-objects and properties). Clearly, though, Leibnizian concept containment and concept inclusion are suppose to connect things of a single kind, namely, concepts. That idea is preserved in the way we have analyzed containment and inclusion in the present work—the notions of containment and inclusion that we use both in the logic and metaphysics of concepts are defined as conditions that apply to concepts (i.e., A-objects). Of course there is a simple thesis that connects the present analysis with the earlier analysis, namely, Theorem 39a. This thesis shows that the former analysis is equivalent to the present one—our Leibnizian analysis of 'Alexander is a king' in [1983] (p. 90) as c_aK is equivalent to our present analysis of this claim as: $c_a \succeq c_K$.

The consistency of the theory of abstract objects immediately establishes the consistency of our Leibnizian logic of concepts, metaphysics of individual concepts, and containment theory of truth. Moreover, the truth of the theory of (Leibnizian) concepts, and its consistency with our naturalized world view, depends on the truth of the principles of the theory of abstract objects and their consistency with naturalism. This latter theory is at least consistent with naturalism, or so we have recently argued.[74] We shall not argue here that the theory itself is true, though the fact that it validates a Leibnizian metaphysics should constitute further evidence in favor of the theory.

Appendix I: A Sketch of the Theory

The metaphysical theory of abstract objects is a stated in terms of a language containing two basic forms of predication. Encoding predication contrasts with the traditional *exemplification* mode of predication. That is, in addition to the traditional form of predication 'x exemplifies F' ('Fx'), we also have the new form of predication 'x encodes F' ('xF'). Whereas ordinary objects only exemplify properties, abstract objects both exemplify and encode properties. And whereas ordinary objects are identified whenever they necessarily exemplify the same properties, abstract objects are identified whenever they necessarily encode the same properties.

To state the theory of abstract objects more precisely, we utilize two primitive predicates: a 1-place predicate $E!$ (to denote the property of being concrete), and a 2-place predicate $=_E$ (to denote the relation of identity on ordinary objects). We then say that x *is ordinary* ('$O!x$') iff it is possible that x is concrete. We also say that x *is abstract* ('$A!x$') iff x is not the kind of object that could be concrete. In formal terms:

$$O!x \;=_{df}\; \Diamond E!x$$

$$A!x \;=_{df}\; \neg\Diamond E!x$$

Now we may state the six basic principles (axioms and definitions) of our theory in terms of these notions:

[74] See Linsky and Zalta [1995].

Principle 1: $O!x \rightarrow \Box\neg\exists F xF$

Principle 2: $x{=}_E y \ \equiv\ O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)$

Principle 3: $\exists x(A!x \,\&\, \forall F(xF \equiv \phi))$, where ϕ has no free xs

Principle 4: $x{=}_A y \ =_{df}\ A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)$

Principle 5: $x{=}y \ =_{df}\ x{=}_E y \vee x{=}_A y$

Principle 6: $x{=}y \equiv (\phi(x,x) \equiv \phi(x,y))$, where $\phi(x,y)$ is the result of substituting y for one or more occurrences of x in $\phi(x,x)$, provided that y is substitutable (in the usual sense) for x at each such occurrence.

Principle 1 tells us that ordinary objects necessarily fail to encode properties, and *Principle 2* axiomatizes the notion of identity that pertains to ordinary objects: ordinary objects x and y are identical$_E$ iff necessarily, they exemplify the same properties. *Principle 3* is the comprehension principle for abstract objects ('A-objects'). For any condition ϕ on properties F, this principle has an instance that asserts that there is an abstract object that encodes just the properties satisfying ϕ. *Principle 4* defines a notion of identity for abstract objects: abstract objects x and y are identical$_A$ iff necessarily, they encode the same properties. *Principle 5* just defines a notion of identity for any objects x and y in the domain of objects (since every object is either ordinary or abstract, this constitutes a completely general definition of object identity). Finally, *Principle 6* asserts that identical objects may be substituted for one another in any context ϕ.[75]

The comprehension principle asserts the existence of a wide variety of abstract objects, some of which are complete with respect to the properties they encode, while others are incomplete in this respect. For example, one instance of comprehension asserts there exists an abstract object that encodes just the properties Clinton exemplifies. This object is complete because Clinton either exemplifies F or exemplifies the negation of F, for every property F. Another instance of comprehension asserts that there

[75]I should mention that this principle can be formulated in more general terms so that identical properties F and G can also be substituted for one another in any context. If we suppose that α and β are either both object variables or both relation variables, then our system includes the following more general formulation of the substitution principle: $\alpha{=}\beta \equiv (\phi(\alpha,\alpha) \equiv \phi(\alpha,\beta))$.

is an abstract object that encodes just the two properties: *being blue* and *being round*. This object is incomplete because for every *other* property F, it encodes neither F nor the negation of F. But though abstract objects may be partial with respect to their encoded properties, they are all complete with respect to the properties they *exemplify*. In other words, the following principle of classical logic is preserved: for every object x and property F, either x exemplifies F or x exemplifies the negation of F (i.e, $\forall F\forall x(Fx \vee \bar{F}x)$).[76]

Abstract objects are simply different in kind from ordinary objects: the latter are not the kind of thing that could encode properties; the former are not the kind of thing that could be concrete. Moreover, abstract objects *necessarily fail to exemplify* the properties of ordinary objects—they necessarily fail to have a shape, they necessarily fail to have a texture, they necessarily fail to reflect light, etc. Consequently, by the classical laws of complex properties, abstract objects necessarily exemplify the negations of these properties. But the properties abstract objects encode are more important than the properties they necessarily exemplify, since the former are the ones by which we individuate them.

The six principles listed above are cast within the framework of a classical modal (S5 with Barcan formulas) second-order predicate logic.[77] Moreover, this logic is extended by the logical axiom for encoding and the axioms that govern the two kinds of complex terms: (rigid) definite descriptions of the form $\imath x\phi$ and λ-predicates of the form $[\lambda y_1 \ldots y_n\ \phi]$. The logical axiom for encoding is:

Logic of Encoding: $\Diamond xF \rightarrow \Box xF$

Intuitively, this says that if an object encodes a property at any possible world, it encodes that property at every world; thus facts about encoded

[76]In this principle, $\bar{F}$ is still defined as $[\lambda y\ \neg Fy]$. This λ-expression is governed by the usual principle (see below in the text):

$$[\lambda y\ \neg Fy]x \equiv \neg Fx$$

Note that encoding satisfies classical bivalence: $\forall F\forall x(xF \vee \neg xF)$. But the incompleteness of abstract objects is captured by the fact that the following is not in general true: $xF \vee x\bar{F}$.

[77]By including the Barcan formulas, this quantified modal logic is the simplest one available—it may interpreted in such a way that the quantifiers $\forall x$ and $\forall F$ range over fixed domains, respectively. See Linsky and Zalta [1994], in which it is shown that this simplest quantified modal logic, with the first and second order Barcan formulas, is consistent with actualism.

properties are not relativized to any circumstance. This axiom and the definition of identity for abstract objects jointly ensure that the properties encoded by an abstract object are in some sense intrinsic to it.

We conclude this summary of the system by describing the axioms governing the complex terms and some simple consequences of the foregoing. A standard free logic governs the definite descriptions, along with a single axiom that captures the Russellian analysis: an atomic or (defined) identity formula ψ containing $\imath x\phi$ is true iff there is something y such that: (a) y satisfies ϕ, (b) anything satisfying ϕ is identical to y, and (c) y satisfies ψ. In formal terms, this becomes:

> *Descriptions*: $\psi_y^{\imath x\phi} \equiv \exists x(\phi \,\&\, \forall z(\phi_x^z \rightarrow z\!=\!x) \,\&\, \psi_y^x)$, for any atomic or identity formula $\psi(y)$ in which y is free.

To keep the system simple, these definite descriptions are construed as rigid designators, and so this axiom is a classic example of a logical truth that is *contingent*. Thus, the Rule of Necessitation may not be applied to any line of a proof that depends on this axiom.[78]

The final element of the logic concerns complex relations. They are denoted in the system by terms of the form $[\lambda y_1 \ldots y_n\, \phi]$ meeting the condition that ϕ have no encoding subformulas. These λ-predicates behave classically:[79]

> λ-*Equivalence*: $[\lambda y_1 \ldots y_n\, \phi]x_1 \ldots x_n \equiv \phi_{y_1,\ldots,y_n}^{x_1,\ldots,x_n}$

In less formal terms: objects $x_1, \ldots, x_n$ exemplify the relation $[\lambda y_1 \ldots y_n \phi]$ iff $x_1, \ldots, x_n$ satisfy ϕ. A comprehension schema for relations is a simple consequence of λ-Equivalence.

In what follows, it proves useful to appeal to some simple consequences of our system. Let us define 'there is a unique x such that ϕ' ('$\exists!x\phi$') in the usual way:

$$\exists!x\phi \;\; =_{df} \;\; \exists x\forall y(\phi_x^y \equiv y\!=\!x)$$

[78] Further examples and discussion of such logical truths that are not necessary may be found in Zalta [1988b].

[79] I should mention here that this logic of relations is supplemented by precise identity principles that permit necessarily equivalent properties, relations, and propositions to be distinct. These principles are expressed in terms of a basic definition of property identity: $F\!=\!G \equiv \Box\forall x(xF \equiv xG)$. For a more detailed explanation of this principle and the definitions of relation identity and proposition identity, the reader may consult the cited works on the theory of abstract objects.

Then we may appeal to *Principles 3* and *5* to prove the following more exact comprehension principle for A-objects:

> *Principle 3′*: $\exists!x(A!x \,\&\, \forall F(xF \equiv \phi))$, where ϕ has no free xs

The proof of this principle is simply this: by *Principle 3*, we know there is an A-object that encodes just the properties satisfying ϕ; but there couldn't be two distinct A-objects that encode exactly the properties satisfying ϕ, since distinct A-objects have to differ by at least one encoded property.

Consequently, for each formula ϕ that can be used to produce an instance of *Principle 3′*, the following is true:

> *Corollary to Principle 3′*: $\exists y\, y\!=\!\imath x(A!x \,\&\, \forall F(xF \equiv \phi))$

We are therefore assured that the following description is always well-defined:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \phi))$$

Such descriptions will be used frequently in what follows to define various Leibnizian concepts. Indeed, they are governed by a simple theorem that plays a role in the proof of most of the theorems which follow:

> *A-Descriptions*: $\imath x(A!x \,\&\, \forall F(xF \equiv \phi))G \equiv \phi_F^G$

In other words, the A-object that encodes just the properties satisfying ϕ encodes property G iff G satisfies ϕ. This theorem is easily derivable from the *Descriptions* axiom described above.

Appendix II: Proofs of Selected Theorems

The proofs in what follows often appeal to the Principles described in Appendix I and to the theorems of world theory discussed in Appendix III.

• **Proof of Theorem 4**: We prove the concepts in question encode the same properties. ($\leftarrow$) Assume $c_G \oplus c_H P$. We need to show:

$$\imath x\forall F(xF \equiv G\!\Rightarrow\!F \;\vee\; H\!\Rightarrow\!F)P$$

So by *A-Descriptions*, we must show: $G\!\Rightarrow\!P \;\vee\; H\!\Rightarrow\!P$. By hypothesis, we know:

$$\imath x\forall F(xF \equiv c_G F \vee c_H F)P$$

But by *A-Descriptions*, it follows that $c_G P \vee c_H P$. Now, for disjunctive syllogism, suppose $c_G P$. Then by definition of of c_G, it follows that:

$$\imath x\forall F(xF \equiv G\Rightarrow F)P$$

So by *A-Descriptions*, we know $G\Rightarrow P$. And by similar reasoning, if $c_H P$, then $H\Rightarrow P$. So by disjunctive syllogism, it follows that $G\Rightarrow P \vee H\Rightarrow P$, which is what we had to show.

($\rightarrow$) Assume:

$$\imath x\forall F(xF \equiv G\Rightarrow F \vee H\Rightarrow F)P$$

We want to show: $c_G\oplus c_H P$. By the definition of real sum, we have to show:

$$\imath x\forall F(xF \equiv c_G F \vee c_H F)P$$

By *A-Descriptions*, we therefore have to show that $c_G P \vee c_H P$. By applying *A-Descriptions* to our hypothesis, though, we know:

$$G\Rightarrow P \vee H\Rightarrow P$$

So, for disjunctive syllogism, suppose $G\Rightarrow P$. Then, by *A-Descriptions*:

$$\imath x\forall F(xF \equiv G\Rightarrow F)P$$

That is, by definition of the concept G, we know: $c_G P$. By similar reasoning, if $H\Rightarrow P$, then $c_H P$. So by our disjunctive syllogism, it follows that $c_G P \vee c_H P$, which is what we had to show. $\bowtie$

• **Proof of Theorem 7**: ($\rightarrow$) Assume $(x\oplus y)\oplus zP$. Then, by definition of $\oplus$, we have:

$$\imath w\forall F(wF \equiv x\oplus yF \vee zF)P$$

This, by *A-Descriptions*, entails:

$$x\oplus yP \vee zP$$

Expanding the left disjunct by the definition of $\oplus$, we have:

$$\imath w\forall F(wF \equiv xF \vee yF)P \vee zP$$

And reducing the left disjunct by applying *A-Descriptions*, we have:

$$(xP \vee yP) \vee zP$$

This, of course, is equivalent to:

$$xP \vee (yP \vee zP)$$

Applying *A-Descriptions* in the reverse direction to this line, we obtain:

$$xP \vee \imath w\forall F(wF \equiv yF \vee zF)P$$

By definition of $\oplus$, this is equal to:

$$xP \vee y\oplus zP$$

And by another application of *A-Descriptions*, this becomes:

$$\imath w\forall F(wF \equiv xF \vee y\oplus zF)P$$

At last, by definition of $\oplus$, we reach:

$$x\oplus(y\oplus z)P$$

So if $(x\oplus y)\oplus z$ encodes P, so does $x\oplus(y\oplus z)$. ($\leftarrow$) To show that $(x\oplus y)\oplus z$ encodes P given that $x\oplus(y\oplus z)$ encodes P, reverse the reasoning. $\bowtie$

• **Proof of Theorem 12i**: Suppose $x \preceq y$ and $x \neq y$. To show that $y \not\preceq x$, we need to find a property F such that $yF \,\&\, \neg xF$. But, if $x\neq y$, either there is a property x encodes y doesn't, or there is a property y encodes that x doesn't. But, since $x \preceq y$, it must be the latter. $\bowtie$

• **Proof of Theorem 15i**: Assume $\forall z(z \preceq x \equiv z \preceq y)$ to show that $x=y$, i.e., that for an arbitrary property P, that $xP \equiv yP$. ($\rightarrow$) Assume xP, and for reductio, assume $\neg yP$. To reach a contradiction, simply consider the concept c_P. $c_P \preceq x$ and $\neg(c_p \preceq y)$, contrary to hypothesis. ($\leftarrow$) By analogous reasoning. $\bowtie$

• **Proof of Theorem 16i**: Assume that $\forall z(x \preceq z \equiv y \preceq z)$ to show that $x=y$, i.e., that for an arbitrary property P, that $xP \equiv yP$. ($\rightarrow$) Assume xP, and for reductio, assume $\neg yP$. To reach a contradiction, consider the fact that y must satisfy our initial hypothesis: $x \preceq y \equiv y \preceq y$. But we know that $y \preceq y$ by the reflexivity of inclusion. So $x \preceq y$, contradicting the fact that $xP \,\&\, \neg yP$. ($\leftarrow$) By analogous reasoning. $\bowtie$

• **Proof of Theorem 25**: ($\Rightarrow$) Assume $x \preceq y$.

a) Suppose $x = y$. By the idempotency of $\oplus$, $x\oplus x = x$, in which case,

$x \oplus x = y$. So, we automatically have $\exists z(x \oplus z = y)$.

b) Suppose $x \neq y$. Then since $x \preceq y$, we know there must be some properties encoded by y which are not encoded by x. Consider, then, the object that encodes just such properties; i.e., consider:

$$\imath w \forall F(wF \equiv yF \;\&\; \neg xF)$$

Call this object 'w' for short (we know such an object exists by the abstraction schema for A-objects). We need only establish that $x \oplus w = y$, i.e., that $x \oplus w$ encodes the same properties as y. ($\rightarrow$) Assume $x \oplus wP$ (to show: yP). By definition of $\oplus$ and *A-Descriptions*, it follows that $xP \vee wP$. If xP, then by the fact that $x \preceq y$, it follows that yP. On the other hand, if wP, then by definition of w, it follows that $yP \;\&\; \neg xP$. So in either case, we have yP. ($\leftarrow$) Assume yP (to show $x \oplus wP$). The alternatives are xP or $\neg xP$. If xP, then $xP \vee wP$, so by *A-Descriptions*:

$$\imath z \forall F(zF \equiv xF \vee wF)P$$

So, by the definition of $\oplus$, we have $x \oplus wP$. Alternatively, if $\neg xP$, then we have $yP \;\&\; \neg xP$. So by definition of w, wP, and by familiar reasoning, it follows that $x \oplus wP$. Combining both directions of our biconditional, we have established that $x \oplus wP$ iff yP, for an arbitrary P. So $x \oplus w = y$, and we therefore have $\exists z(x \oplus z = y)$.

($\Leftarrow$) Assume $\exists z(x \oplus z = y)$. Call such an object '$w$'. To show $x \preceq y$, assume xP (to show yP). Then, $xP \vee wP$, which by *A-Descriptions* and the definition of $\oplus$, entails that $x \oplus wP$. But by hypothesis, $x \oplus w = y$. So yP. $\bowtie$

• **Proof of Theorem 26**: ($\Rightarrow$) Assume $x \preceq y$. So $\forall F(xF \rightarrow yF)$. To show that $x \oplus y = y$, we need to show that $x \oplus y$ encodes a property P iff y does. ($\rightarrow$) So assume $x \oplus y$P. Then, by definition,

$$\imath z \forall F(zF \equiv xF \vee yF)P$$

By *A-Descriptions*, it then follows that $xP \vee yP$. But if xP, then by the fact that $x \preceq y$, it follows that yP. So both disjuncts lead us to conclude yP. ($\leftarrow$) Assume yP. Then $xP \vee yP$. So by *A-Descriptions*,

$$\imath z \forall F(zF \equiv xF \vee yF)P$$

In other words, $x \oplus yP$.

($\Leftarrow$) Assume that $x \oplus y = y$.[80] To show that $x \preceq y$, assume, for an arbitrary property P, that xP (to show: yP). Then $xP \vee yP$. By *A-Descriptions*, it follows that:

$$\imath z \forall F(zF \equiv xF \vee yF)P$$

By definition of $\oplus$, it follows that $x \oplus yP$. But given that $x \oplus y = y$, it follows that yP. $\bowtie$

• **Proof of Theorem 7 of World Theory** (See Appendix III): The proof is simplified by citing the following theorem of object theory:[81]

Consequence of λ-Equivalence: $\Box([\lambda y\, p]x \equiv p)$

Now to prove the theorem, consider an arbitrary proposition r, world w, and object a. Then, by the *Consequence of λ-Equivalence*, $\Box([\lambda y\, r]a \equiv r)$. Now, for the left-right direction, suppose $\models_w r$. Then, since $\Box(r \rightarrow [\lambda y\, r]a)$, it follows by Theorem 4 of World Theory (Appendix III), that $\models_w [\lambda y\, r]a$. Analogous reasoning establishes the theorem in the right-to-left direction. $\bowtie$

• **Proof of Theorem 28**: Consider an arbitrary concept of an individual, say, c_a, for some ordinary individual a. To show that c_a is an individual concept, we show that c_a appears at the actual world w_α.[82] Pick an arbitrary property, say P. By the Lemma just stated in the text, $c_aP \equiv Pa$. But, it is a consequence of the definition of the actual world w_α that $\models_{w_\alpha} Pa \equiv Pa$. So by properties of the biconditional, $\models_{w_\alpha} Pa \equiv c_aP$. Since P was arbitrary, it follows that $RealizesAt(a, c_a, w_\alpha)$. Generalizing on a, it follows that c_a appears at w_α. $\bowtie$

• **Proof of Theorem 30**: Suppose x appears at w. So some ordinary object, say b, realizes x at w; i.e., $\forall F(\models_w Fb \equiv xF)$. We want to show, for an arbitrary proposition q, that $x[\lambda y\, q]$ iff $w[\lambda y\, q]$. ($\rightarrow$) Assume $x[\lambda y\, q]$.

[80] If we allow ourselves an appeal to Theorem 25, we are done. For it follows from this that $\exists z(x \oplus z = y)$, which by Theorem 25, yields immediately that $x \preceq y$.

[81] The logical axiom *λ-Equivalence* is:

$$[\lambda y_1 \ldots y_n\, \phi]x_1 \ldots x_n \equiv \phi^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$$

The consequence of this logical axiom just cited in the text is derived by: letting $n = 1$, letting ϕ be the propositional variable 'p', and applying the Rule of Necessitation to the result. Notice that the variable 'y' doesn't appear free in the simple variable 'p', and so the result of substituting the variable 'x' for 'y' in ϕ (i.e, ϕ^x_y) is just ϕ itself.

[82] The definition of w_α is: $\imath z \forall F(zF \equiv \exists p(p \;\&\; F = [\lambda y\, p]))$. It follows that a proposition p is true in w_α iff p is true; i.e., that $\models_{w_\alpha} p \equiv p$.

So, by definition of b, $\models_w [\lambda y\, q]b$. And by Theorem 7 of World Theory (Appendix III), it follows that $\models_w q$, i.e., $w[\lambda y\, q]$. (←) Reverse the reasoning. ⋈

• **Proof of Theorem 31**: Assume c is an individual concept. So it appears at some world, say, w_1. For reductio, assume that c also appears at w_2, $w_2 \neq w_1$. Since the worlds are distinct, there must be some proposition true at one but not the other (by the definition of A-object identity and the fact that worlds only encode properties of the form $[\lambda y\, p]$). So without loss of generality, assume that $\models_{w_1} p$ and $\not\models_{w_2} p$. Since worlds are maximal (by Theorem 2 of World Theory) (Appendix III), it follows that $\models_{w_2} \neg p$. But, by the previous theorem, c mirrors w_1, since it appears there. So since $\models_{w_1} p$, we know $c[\lambda y\, p]$. But c also mirrors w_2, since it appears there as well. So, from our last fact, it follows that $\models_{w_2} p$. This contradicts the possibility of w_2 (Theorem 3 of World Theory). ⋈

• **Proof of Theorem 36**: Suppose c is a individual concept. Then c appears at some world, say w_1, and some ordinary object, say b, realizes c at w_1. Consider an arbitrary property P. By a theorem of logic: $\Box\forall y(Py \vee \neg Py)$. Given $\Box\forall y(\bar{P}y \equiv \neg Py)$, we know that $\Box\forall y(Py \vee \bar{P}y)$. So by Theorem 5 of World Theory (Appendix III), $\forall w\forall y(\models_w (Py \vee \bar{P}y))$. Instantiating to w_1 and b, we therefore know: $\models_{w_1} (Pb \vee \bar{P}b)$. Now if $Pb \vee \bar{P}b$ is true at w_1, it is a simple exercise to show that either $\models_{w_1} Pb \vee \models_{w_1} \bar{P}b$. But since b realizes c at w_1 (i.e., c encodes all and only the properties that b exemplifies at w_1), it follows that $cP \vee \neg c\bar{P}$. And since P was arbitrary, c is complete. ⋈

• **Proof of Theorem 37**: Suppose c is an individual concept. To show that c is singular, pick an arbitrary world, say w_1, and assume that there are ordinary objects, say a and b, which both realize c at w_1. For reductio, suppose that $b \neq a$. Then since a and b are distinct ordinary objects, we know that both $a =_E a$ and $b =_E b$.[83] Now it is a fact of object theory that:[84]

$$u =_E u \rightarrow \Box(u =_E u)$$

[83]It is an immediate consequence of the principle defining $=_E$ described at the outset of the paper that, for any ordinary object u, $u =_E u$.

[84]If $u =_E v$, then given that $O!u$ is defined as $\Diamond E!u$, one can, in S5, derive the necessitation of each conjunct in the definition of $=_E$ (*Principle 2* in Appendix I). So, since all three conjuncts are necessary, the conjunction is necessary. Thus, $\Box(u =_E v)$.

It therefore follows that $\Box(a =_E a)$ and $\Box(b =_E b)$. Now the following is a logical theorem of object theory:[85]

Consequence of λ-Equivalence: $\Box([\lambda y\, y =_E u]u \equiv u =_E u)$

So, in particular, $\Box([\lambda y\, y =_E a]a \equiv a =_E a)$, and a similar claim holds for b. So by a simple principle of modal logic, we know both $\Box[\lambda y\, y =_E a]a$ and $\Box[\lambda y\, y =_E b]b$. So by Theorem 5 of World Theory (Appendix III), it follows that $\models_{w_1} [\lambda y\, y =_E a]a$ and $\models_{w_1} [\lambda y\, y =_E b]b$. But, if the former, then since a realizes c at w_1, we know that $c[\lambda y\, y =_E a]$. But since b realizes c at w_1, $\models_{w_1} [\lambda y\, y =_E a]b$. Now by Theorem 5 of World Theory and the above Consequence of λ-Equivalence, we also know that $\models_{w_1} ([\lambda y y =_E a]b \equiv b =_E a)$. So since worlds are modally closed (Theorem 4 of World Theory), it follows that $\models_{w_1} b =_E a$. So, $\Diamond b =_E a$, and by facts governing $=_E$, it follows that $b =_E a$. But, by the definition of identity (Principle 5, Appendix I), if $b =_E a$, then $b = a$, which contradicts our hypothesis. ⋈

• **Proof of Theorem 38**: (→) Assume Qa, where a is an ordinary object and Q an arbitrary property. To show $c_a \succeq c_Q$, we need to show $\forall F(c_Q F \rightarrow c_a F)$. So assume $c_Q P$, for an arbitrary property P. Then by definition of c_Q and *A-Descriptions*, we know $Q \Rightarrow P$. So from Qa and $Q \Rightarrow P$ it follows that Pa. So, by the definition of c_a and *A-Descriptions*, it follows that $c_a P$.

(←) Assume $c_a \succeq c_Q$ (to show: Qa). But by *A-Descriptions* and the fact that $Q \Rightarrow Q$, it follows that

$$\imath z\forall F(zF \equiv Q \Rightarrow F)Q$$

So, by definition of c_Q, $c_Q Q$. But, by hypothesis, $c_a \succeq c_Q$, and so it follows that $c_a Q$. But by definition of c_a and *A-Descriptions*, it now follows that Qa. ⋈

• **Proof of Theorem 39b**: Suppose a is an individual concept. Then a appears at some world, say w_1. So some ordinary object, say b, realizes a at w_1; i.e., $\forall F(\models_{w_1} Fb \equiv aF)$. Now we want to show, for an arbitrary property, say Q, that $aQ \equiv a \succeq c_Q$.

(→) Assume aQ. We want to show $\forall F(c_Q F \rightarrow aF)$, so where P is arbitrary, assume $c_Q P$ (to show: aP). By the definition of c_Q, it follows

[85]Since $=_E$ is a primitive relation symbol, we may use it to construct complex relations. So $[\lambda y\, y =_E u]$ is an acceptable λ-expression.

that $Q \Rightarrow P$, i.e., $\Box\forall x(Qx \rightarrow Px)$. Now by Theorem 5 of World Theory (Appendix III), it follows that $\forall w(\models_w \forall x(Qx \rightarrow Px))$. And in particular, it follows that $\models_{w_1} \forall x(Qx \rightarrow Px)$. Now since a encodes precisely what b exemplifies at w_1, we know in particular that $\models_{w_1} Qb \equiv aQ$. But since we have assumed, aQ, it follows that $\models_{w_1} Qb$. Now since Qb is true at w_1, and it is also true at w_1 that $\forall x(Qx \rightarrow Px)$, it follows by the modal closure of worlds, that $\models_{w_1} Pb$. In which case, we know that aP, which is what we had to show.

($\leftarrow$) Simple exercise. (Hint: Consider the proof of Theorem 38.) $\bowtie$

Appendix III: The Basic Theorems of World Theory

In this Appendix, we review the notion of a possible world that is definable in the theory of abstract objects and review the theorems governing this notion. The notions of 'possible world' and 'truth at a world' have been made formally precise elsewhere in work on the theory of abstract objects.[86] So we only sketch the basic results (definitions and theorems) that are needed to formally represent Leibniz's modal metaphysics of concepts.

We begin by extending the notion of an object encoding a property to that of an object encoding a proposition. If we treat propositions as 0-place properties, then an object x may encode the proposition p in virtue of encoding the complex propositional property *being such that* p. We will symbolize such a propositional property as '$[\lambda y\, p]$' and so to represent the fact that object x encodes proposition p, we write: $x[\lambda y\, p]$.[87] We then say that a *world* is any A-object x which might have encoded all and only true propositions.[88] Using the variable 'w' to range over A-objects that satisfy the definition of a world, we may then say that that a proposition p is *true at* world w ('$\models_w p$') just in case w encodes p:

$$\models_w p \ =_{df} \ w[\lambda y\, p]$$

The consequences of these definitions constitute 'world theory'. The theorems of world theory that play a role in what follows are:[89]

[86]See Zalta [1993] and [1983], Chapter IV.

[87]The propositional property $[\lambda y\, p]$ is logically well-behaved despite the vacuously bound λ-variable y. It is constrained by the ordinary logic of complex predicates, which has the following consequence: x *exemplifies* $[\lambda y\, p]$ iff p, i.e., $[\lambda y\, p]x \equiv p$.

[88]In formal terms: $World(x) =_{df} \Diamond\forall p(x[\lambda y\, p] \equiv p)$.

[89]In what follows, we always give the symbol $\models_w$ the narrowest possible scope; for example, '$\models_w p \rightarrow p$' is to be read as: $(\models_w p) \rightarrow p$.

1. There is a unique actual world.
   $\exists! w\forall p(\models_w p \equiv p)$

2. Every world is maximal.
   $\forall p(\models_w p \vee \models_w \neg p)$

3. Every world is possible.
   $\neg\exists p, q(\neg\Diamond(p \,\&\, q) \;\&\; \models_w p \;\&\; \models_w q)$

4. Every world is modally closed.
   $\models_w p \,\&\, \Box(p \rightarrow q) \ \rightarrow \models_w q$

5. A proposition is necessarily true iff true at all worlds.
   $\Box p \equiv \forall w(\models_w p)$

6. A proposition is possible iff there is a world where p is true.
   $\Diamond p \equiv \exists w(\models_w p)$.

7. For any object x: a proposition p is true at world w if and only if at w, x exemplifies being such that p.
   $\forall x(\models_w p \ \equiv \ \models_w [\lambda y\, p]x)$

The proof of Theorems 1 – 6 may be found elsewhere.[90] The proof of Theorem 7 is given in Appendix II.

The picture that emerges from this theory of worlds can be described in Leibnizian language. Each of these possible worlds exists (i.e., our quantifier '$\exists$' ranges over them all). God entertained them all and decided which one was the best. Then to 'actualize' the best possible world, God made it the case that p, for each proposition p encoded in that world.

Appendix IV: Original Texts of the Cited Passages

In this Appendix, we provide the original texts for those scholars who wish to check and confirm the accuracy of the translations and citations which appear in our main text. We begin with the sources in the Gerhardt volumes and end with the sources in the Couturat volume.

[90]See Zalta [1983] and [1993].

Gerhardt

G.ii 18:

Ces dernieres paroles doivent contenir proprement la preuve de la consequence, mais il est tres manifeste, qu'elles confondent *necessitatem ex hyposthesi* avec la necessité absolue.

G.ii 20:

Car par la notion individuelle d'Adam j'entends certes une parfaite representation d'un tel Adam qui a de telles conditions individuelles et qui est distingué par là d'une infinité d'autres personnes possibles fort semblables, mais pourtant differentes de Luy ... Il y a un Adam possible dont la posterité est telle, et une infinité d'autres dont elle seroit autre n'est il pas vray que ces Adams possibles (si on les peut appeller ainsi) sont differens entre eux, et que Dieu n'en a choisi qu'un, qui est justement le nostre?

G.ii 44:

Car tous les predicats d'Adam dependent d'autres predicats du même Adam, ou n'en dependent point. Mettant donc à part ceux qui dependent d'autres, on n'a qu'à prendre ensemble tous les predicats primitifs pour former la notion complete d'Adam suffisante à en deduire tout ce qui luy doit jamais arriver, ...

G.ii 56:

[...et ce n'est que dans ce sens que] je dis que la notion de la substance individuelle enferme tous ses evenemens et toutes ses denominations, même celles qu'on appelle vulgairement extrinseques (c'est à dire qui ne luy appartiennent qu'en vertu de la connexion generale des choses et de ce qu'elle exprime tout l'univers à sa maniere), ...

G.iv 432:

Il est bien vray, que lorsque plusieurs predicats s'attribuent à un même sujet, et que ce sujet ne s'attribue plus à aucun autre, on l'appelle substance individuelle.

G.iv 433:

Il faut donc considerer ce que c'est que d'estre attribué veritablement à un certain sujet. ... Ainsi il faut que le terme du sujet enferme tousjours celuy du predicat, en sorte que celuy qui entendroit parfaitement la notion du sujet, jugeroit aussi que le predicat luy appartient.

...

Cela estant, nous pouvons dire que la nature d'une substance individuelle ou d'un estre complet, est d'avoir une notion si accomplie qu'elle soit suffisante à comprendre et à en faire deduire tous les predicats du sujet à qui cette notion est attribuée. Au lieu, que l'accident est un estre dont la notion n'enferme point tout ce qu'on peut attribuer au sujet à qui on attribue cette notion. Ainsi la qualité de Roy que appartient à Alexandre le Grand, faisant abstraction du sujet n'est pas assez determinée à un individu, et n'enferme point les autres qualités du même sujet, ny tout ce que la notion de ce Prince comprend, ...

G.iv 434:

De plus toute substance est comme un monde entier et comme un miroir de Dieu ou bien de tout l'univers, qu'elle exprime chacune à sa façon...

...

Ainsi l'univers est en quelque façon multiplié autant de fois qu'il y a de substances, et la gloire de Dieu est redoublée de même par autant de representations toutes differentes de son ouvrage.

G.iv 437:

...pour y satisfaire solidement, je dis que la connexion ou consecution est de deux sortes, l'une est absolument necessaire, dont le contraire implique contradiction, et cette deduction a lieu dans les verités éternelles, comme sont celles de Geometrie; l'autre n'est necessaire qu'*ex hypothesi*, et pour ainsi dive par accident, et celle est contingente en elle même, lors que le contraire n'implique point.

G.vi 107:

... et qu'il y a une infinité de Mondes possibles, dont il faut que Dieu ait choisi le meilleur, ...

G.vi 363:

Je vous en montreray, où se trouvera, non pas tout à fait le même Sextus que vous avés vu (cela ne se peut, il porte tousjours avec luy ce qu'il sera) mais des Sextus approchans, qui auront tout ce que vous connoissés déja du veritable Sextus, mais non pas tout ce qui est déja dans luy, sans qu'on s'en apper coive, ny par consequent tout ce qui luy arrivera encor. Vouis trouverés dans un monde, un Sextus fort heureux et elevé, dans un autre un Sextus content d'un etat mediocre, ...

G.vi 615-616:

Or, comme il y a une infinité des Univers possibles dans les Idées de Dieu et qu'il n'en peut exister qu'un seul, il faut qu'il y ait une raison suffisante du choix de Dieu, qui le determine à l'un plustôt qu'a l'autre.

G.vii 21:

Car si nous l'avions telle que je la conçois, nous pourrions raisonner en metaphysique et en morale à pue pres comme en Geometrie et en Analyse, parce que les Caracteres fixeroient nos pensées trop vagues et trop volatiles en ces matieres, où l'imagination ne nous aide point, si ce ne seroit par le moyen de caracteres.

G.vii 200:

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo (accito si placet amico) dicere: calculemus.

G.vii 218:

ab est *a* sive (omne) animal rationale est animal.

a est *a* sive (omne) animal est animal.

G.vii 222:

... nihil referre sive dicas *ab* sive dicas *ba*,...

...Repetitio alicujus literae in eodem termino inutilis est et sufficit eam reineri semel, exempli causa *a a* seu homo homo.

G.vii 223:

d est *c*, ergo *bd* est *bc* rursus per priora.

G.vii 230:

Axioma 1. Si idem secum ipso sumatur, nihil constituitur novum, seu A⊕A ∞ A.

G.vii 236:

Defin. 1. Eadem seu coincidentia sunt quorum alterutrum ubilibet potest substitui alteri salva veritate. Exempli gratia, Triangulum et Trilaterum, ... A ∞ B significat A et B esse eadem, ...

Propos. 1. Si A sit ∞ B, etiam B erit ∞ A.

Prop. 2. Si A non ∞ B, etiam erit B non ∞ A.

Prop. 3. Si A ∞ B, et B ∞ C, erit A ∞ C.

Coroll. Si A ∞ B et B ∞ C et C ∞ D, erit A ∞ D.

G.vii 237:

Prop. 4. Si A ∞ B et B non ∞ C, erit A non ∞ C.

Def. 3. A inesse in L seu L continere A idem est ac pro pluribus inter quae est A simul sumtis coincidens poni L.

Axiom 1. A⊕A ∞ A.

Axiom 2. B⊕N ∞ N⊕B.

Prop. 5. Si A est in B, et sit A ∞ C, etiam C est in B.

Prop. 6. Si C est in B, et sit A ∞ B, etiam C erit in A.

G.vii 238:

Prop. 7. A est in A.

Prop. 8. A est in B, si A ∞ B.

Prop. 9. Si A ∞ B, erit A⊕C ∞ B⊕C.

Scholium: Haec propositio converti non potest, multoque minus duae sequentes, et infra in probl. quod est prop. 23 docebitur modus instantiam reperiendi.

Prop. 10. Si A ∞ L et B ∞ M, erit A⊕B ∞ L⊕M.

Scholium: Haec propositio converti non potest, neque enim si sit A⊕B ∞ L⊕M et A ∞ L, sequitur statim esse B ∞ M; ...

G.vii 239:

Prop. 12. Si B est in L, erit A⊕B in A⊕L.

Prop. 13. Si L⊕B ∞ L, erit B in L.

Prop. 14. Si B est in L, erit L⊕B ∞ L.

G.vii 240:

Prop. 15. Si A est in B et B est in C, etiam A est in C.

Corollary to Prop. 15: Si A⊕N est in B, etiam N est in B.

Prop. 17. Si A est in B et B est in A, erit A ∞ B.

G.vii 241:

Prop. 18. Si A est in L et B est in L, etiam A⊕B erit in L.

Prop. 20. Si A est in M et B est in N, erit A⊕B in M⊕N.

Couturat

C 51:

Omnem propositionem veram categoricam ⟨affirmativam ⟨universalem⟩⟩, nihil aliud significare quam connexionem quandam inter Prædicatum et subjectum ⟨in casu recto de quo hic semper loquar⟩, ita scilicet ut prædicatum dicatur inesse subjecto ⟨vel in subjecto contineri, eoque vel absolute et in se spectato, vel certe [in aliquo casu] seu in aliquo exemplo⟩, seu ut subjectum dicto modo dicatur continere prædicatum : hoc est ut notio subjecti ⟨vel in se, vel cum addito⟩ involvat notionem prædicati, . . .

C 235:

(6) AA ∞ A

(7) AB ∞ BA

C 260:

(14) AA idem est in hoc calculo quod A.

C 262:

(7) AA idem est quod A.

C 366:

(18) Coincidunt A et AA, . . .

C 396:

189. Principia ergo hæc erunt: ⟨primò⟩ $aa = a$ (unde patet etiam non $b =$ non b, si ponamus non $b = a$).

C 421:

(3) A ∞ AA.

(4) AB ∞ BA seu transpositio nil nocet.

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