HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

AND GEOMETRIC OPTICS

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Preface

§P.1. How the book came to be and its peculiarities

This book presents an introduction to hyperbolic partial differential equations. A major subtheme is geometric optics linear and nonlinear. The two central results of linear microlocal analysis are derived from geometric optics. The nonlinear geometric optics presents an introduction to methods developed within the last twenty years including a rethinking of the linear case.

Much of the material has grown out of courses that I have taught. The crucial step was a series of eight lectures on nonlinear geometric optics at the Institute for Advanced Study Park City Mathematics Institute in July 1995. The Park City notes were prepared with the assistance of Markus Keel and were published in Volume 5 of the IAS/Park City Math Series. The notes presented a straight line path to some theorems in nonlinear geometric optics. Two graduate courses at the University of Michigan in 1993 and 2008 were also important. Much of the material was refined in inivited minicouses;

• Ecole Normale Superieure de Cachan, 1997,

• Nordic Conference on Conservation Laws at the Mittag-Leffler Institute and KTH in Stockholm, December 1997 (chapters 9-11).

• Centro DiGiorgi of the Scuola Normale Superiore in Pisa, February 2004,

• Univeristé de Provence, Marseille, March 2004 (§5.4.2-5.4.4, appendix II.1),

• Università di Pisa, February-May 2005, March-April 2006 (chapter. 3, §6.7,6.8), March-April 2007(chapters 9-11).

• Université de Paris Nord, February 2006, 2007, 2008 (§1.4-1.7),

The auditors included many at the beginnings of their careers and I would like to thank in particular R. Carles, E. Dumas, J. Bronski, J. Colliander, M. Keel, L. Miller, K. McLaughlin, R. McLaughlin, H. Zag, G. Crippa and A. Figalli, and N. Visciglia, for many interesting questions and comments.

The book is aimed at the level of graduate students who have studied one hard course in partial differential equations. Following the lead of the book of Guillemin and Pollack, there are exercises scattered throughout the text. The goal is to teach graduate students the habit of reading with a pencil, filling in and verifying as you go. Experience shows that passing from passive reading to active acquisition is a difficult transitions. Instructor's corrections of exercises offer the opportunity to teach the writing of mathematics, a skill vital for theses.

The choice of subject matter is guided by several principles. By restricting to symmetric hyperbolic systems, the basic energy estimates come from integration by parts. The majority of examples from applications fall under this umbrella.

The treatment of constant coefficient problems does not follow the usual path of describing classes of operators for which the Cauchy problem is weakly well posed. Such results are described in a brief appendix along with the Kreiss matrix theorem. Rather, Fourier transform methods are used to analyse the dispersive properties of constant coefficient symmetric hyperbolic equations including Brenner's theorem and Strichartz estimates.

Pseudodifferential operators are neither presented nor used. This is not because they are in any sense vile, but to get to the core without too many pauses to develop machinery. There are several good sources on pseudodifferential operators and the reader is encouraged to consult them to get alternate viewpoints on some of the material. It is interesting to see how far one can go without them. In a sense, the expansions of geometric are a natural replacement for that machinery. The Lax parametrix requires the analysis of oscillatory integrals as in the theory of Fourier integral operators . The results require only the method of nonstationary phase and are included.

The topic of caustics and caustic crossing is not treated. The sharp linear results use more microlocal machinery and the nonlinear analogues are topics of current research. The same is true for supercritical nonlinear geometric optics which is not discussed. The subjects of dispersive and diffractive nonlinear geometric optics in contrast have reached a mature state. Readers of this book should be in a position to readily attack the papers describing that material.

There is no discussion of modeling of practical applications. There is no discussion of mixed initial boundary value problems a subject with many interesting applications and a solid theory with many difficult challenges. There is no discussion of the geometric optics approach to shocks.

I have omitted several areas where there are already good sources. For example, the books of Smoller, Serre, Dafermos, Majda, and Bressan on conservation laws, and the books of Hörmander and Taylor on the use of pseudodifferential techniques in nonlinear problems. Other books on hyperbolic partial differential equations include those of Hadamard, Leray, Gårding, and Mizohata and, Benzoni-Gavage and Serre. Lax's 1963 Stanford notes occupy a special place in my heart. A revised and enlarged version is his book *Hyperbolic Partial Differential Equations.* I owe a great intellectual debt to the notes, and to all that Peter Lax has taught me through the years.

The book represent a first step aimed at a large and rich subject and I hope that readers are sufficiently attracted to probe further.

§**P.2. A bird's eye view of hyperbolic equations**

The central theme of this book is hyperbolic partial differential equations. These equations are important for a variety of reasons. It is worth while to keep these ideas in mind while reading. They have many different expressions in the computations and theorems in the book.

The first encounter with the concept is usually in considering scalar real linear second order partial differential operators in two variables,

$$a\,u_{x_1x_1} \;+\; b\,u_{x_1x_2} \;+\; c\,u_{x_2x_2} \;+\; \text{lower order terms}\,.$$

It is assumed that at least one of a, b, c is nonzero. The operator is called elliptic, (strictly) hyperbolic, or parabolic (this last is an incorrect definition) when the associated quadratic form

$$a\,\xi_1^2 \;+\; b\,\xi_1\xi_2 \;+\; c\,\xi_2^2$$

is positive or negative definite, nondegenerate and indefinite, or degenerate. In the elliptic cases one has strong local regularity theorems and solvability of the Dirichlet problem on small discs. In the hyperbolic cases, the initial value problem is locally well set at noncharacteristic surfaces and there is finite speed of propagation. Singularities or oscillations in Cauchy data propagate along characteristic curves. The defining properties of hyperbolic problems include well posed Cauchy problems, finite speed of propagation, and the existence of wave like structures with infinitely varied form. To see the latter, consider initial data with the form of a short wavelength wave packet localized near a point p on a noncharateristic surface. The solution will launch wave packets along each of two characteristic curves. The envelopes are computed from those of the intitial data and can take any form. One can send essentially arbitrary amplitude modulated signals. The infinite variety of wave forms make hyperbolic equations the preferred mode for sending information for example in hearing, sight, television, and radio. The model equations for the first is linearized compressible invsicid fluid dynamics, a.k.a. acoustics. For the latter three it is Maxwell's equations and most particularly Maxwell's equations in vacuum, since the atmosphere is nearly indistinguishable from the vacuum at the low intensities considered. The last two examples illustrate the fourth key characteristic of some hyperbolic problems which is propagation with no, or very small, losses over large distances. With these origins in mind, the importance of short wavelength and of large time analyses are central problems.

Well posed Cauchy problems with finite speed lead to hyperbolic equations.† Since the fundamental laws of physics must respect the principles of relativity, finite speed is required. This together with causality require hyperbolicity. Thus there are many equations from Physics. Those which are most fundamental tend to have close relationships with Lorentzian geometry. D'Alembert's wave equation and the Maxwell equations are two such fundamental equations.

Problems with origins in general relativity are of increasing interest in the mathematical community and it is the hope of hyperbolicians, that the wealth of geometric applications of elliptic equations in Riemannian geometry will one day be paralled by Lorenzian cousins of hyperbolic type.

A source of countless mathematical and technnological problems of hyperbolic type are the equations of inviscid compressible fluid dynamics. Linearization of those equations yields linear acoustics. It is common that viscous forces are important only near boundaries, and therefore for many phenomena inviscid theories suffice. Inviscid models are often easier to compute numerically. This is easily understood as a small viscous term $\epsilon^2 \partial^2/\partial x^2$ introduces a length scale $\sim \epsilon$ and accurate numerics require a discretization small enough to resolve this scale say $\sim 1/10\epsilon$. In dimensions $1 + d$ discretization of a unit volume for times of order 1 on such a scale requires $10^4 \epsilon^{-4}$ mesh points. For ϵ only modestly small, this drives computations beyond the practical. Faced with this one can employ meshes which are only locally fine, or try to construct, numerical schemes which resolve features on longer scales without resolving the short scale structures. Alternatively one can use asymptotic methods like those in this book to describe the boundary layers where the viscosity can not be neglected (see for example [Grenier-Gues], [Gerard-Varet]) All of these are active areas of research.

One of the key features of inviscid fluid dynamics is that smooth large solutions often break down in finite time. The continuation of such solutions as nonsmooth solutions containing shock waves satisfying suitable conditions (often called entropy conditions) is an important subarea of hyperbolic theory which is not discussed at all in this book. The interested reader is referred to the conservation law references cited earlier. An interesting counterpoint is that for suitably dispersive equations in high dimensions, small smooth data yield global smooth (hence shock free) solutions (see §6.7).

The subject of geometric optics is a major theme of this book. The subject begins with the earliest understanding of the propagation of light. Simple observation of sun beams streaming through a partial break in clouds, or a flashlight beam in a dusty room gives the impression that light travels in straight lines. At mirrors the lines reflect with the usual law of equal angles of incidence and reflection. Passing from air to water the lines are bent. These phenomena are described by the three fundamental principals of a physical theory called *geometric optics*. They are, rectilinear propagation, and the laws of reflection and refraction.

All three phenomena are explained by Fermat's principal of least time. The rays are locally paths of least time. Refraction at an interface is explained by positing that light travels at different speeds in the two media. This description is purely geometrical involving only broken rays and times of transit. The appearance of a minimum principal had important philosophical impact, since it was consistent with a world view holding that nature acts in a best possible way. Fermat's prinicpal was enunciated twenty years before Römer demonstrated the finiteness of the speed of light based on observations of the moons of jupiter.

† See [Lax, 1963, 2006] for a proof in the constant coefficient linear case. The necessity of hyperpolicity in the variable coefficient case dates to [Lax, 1957] and extensions are discussed [Mizohata]. See [Nishitani] for the state of the art.

Today light is understood as an electromagnetic phenomenon. It is described by the time evolution of electromagnetic fields which are solutions of a system of partial differential equations. When quantum effects are important, this theory must be quantized. A mathematically solid foundation for the quantization of the electromagnetic field in 1+3 dimensional space time has not yet been found.

The reason that a field theory involving partial differential equations can be replaced by a geometric theory involving rays is that visible light has very short wavelength compared to the size of human sensory organs and common physical objects. Thus, much observational data involving light occurs in an asymptotic regime of very short wavelength. The short wavelength asymptotic study of systems of partial differential equations often involves significant simplifications. In particular there are often good descriptions involving rays. We will use the phrase *geometric optics* to be synonymous with short wavelength asymptotic analysis of solutions of systems of partial differential equations.

In optical phenomena, not only is the wavelength short but the wave trains are long. The study of structures which have short wavelength and are in addition very short, say a short pulse, also yields a geometric theory. Long wavetrains have a longer time to allow nonlinear interactions which makes nonlinear effects more important. Long propagation distances also increase the importance of nonlinear effects. An extreme example is the propagation of light across the ocean in optical fibers. The nonlinear effects are very weak, but over 5000 kilometers, the cumulative effects can be large. To control signal degradation in such fibers the signal is treated about every 30 kilometers. Still, there is free propagation for 30 kilometers which needs to be understood. This poses serious analytic, computational, and engineering challenges.

A second way to bring nonlinear effects to the fore is to increase the amplitude of disturbances. It was only with the advent of the laser that sufficiently intense optical fields are produced so that nonlinear effects are routinely observed. The conclusion is that for nonlinearity to be important, either the fields or the propagation distances must be large. For the latter, dissipative losses must be small.

The ray description as a simplification of the Maxwell equations is analogous to the fact that classical mechanics gives a good approximation to solutions of the Schrödinger equation of quantum mechanics. The associated method is called the quasiclassical approximation. The role of rays in optics is played by the paths of classical mechanics. There is an important difference in the two cases. The Schrödinger equation has a small parameter, Planck's constant. The quasiclassical approximation is an approximation valid for small Planck's constant. The mathematical theory involves the limit as this constant tends to zero. Maxwell's equations apparently have a small parameter too, the inverse of the speed of light. One might guess that rays occur in a theory where this speed tends to infinity. This is not the case. For Maxwell's equations in vacuum the small parameter which appears is the wavelength which is introduced via the initial data. It is not in the equation. The equations describing the dispersion of light when it interacts with matter do have a small parameter, the inverse of the resonant frequencies of the material, and the analysis involves data tuned to this frequency just as the quasiclassical limit involves data tuned to Planck's constant. This topic is one of my favorites and interested readers are referred to the articles [Donnat-Rauch], [Rauch, A travers un prism].

Short wavelength phenomena cannot simply be studied by numerical simulations. If one were to discretize a cubic meter of space with mesh size 10^{-5} cm. so as to have five mesh points per wavelength, there would be 10^{21} data points in each time slice. Since this is nearly as large as the number of atoms per cubic centimeter, there is no chance for the memory of a computer to be sufficient to store enough data, let alone make calculations. Such brute force approaches are doomed to fail. A more intelligent approach would be to use radical local mesh refinement so that the fine mesh was used only when needed. Still this falls far outside the bounds of present computing power. The asymptotic analysis offers an alternative approach which is not only powerful but is mathematically elegant. In the scientific literature it also embraced because the resulting equations sometimes have exact solutions and scientists are well versed in understanding phenomena from small families of exact solutions.

Short wavelength asymptotics can be used to great advantage in many disparate domains. They explain and extend the basic rules of linear geometric optics. They explain the dispersion and diffraction of linear electromagnetic waves. There are nonlinear optical effects, generation of harmonics, rotation of the axis of elliptical polarization, and self focussing which are also well described. These topics can be pursued by consulting the references.

Geometric optics has many applications within the subject of partial differential equations. They play a key role in the problem of solvability of linear equations via results on propagation of singularities as presented in §5.5. They are used in deriving necessary conditions, for example for hypoellipticity and hyperbolicity. They are used by Ralston to prove necessity in the conjecture of Lax and Phillips on local decay via propagation of singularities they play the central role in the proof of sufficiency. Propagation of singularities plays a central role in problems of observability and controlability (see §5.6). The microlocal elliptic regularity theorem and the propagation of singularities for symmetric hyperbolic operators of constant multiplicity is treated in this book. These are the two basic results of linear microlocal analysis. These notes are not an introduction to that subject, but present an important piece *en passant*.

Chapters 9 and 10 are devoted to the phenomenon of *resonance* whereby waves with distinct phases can interact nonlinearly. They are preparatory for Chapter 11 That chapter 11 constructs a family of solutions of the compressible $2d$ Euler equations exhibiting three incoming wave packets interacting to generate an infinite number of oscillatory wave packets whose velocities are dense in the unit circle.

Because of the central role played by rays and characteristic hypersurfaces, the analysis of conormal waves is a closely related to geometric optics. The reader is referred to Lax's treatment of progressing waves and the book of M. Beals for this material.

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Chapter 1. Simple Examples of Propagation.

This chapter presents examples of wave propagation governed by hyperbolic equations. The ideas of propagation of singularities, group velocity, and short wavelength asymptotics are introduced in simple situations. The method of characteristics for problems in dimension $d = 1$ is presented as well as the method of nonstationary phase. The latter is a fundamental tool for estimating oscillatory integrals. The examples are elementary. They could each be part of an introductory course in partial differential equations, but often are not. This material can be skipped. If later needed the reader may return to this chapter.* In sections 1.3, 1.5, 1.6 we derive in simple situations the three basic laws of .

Wave like solutions of partial differential equations have spatially localized structures whose evolution in time can be followed. The most common are solutions with propagating singularities and solutions which are modulated wave trains also called wave packets. The latter have the form

$$a(t,x)\, e^{i\phi(t,x)/\epsilon}$$

with smooth profile a, real valued smooth phase, ϕ, with $d\phi \neq 0$ on the support of a. The parameter ϵ is small compared to the scale on which a and ϕ vary. They are radically different spatial scales, the scale on which the profile varies and the much smaller wavelength. The classic example is light with a wavelength on the order of 5×10^{-5} centimeter. Singularities are often restricted to varieties of lower codimension, hence of width equal to zero which is infinitely small compared to the scales of their other variations. Real world waves modeled by such solutions have the singular behavior spread over very small lengths, not exactly zero.

The path of a localized structure in space time is curvelike, and such curves are often called **rays**. When phenomena are described by partial differential equations, linking the above ideas with the equation means finding solutions whose salient features are localized and in simple cases are described by transport equations along rays. In the case of, such results appear in an asymptotic analysis as $\epsilon \to 0$.

In this chapter some introductory examples are presented that illustrate propagation of singularities, propagation of energy, group velocity and short wavelength asymptotics. That energy and singularities may behave very differently is a consequence of the dichotomy that up to an error as small as one likes in energy, the data can be replaced by data with compactly supported Fourier transform. In contrast, up to an error as smooth as one likes the data can be replaced by data with Fourier transform vanishing on $|\xi| \leq R$ with R as large as one likes. Propagation of singularities is about short wavelengths while propagation of energy is about wavelengths bounded away from 0. When most of the energy is carried in short wavelengths, for example the wave packets above, the two tend to propagate in the same way.

§1.1. The method of characteristics.

The method of characteristics reduces many questions concerning solutions of hyperbolic partial differential equations *when the space dimension is equal to one* to the integration of ordinary differential equations. The central idea is the following. Suppose that $c(t,x)$ is a smooth real valued function and introduce the ordinary differential equation

$$\frac{dx}{dt} \;=\; c(t,x)\,. \tag{1.1.1}$$

* Some ideas are used which are not formally presented till later, for example the Soboev spaces $H^s(\mathbb{R}^d)$ and Gronwall's lemma.

Solutions $x(t)$ satsify

$$\frac{dx(t)}{dt} \;=\; c(t,x(t)).$$

For a smooth function u,

$$\frac{d}{dt}\,u(t,x(t)) \;=\; \partial_t u + c(t,x)\,\partial_x u\,.$$

Therefore, solutions of the homogeneous linear equation

$$\partial_t u + c(t,x)\,\partial_x u \;=\; 0,$$

are exactly the functions u which are constant on the integral curves $(t,x(t))$ which are called **characteristic curves** or simply **characteristics**.

Example. *If $c \in \mathbb{R}$ is constant then $u \in C^\infty([0,T]\times\mathbb{R})$ satisfies*

$$\partial_t u + c\,\partial_x u \;=\; 0, \tag{1.1.2}$$

if and only if there is an $f \in C^\infty(\mathbb{R})$ so that $u = f(x-ct)$. The function f is uniquely determined.

Proof. For constant c the characteristics along which u is constant are the lines $(t, x+ct)$. Therefore, $u(t,x) = u(0,x-ct)$ proving the result with $f(x) := u(0,x)$. ∎

This shows that the Cauchy problem consisting of (1.1.2) together with the initial condition $u|_{t=0} = f$ is uniquely solvable with solution $f(x-ct)$. The solutions are waves translating rigidly with velocity equal to c.

Exercise 1.1.1. *Find an explicit solution formula for the solution of the Cauchy problem*

$$\partial_t u + c\,\partial_x u + z(t,x)u \;=\; 0, \qquad u|_{t=0} = g,$$

where $z \in C^\infty$.

Example. D'Alembert's formula. *If $c \in \mathbb{R}$ then $u \in C^\infty([0,T]\times\mathbb{R})$ satisfies*

$$u_{tt} \;-c^2\; u_{xx} \;=\; 0 \tag{1.1.3}$$

if and only if there are smooth $f, g \in C^\infty(\mathbb{R})$ so that

$$u \;=\; f(x-ct) \;+\; g(x+ct)\,. \tag{1.1.4}$$

The set of all pairs $\tilde f, \tilde g$ so that this is so is of the form $\tilde f = f + b$, $\tilde g = g - b$ with $b \in \mathbb{C}$.

Proof. Factor

$$\partial_t^2 - c^2\partial_x^2 \;=\; (\partial_t - c\partial_x)\,(\partial_t + c\partial_x) \;=\; (\partial_t + c\partial_x)\,(\partial_t - c\partial_x)\,.$$

Conclude that

$$u_+ \;:=\; \partial_t u - c\,\partial_x u\,, \qquad \text{and,} \qquad u_- \;:=\; \partial_t u + c\,\partial_x u$$

satisfy

$$\big(\partial_t \;\pm\; c\,\partial_x\big)u_\pm \;=\; 0\,. \tag{1.1.5}$$

Therefore there are smooth $F_\pm$ so that

$$u_\pm \;=\; F_\pm(x \mp ct)\,. \tag{1.1.6}$$

In order for (1.1.4) and (1.1.6) to hold, one must have

$$F_+ = (\partial_t - c\partial_x)u \;=\; (1+c^2)f', \qquad F_- = (\partial_t + c\partial_x)u \;=\; (1+c^2)g' \tag{1.1.7}$$

Thus if $G_\pm$ are primitives of $F_\pm$ which vanish at the origin, then one must have

$$f \;=\; \frac{G_+}{(1+c^2)} + C_+, \qquad g \;=\; \frac{G_+}{(1+c^2)} + C_-\,, \qquad C_+ + C_- = u(0,0)\,.$$

Reversing the process shows for such f, g, defining $\tilde{u} := f(x-ct) + g(x+ct)$ yields a solution of D'Alembert's equation with

$$(\partial_t \mp c\,\partial_x)\tilde{u} \;=\; F_\pm, \qquad \text{so,} \quad (\partial_t \mp c\,\partial_x)(u - \tilde{u}) \;=\; 0\,.$$

Adding and subtracting this pair of equations shows that

$$\nabla_{t,x}(u \;-\; \tilde{u}) \;=\; 0.$$

Since $u(0,0) = \tilde{u}(0,0)$, it follows by connectedness of $[0,T]\times\mathbb{R}$ that $u = \tilde{u}$ and the proof is complete. ∎

For speeds $c(t,x)$ which are not bounded, it is possible that characteristics escape to infinity with interesting consequences.

Example of nonuniqueness in the Cauchy problem. Consider $c(t,x) := x^2$. The characteristic through $(0, x_0)$ is the solution of

$$x' = x^2, \qquad x(0) = x_0,$$

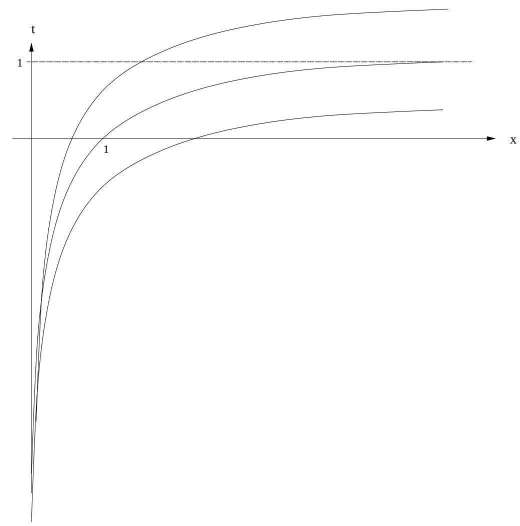
Then,

$$1 \;=\; \frac{x'}{x^2} \;=\; \frac{d}{dt}\Big(\frac{-1}{x}\Big).$$

Integrating from $t = 0$ yields

$$\frac{-1}{x(t)} \;-\; \frac{-1}{x_0} \;=\; t\,, \qquad \text{and therefore,} \qquad x(t) \;=\; \frac{x_0}{1 - x_0\,t}\,.$$

Through each point t, x with $t \geq 0$ there is a unique charateristic tracing backward to $t = 0$. Therefore, given initial data $u(0,x) = g(x)$, the solution $u(t,x)$ is uniquely deterimined in $t \geq 0$ by requiring u to be constant on characteristics.



Characteristics diverge in finite time

The characteristics through $(0, \pm 1)$ diverge to $\pm\infty$ at time $t = 1$. Thus all the backward characteristics starting in $t \geq 1$ meet $\{t = 0\}$ in the interval $]-1, 1[$. The data for $|x| \geq 1$ does not influence the solution in $t \geq 1$. There has been a loss of information. Another manifestation of this is that the initial values do not uniquely determine a solution in $t < 0$.

The characteristics starting at $t = 0$ meet $\{t = -1\}$ in the interval $]-1, 1[$. Outside that interval, the values of a solution are not determined, not even influenced by the initial data. There are many solutions in $t < 0$ which have the given Cauchy data. They are constant on characteristics which diverge to infinity but their values on these characteristics is otherwise arbitrary.

To avoid this phenomenon we make the following strong assumption.

Hypothesis 1.1.1. *Suppose that for all* $T > 0$

$$\partial^\alpha_{t,x} c \ \in \ L^\infty([0, T] \times \mathbb{R}) \,.$$

The coefficient $d(t, x)$ *satisfies analogous bounds.*

For arbitrary $f \in C^\infty(\mathbb{R}^2)$ and $g \in C^\infty(\mathbb{R})$ there is a unque solution of the Cauchy problem

$$\Big(\partial_t \ + \ c(t,x)\,\partial_x \ + \ d(t,x)\Big)u \ = \ f, \qquad u(0,x) \ = \ g.$$

Its values along the characteristic $(t, x(t))$ is determined by integrating the nonhomogeneous linear ordinary differential equation

$$\frac{d}{dt}\,u(t,x(t)) + d(t,x(t))\,u(t,x(t)) \ = \ f(t,x(t))\,. \tag{1.1.8}$$

There are finite regularity results too. If f, g are k times differentiable with $k \geq 1$ then so is u. Though the equation is first order, u is in general not smoother than f. This is in contrast to the elliptic case.

The method of characteristics also applies to systems of hyperbolic equations. Consider vector valued unknowns $u(t, x) \in \mathbb{C}^N$. The simplest generalization is diagonal real systems

$$u_t \ + \ \mathrm{diag}(c_1(t,x), \ldots, c_N(t,x))\,u \ = \ 0.$$

Here u_j is constant on characteristics with speed $c_j(t,x)$. This idea extends naturally to systems

$$L \;:=\; \partial_t \;+\; A(t,x)\,\partial_x \;+\; B(t,x),$$

where A, B are smooth matrix valued functions so that

$$\forall T,\ \forall \alpha, \quad \partial^\alpha_{t,x}\{A,B\} \;\in\; L^\infty([-T,T]\times\mathbb{R})\,.$$

The method of characteristics applies when the following hypothesis is satisfied. It says that the matrix A has real eigenvalues and is smoothly diagonalisable. The real spectrum as well as the diagonalisability are understood as part of the general theory of constant coefficient hyperbolic systems sketched in the appendix to chapter 2.

Hypothesis 1.1.2. *There is a smooth matrix valued function, $M(t,x)$, so that*

$$\forall T,\ \forall \alpha, \quad \partial^\alpha_{t,x}M \ \text{ and } \ \partial^\alpha_{t,x}(M^{-1}) \ \text{ belong to } \ L^\infty([0,T]\times\mathbb{R})\,,$$

and,

$$M^{-1}\,A\,M \;=\; \text{diagonal and real.} \tag{1.1.9}$$

Examples. 1. The hypothesis is satisfied if for each t,x the matrix A has N distinct real eigenvalues $c_1(t,x) < c_2(t,x) < \ldots < c_N(t,x)$. Such sytems are called **strictly hyperbolic**. To guarantee that the estimates on M, M^{-1} are uniform as $|x|\to\infty$ it suffices to assume that,

$$\inf_{(t,x)\in[0,T]\times\mathbb{R}} \ \min_{2\le j\le N} \ c_j(t,x) \;-\; c_{j-1}(t,x) \;>\; 0\,.$$

2. More generally the hypothesis is satisfied if for each (t,x), A has uniformly distinct real eigenvalues and is is diagonalisable. It follows that the multiplicity of the eigenvalues is independent of t,x. **4.** If A_1 and A_2 satisfy Hypothesis 1.1.2. then so does

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad \text{with} \qquad M \;:=\; \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

In this way one constructs examples with variable multiplicity.

Since $A = MDM^{-1}$ with $D = \operatorname{diag}(c_1,\ldots,c_N)$

$$L \;=\; \partial_t + MDM^{-1}\partial_x + B\,.$$

Define v by $u = Mv$ so

$$M^{-1}Lu \;=\; M^{-1}\Big(\partial_t \;+\; MDM^{-1}\partial_x \;+\; B\Big)M\,v\,.$$

When the derivatives on the right fall on v the product $M^{-1}M = I$ simplifies. This shows that,

$$M^{-1}Lu \;=\; \Big(\partial_t + D\,\partial_x + \widetilde{B}\Big)u \;:=\; \widetilde{L}\,v$$

where

$$\widetilde{B} \;:=\; M^{-1}\,B\,M \;+\; M^{-1}\,M_t \;+\; M^{-1}\,A\;M_x\,.$$

This change of variable converts the equation $Lu = f$ to $\widetilde{L}v = M^{-1}f$ where $\widetilde{L}$ has the same form as L but with leading part which is a set of directional derivatives.

Theorem 1.1.1. *If $f \in C^k([0,T]\times\mathbb{R})$ and $g \in C^k(\mathbb{R})$ with $k \ge 1$ and for all α, β with $|\alpha| \le k$ and $|\beta| \le k$,*

$$\partial^\alpha_{t.x} f \ \in\ L^\infty([0,T]\times\mathbb{R}), \qquad \text{and} \qquad \partial^\beta_x g \ \in\ L^\infty([0,T]\times\mathbb{R}),$$

then there is a unique solution $u \in C^k([0,T]\times\mathbb{R})$ to the initial value problem $Lu = f$, $u|_{t=0} = g$ so that all partial derivatives of u of order $\le k$ are in $L^\infty([0,T]\times\mathbb{R})$.

The crux is the following estimate called *Haar's inequality.* For a vector valued function $w(x) = (w_1(x), \dots, w_N(x))$ on $\mathbb{R}$ the L^∞ norm is taken to be

$$\|w\|_{L^\infty(\mathbb{R})} \ :=\ \max_{1\le j\le N}\ \|w_j(x)\|_{L^\infty(\mathbb{R})}\,.$$

Haar's Lemma 1.1.2. i. *There is a constant $C = C(T,L)$ so that if u and Lu are bounded continuous functions on $[0,T]\times\mathbb{R}$ then for $t\in[0,T]$,*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \ \le\ C\,\Big(\|u(0)\|_{L^\infty(\mathbb{R})} \ +\ \int_0^t \|Lu(\sigma)\|_{L^\infty(\mathbb{R})}\,d\sigma\Big)\,.$$

ii. *More generally, there is a constant $C(k,T,L)$ so that if for all $|\alpha| \le k$, $\partial^\alpha_{t,x}u$ and $\partial^\alpha_{t,x}Lu$ are bounded continuous funtions on $[0,T]\times\mathbb{R}$ then*

$$m_k(u,t) \ :=\ \sum_{|\alpha|\le k}\ \|\partial^\alpha_{t,x}u(t)\|_{L^\infty(\mathbb{R})}$$

satisfies for $t\in[0,T]$,

$$m_k(u,t) \ \le\ C\,\Big(m_k(u,0) \ +\ \int_0^t m_k(Lu,\sigma)\,d\sigma\Big)\,.$$

Proof of Lemma. The change of variable shows that it suffices to consider the case of a real diagonal matrix $A = \operatorname{diag}(c_1(t,x),\dots,c_N(t,x))$.
i. For $\underline{t}\in[0,T]$ and $\epsilon > 0$ choose j and $\underline{x}$ so that

$$\|u(\underline{t})\|_{L^\infty(\mathbb{R})} \ \le\ \|u_j(\underline{t},\underline{x})\|_{L^\infty(\mathbb{R})} \ +\ \epsilon\,.$$

Choose $(t, x(t))$ the integral curve of $x' = c_j(t,x)$ passing through $\underline{t}, \underline{x}$. Then

$$u_j(\underline{t},\underline{x}) \ =\ u_j(0,x(0)) \ +\ \int_0^{\underline{t}} (\partial_t + c_j(t,x)\partial_x)u_j(\sigma,x(\sigma))\ d\sigma\,.$$

Therefore

$$\|u(\underline{t})\|_{L^\infty(\mathbb{R})} \ \le\ \|u(0)\|_{L^\infty(\mathbb{R})} \ +\ \int_0^{\underline{t}} \|(Lu \ +\ B\,u)(\sigma)\|_{L^\infty(\mathbb{R})}\,d\sigma \ +\ \epsilon\,,$$

Since this is true for all ϵ one has,

$$\|u(\underline{t})\|_{L^\infty(\mathbb{R})} \;\le\; \|u(0)\|_{L^\infty(\mathbb{R})} \;+\; \int_0^{\underline{t}} \|(Lu)(\sigma)\|_{L^\infty(\mathbb{R})} \;+\; C\,\|u(\sigma)\|_{L^\infty(\mathbb{R})}\; d\sigma\,,$$

and **i** follows using Gronwall's Lemma 2.1.3.

ii. Apply the inequality of **i** to $\partial^\alpha_{t,x} u$ with $|\alpha| \le k$.

$$\|\partial^\alpha u(t)\|_{L^\infty(\mathbb{R})} \;\le\; C\Big(\|\partial^\alpha u(0)\|_{L^\infty(\mathbb{R})} \;+\; \int_0^t \|L\partial^\alpha u(\sigma)\|_{L^\infty(\mathbb{R})}\Big)\, d\sigma.$$

Compute

$$L\,\partial^\alpha u \;=\; \partial^\alpha L u \;+\; [L, \partial^\alpha]\, u\,.$$

The commutator is a differential operator of order k with bounded coefficients so

$$\|[L, \partial^\alpha]\, u(\sigma)\|_{L^\infty(\mathbb{R})} \;\le\; C\, m_k(u, \sigma)\,.$$

Therefore,

$$\|\partial^\alpha u(t)\|_{L^\infty(\mathbb{R})} \;\le\; C\Big(\|\partial^\alpha u(0)\|_{L^\infty(\mathbb{R})} \;+\; \int_0^t \|\partial^\alpha L u(\sigma)\|_{L^\infty(\mathbb{R})} \;+\; m_k(u,\sigma)\Big)\,.$$

Sum over $|\alpha| \le k$ to find

$$m_k(u,t) \le C\Big(m_k(u,0) + \int_0^t m_k(u,\sigma) \;+\; m_k(Lu,\sigma)\, d\sigma\Big)\,.$$

Gronwall's Lemma implies **ii.** ∎

Proof of Theorem. The change of variable shows that it suffices to consider the case of $A = \mathrm{diag}(c_1, \dots, c_N)$

The solution u is constructed as a limit of approximate solutions u^n. The solution u^0 is defined as the solution of the initial value problem

$$\partial_t u^0 \;+\; A\,\partial_x u^0 \;=\; f, \qquad u^0|_{t=0} = g\,.$$

The solution is explicit by the method of characteristics and Haar's inequality yields

$$\exists C_1, \quad \forall t \in [0,T], \quad m_k(u^0, t) \;\le\; C_1\,. \tag{1.1.10}$$

For $n > 0$ the solution u^n is again explicit by the method of characteristics in terms of u^{n-1},

$$\partial_t u^n \;+\; A\,\partial_x u^n \;+\; B\,u^{n-1} \;=\; f \qquad u^{n-1}|_{t=0} = g\,. \tag{1.1.11}$$

Usiing (1.1.10) and Haar's inequality yields,

$$\exists C_2, \quad \forall t \in [0,T], \quad m_k(u^1, t) \;\le\; C_2\,. \tag{1.1.12}$$

For $n \geq 2$ estimate $u^n - u^{n-1}$ by applying Haar's inequality to

$$\tilde{L}\,(u^n - u^{n-1}) \;+\; B\,(u^{n-1} - u^{n-2}) \;=\; 0, \qquad (u^n - u^{n-1})\big|_{t=0} \;=\; 0\,,$$

to find

$$m_k(u^n - u^{n-1}, t) \;\leq\; C \int_0^t m_k(u^{n-1} - u^{n-2}, \sigma)\; d\sigma\,. \tag{1.1.13}$$

For $n = 2$ this together with (1.1.10) and (1.1.12) yields,

$$m_k(u^2 - u^1, t) \;\leq\; (C_1 + C_2)Ct\,.$$

Injecting this in (1.1.13) yields

$$m_k(u^3 - u^2, t) \;\leq\; (C_1 + C_2)C^2t^2/2\,.$$

Continuing yields,

$$m_k(u^n - u^{n-1}, t) \;\leq\; (C_1 + C_2)C^{n-1}t^{n-1}/(n-1)!\,. \tag{1.1.14}$$

The summability of right hand side implies that u^n and all of its partials of order $\leq k$ converge uniformly on $[0, T] \times \mathbb{R}$. The limit u is C^k with bounded partials and passiing to the limit in (1.1.11) shows that u solves the initial value problem.

To prove uniqueness, suppose that u and v are solutions. Haar's inequality applied to $u - v$ implies that $u - v = 0$. ∎

The proof yields also the fact that there is finite speed of propagation of signals. Define $\lambda_{\min}(t, x)$ and $\lambda_{\max}(t, x)$ to the the smallest and largest eigenvalues of $A(t, x)$. Then the functions λ are uniformly Lipschitzean on $[0, T] \times \mathbb{R}$. The characteristics have speeds bounded below by $\lambda_{\min}$ and above by $\lambda_{\max}$. The next result shows that these are respectively lower and upper bounds for the speeds of propagation of signals.

Corollary 1.1.3. *Suppose that $-\infty < x_l < x_r < \infty$ and γ_l (respectively γ_r) is the integral curve of $\partial_t + \lambda_{\min}\partial_x$ (resp. $\partial_t + \lambda_{\max}\partial_x$) passing through x_l (resp. x_r). Denote by Q the four sided region in $0 \leq t \leq T$ bounded on the left by γ_l and the right by γ_r. If g is supported in $[x_l, x_r]$ and for $0 \leq t \leq T$, f is supported in Q, then the solution u is also supported in Q.*

Proof. The explicit formulas of the method of characteristics show that the approximate solutions u^n are supported in Q. Passing to the limit proves the result. ∎

Consider next the case of $f = 0$ and $g \in C^1(\mathbb{R})$ whose restrictions to $]-\infty, \underline{x}[$ and $]\underline{x}, \infty[$ is each smooth with uniformly bounded derivatives of every order. Such a function is called **piecewise smooth**.

The simplest case is that of an operator $\partial_t + c(t, x)\partial_x$. Denote by γ the characteristic through $\underline{x}$. The values of u to the left of γ is determined by g to the left of $\underline{x}$. Choose a $\tilde{g} \in C^\infty(\mathbb{R})$ which agrees with g to the left. The solution $\tilde{u}$ then agrees with u to the left of γ and $\tilde{u}$ has bounded partials of all orders for $0 \leq t \leq T$. An analogous argument works for the right hand side and one sees that u is piecewise C^∞ in the decompostion of $[0, T] \times \mathbb{R}$ into two pieces by γ.

Suppose now that A satisfies Hypothesis 1.1.2 and for all $(t, x) \in [0, T] \times \mathbb{R}$ has N distinct real eigenvalues ordered so that $c_j < c_{j+1}$. If one repeats the above for the system $\partial_t + \operatorname{diag}(c_1, \ldots, c_N)\,\partial_x$ one finds a solution u which is piecewise C^∞ on the pie shaped decomposition of $[0, T] \times \mathbb{R}$ into $N + 1$ wedges by the N characteristics through $(0, \underline{x})$.

Denote by γ_j the corresponding characteristics through $\underline{x}$. Define open wedges,

$$W_1 \ := \ \big\{(t,x) \ : \ 0<t<T, \quad -\infty < x < \gamma_1(t)\big\},$$

$$W_{N+1} \ := \ \big\{(t,x) \ : \ 0<t<T, \quad \gamma_N(t) < x < \infty\big\},$$

and for $2<j<N$,

$$W_j \ := \ \big\{(t,x) \ : \ 0<t<T, \quad \gamma_{j-1}(t) < x < \gamma_j(t)\big\}.$$

They decompose $[0,T]\times\mathbb{R}$ with the wedges numbered from left to right.

Definition. *For $k\ge 1$, the set PC^k consists of functions which are piecewise C^k as the set of bounded continuous functions u on $[0,T]\times\mathbb{R}$ so that for $\alpha\le k$ and $1\le j\le N+1$,*

$$\partial^\alpha_{t,x}\big(u\big|_{W_j}\big) \ \in \ L^\infty(W_j)\,.$$

It is a Banach space with norm

$$\big\|u\big\|_{L^\infty([0,T]\times\mathbb{R})} \ + \sum_{|\alpha|\le k}\ \sum_{1\le j\le N+1} \big\|\partial^\alpha_{t,x}\big(u\big|_{W_j}\big)\big\|_{L^\infty(W_j)}\,.$$

The next result asserts that for piecewise smooth data with singularity at $\underline{x}$ the solution is piecewise smooth with its singularities restricted to the characteristics through $\underline{x}$.

Theorem 1.1.4. *Suppose in addition to Hypothesis 1.1.2, that A has N distinct real eigenvalues for all (t,x) If $f\in PC^k$ and $g\in C(\mathbb{R})$ has bounded continuous derivatives up to order k on each side of $\underline{x}$ then the solution u belongs to PC^k.*

Sketch of Proof. Repeat the construction of u. In addition to the $L^\infty([0,T]\times\mathbb{R})$ estimates one needs to estimate the derivatives of order k on the wedges W_j. Introduce

$$\mu_k(u,\sigma) \ := \ \|u(\sigma)\|_{L^\infty(\mathbb{R})} \ + \sum_{2\le|\alpha|\le k}\ \sum_{1\le j\le N+1} \big\|\partial^\alpha_{t,x}\big(u\big|_{W_j}\big)(\sigma)\big\|_{L^\infty(W_j\cap\{t=\sigma\})}\,.$$

To estimate u^n-u^{n-1} use the following Lemma. ∎

Lemma 1.1.5. *Assume the hypotheses of the Theorem and that $c_j(t,x)$ is one of the eigenvalues of $A(t,x)$. Then, there is a constant $C(j,T,L)$ so that if $f\in PC^k$ and*

$$\big(\partial_t \ + \ c_j(t,x)\,\partial_x\big)w \ = \ f, \qquad w\big|_{t=0} \ = \ 0,$$

then $w\in PC^k$ and,

$$\mu_k(w,t) \ \le \ C\Big(\mu_k(w,0) \ + \int_0^t \mu_k(f,\sigma)\ d\sigma\Big)\,.$$

Exercise 1.1.2 *Prove the Lemma. Then finish the proof of the Theorem.*

Exercise 1.1.3. *Suppose that u is as in the Theorem, $f=0$, and that for some $\epsilon>0$ and j, the derivatives of u of order $\le k$ are continuous across $\gamma_j\cap\{0\le t<\epsilon\}$. Prove that they are* *continuous across* $\gamma_j \cap \{0 \le t \le T\}$. **Hint.** Show that the set of times $\underline{t}$ for which the solution is C^k on $\gamma_j \cap \{0 \le t \le \underline{t}\}$ is both open and closed. Use finite speed.

Denote by $\Phi_j(t,x)$ the flow of the ordinary differential equation $x' = c_j(t,x)$. That is $x(t) = \Phi_j(t, \underline{x})$ is the solution with $x(0) = \underline{x}$. The solution operator for the pure transport equation $(\partial_t + c_j \partial_x)u = 0$ with initial value g is then

$$u(t) \;=\; g(\Phi_j(-t,x))\,.$$

The values at time t are the rearrangements by the diffeomorphism $\Phi(-t,\cdot)$ of the initial function. Because of the uniform boundedness of the derivatives of c_j on slabs $[0,T]\times\mathbb{R}$ one has

$$\partial^\alpha_{t,x}\Phi \;\in\; L^\infty([0,T]\times\mathbb{R})\,.$$

The derivative $\partial_x\Phi$ measures the expansion or contraction by the flow. It is the the length of the image of an infinitesimal interval divided by the original length. In particular Φ can at most expand lengths by bounded quantity. The inverse of $\Phi(t,.)$ is the flow by the ordinary differential equation from time t to time 0 the inverse also cannot expand by much. This is equivalent to a lower bound,

$$\big(\partial_x\Phi\big)^{-1} \;\in\; L^\infty([0,T]\times\mathbb{R})\,.$$

The diffeomorphism $\Phi(t,.)$ can neither increase nor decrease length by much and the maps $u(0)\mapsto u(t)$ are uniformly bounded maps from $L^p(\mathbb{R})$ to itself for all $p\in[1,\infty]$. The case $p=\infty$ leads to the Haar inequalities but there are analogous estimates

$$\|u(t)\|_{L^p(\mathbb{R})} \;\le\; C\left(\|u(t)\|_{L^p(\mathbb{R})} \;+\; \int_0^t \|Lu(\sigma)\|_{L^p(\mathbb{R})}\,d\sigma\right),$$

with constant independent of p. This in turn leads to an existence theory like that just recounted but with $m_k(u,t)$ is replaced by $\sum_{|\alpha|\le k}\|\partial^\alpha_{t,x}u(t)\|_{L^p(\mathbb{R})}$. For the one dimensional case there is a wide class of spaces for which the evolution is well posed. The case of $p=1$ is particularly important for the theory of shock waves while it is only the case $p=2$ which remains valid for typical hyperbolic equations in dimension $d>1$.

§1.2. Examples of propagation of singularities using progressing waves.

D'Alembert's general solution of the one dimensional wave equation,

$$u_{tt} - u_{xx} = 0\,, \tag{1.2.1}$$

is the sum of progressing waves

$$f(x-t) + g(x+t)\,. \tag{1.2.2}$$

The rays are the integral curves of

$$\partial_t \pm \partial_x\,. \tag{1.2.3}$$

Structures are rigidly transported at speeds ±1.

There is an energy law. If u is a smooth solutions whose support intersects each time slab $a\le t\le b$ in a compact set, one has

$$\frac{d}{dt}\int_{\mathbb{R}} u_t^2 + u_x^2\,dx \;=\; \int \partial_t(u_t^2+u_x^2)\,dx \;=\; \int 2u_t(u_{tt}-u_{xx}) + \partial_x(2u_t\,u_x)\;dx \;=\; 0\,,$$

since the first summand vanishes and the second is the x derivative of a function vanishing outside a compact set.

The fundamental solution which solves (1.2.3) together with the initial values

$$u(0,x) = 0\,, \qquad u_t(0,x) = \delta(x)\,, \tag{1.2.4}$$

is given by the explicit formula

$$u(t,x) \;=\; \frac{\operatorname{sgn} t}{2}\,\chi_{[-t,t]} \;=\; \frac{1}{2}\,\Big(h(x-t) - h(x+t)\Big)\,, \tag{1.2.5}$$

where h denotes Heaviside's function, the characteristic function of $]0,\infty[$.

Exercise 1.2.1. i. *Derive (1.2.5) by solving the initial value problem using the Fourier transform in x.* **Hint.** You will likely decompose an expression regular at $\xi = 0$ into two which are not. Use a principal value to justify this step.

ii. *The proof of D'Alembert's formula (1.2.2) shows that every distribution solution of (1.2.1) is given by (1.2.2) for f, g distributions on $\mathbb{R}$. Derive (1.2.5) by finding the f, g which yield the solution of (1.2.4).* **Hint.** You will need to find the solutions of $du/dx = \delta(x)$.

The singularities of the solution (1.2.5) lie on the characteristic curves through $(0,0)$. This is a consequence of Theorem 1.1.4. In fact, define v as the solution of

$$v_{tt} - v_{xx} = 0, \qquad v(0,x) = 0, \qquad v_t(0,x) = x_+^2/2, \qquad x_+ := \max\{x, 0\}\,.$$

Introduce

$$V \;:=\; (v_1, v_2, v), \qquad v_1 := \partial_t v - \partial_x v, \quad v_2 := \partial_t v + \partial_x v\,.$$

to find

$$\partial_t V \;+\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \partial_x V \;+\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V \;=\; 0\,.$$

The Cauchy data, $V(0,x)$ are continuous, piecewise smooth, and singular only at $x = 0$. Theorem 1.1.4 shows that V is piecewise smooth with singularities only on the characteristics through $(0,0)$. In addition $u = \partial_x^3 v$ (in the sense of distributions) since they both satisfy the same initial value problem. Thus v and $u = \partial_x^3 v$ have singular support only on the characteristics through $(0,0)$.

Interesting things happen if one adds a lower order term. For example, consider the Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0\,. \tag{1.2.6}$$

In sharp contrast with (1.2.2), there are hardly any undistorted progressing wave solutions.

Exercise 1.2.3. *Find all solution of (1.2.6) of the form $f(x - ct)$ and all solutions of the form $e^{i(\tau t - x\xi)}$.* **Discussion.** *The solutions $e^{i(\tau t - x\xi)}$ with $\xi \in \mathbb{R}$ are particularly important since the general solution is a Fourier superposition of these special* **plane waves***. The equation $\tau = \tau(\xi)$ defining such solutions is called the* **dispersion relation** *of (3.1.6).*

There is an energy conservation law. Denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing smooth functions. That is, functions such that for all α, β

$$\sup_{x \in \mathbb{R}^d} \left| x^\beta \partial_x^\alpha \psi(x) \right| \;<\; \infty\,.$$

Exercise 1.2.4. *Prove that if* $u \in C^\infty(\mathbb{R} : \mathcal{S}(\mathbb{R}))$ *is a real valued solution of the Klein-Gordon equation, then*

$$\int u_t^2 + u_x^2 + u^2 \; dx$$

is independent of t. *This quantity is called the* **energy**. **Hint.** Justify carefully differentiation under the integral sign and integration by parts. If you find weaker hypotheses which suffice that would be good.

The solution of the Klein-Gordon equation with initial data (1.2.4), is not as simple as in the case of the wave equation. As for the wave equation, Theorem 1. A.4 implies that the singular support lies on $\{x = \pm t\}$. This proof is as for the wave equation except that the zeroth order term in the equation for V is replaced by

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} V .$$

The singularities are computed by the method of progressing waves. Introduce

$$h_n(x) \;:=\; \begin{cases} x^n/n! & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x \leq 0 \end{cases} . \tag{1.2.7}$$

Then

$$\frac{d}{dx}\, h_{n+1} = h_n \,, \qquad \text{for} \quad n \geq 0 \,. \tag{1.2.8}$$

Exercise 1.2.5. *Show that there are uniquely determined functions* $a_n(t)$ *satisfying*

$$a_0(0) = 1/2, \quad \text{and} \quad a_n(0) = 0 \quad \text{for} \quad n \geq 1 \,,$$

and so that for all $N \geq 2$,

$$\Big(\partial_t^2 - \partial_x^2 + 1\Big) \sum_{n=0}^{N} a_n(t)\, h_n(x-t) \in C^{N-2}(\mathbb{R}^2)\,. \tag{1.2.9}$$

In this case, we say that the series

$$\sum_{n=0}^{\infty} a_n(t)\, h_n(t-x)$$

is a formal solution of $(\partial_t^2 - \partial_x^2 + 1)u \in C^\infty$. **Hint.** Pay special attention to the most singular term(s). In particular show that, $\partial_t a_0 = 0$.

Exercise 1.2.6. *Suppose that* u *is the fundamental solution of the Klein-Gordon equation and* $M \geq 0$. *Find a distribution* w_M *such that* $u - w_M \in C^M(\mathbb{R}^2)$. *Show that the fundamental solution of the wave equation and that of the Klein-Gordon equation differ by a Lipschitz continuous function. Show that the singular supports of the two fundamental solutions are equal.* **Hint** Add (1.2.9) to its spatial reflection and choose initial values for the two solutions to match the initial data.

Exercise 1.2.7. *Study the fundamental solution for the dissipative wave equation*

$$u_{tt} - u_{xx} + 2u_t = 0\,. \tag{1.2.10}$$

Use Theorem 1.1.4 to show that the singular support is contained in the characteristics through $(0, 0$. *Show that it is not a continuous perturbation of the fundamental solution of the wave equation.* **Hint.** Find solutions of $(\partial_t^2 - \partial_x^2 + 2\partial_t)u \in C^\infty$ of the form $\sum_n b_n(t)\, h_n(t-x)$ as in Exercises 1.2.3 and 1.2.4 Use two such solutions as in Exercise 1.2.6.

The method **progressing wave expansions** from these examples is discussed in more generality in chapter 6 of Courant-Hilbert Vol. 2, and in Lax's *Lectures on Hyperbolic Partial Differential Equations*. The higher dimensional analogue of these solutions are singular along codimension one characteristic hypersurfaces in space time. The singularities propagate satisfying transport equations along rays generating the hypersurface. The general class goes under the name *conormal solutions*. They are discussed, for example, in M. Beals' book. They describe propagating wave-fronts. Luneberg [Lun] recognized that the propagation laws for fronts of singularities coincide with the classical laws of geometric optics.

§1.3. Group velocity and the method of nonstationary phase.

The Klein-Gordon equation has constant coefficients so can be solved explicitly using the Fourier transform. The computation of the singularities of the fundamental solution of the Klein-Gordon equation suggests that the main part of solutions travel with speed equal to 1. One might expect that the energy in a disk growing linearly in time at a speed slower than one would be small. For compactly supported data, such a disk would contain no singularities for large time. Thus it is not unreasonable to guess that for each $\sigma < 1$ and $R > 0$,

$$\limsup_{t\to\infty} \int_{|x|<R+\sigma t} u_t^2 + u_x^2 + u^2\, dx \;=\; 0. \tag{1.3.1}$$

The energy method shows that speeds are no larger than one. The idea about the main part of the solution expressed in (1.3.1) is dead wrong for the Klein-Gordon equation. The main part of the energy travels strictly slower than speed 1, even though singularities travel with speed exactly equal to 1.

The solution of the Cauchy problem for the Klein-Gordon equation in dimension d

$$u_{tt} \;-\; \Delta\, u \;+\; u \;=\; 0\,, \qquad (t,x) \;\in\; \mathbb{R}^{1+d}\,,$$

is given by,

$$u \;=\; \sum_{\pm} (2\pi)^{-d/2} \int a_\pm(\xi)\; e^{i(\pm\langle\xi\rangle t + x.\xi)}\; d\xi\,, \qquad \langle\xi\rangle := \big(1+|\xi|^2\big)^{1/2},$$

$$\hat u(0,\xi) \;=\; a_+(\xi) + a_-(\xi)\,, \qquad \hat u_t(0,\xi) \;=\; i\,\langle\xi\rangle\, \big(a_+(\xi) - a_-(\xi)\big)\,.$$

The energy is equal to

$$\frac12 \int u_t^2 \;+\; |\nabla_x u|^2 \;+\; u^2\; dx \;=\; \int \langle\xi\rangle^2 \left(|a_+(\xi)|^2 \;+\; |a_-(\xi)|^2\right)\, d\xi\,.$$

Exercise 1.3.1. *Verify these formulas. Verify conservation of energy by an integration by parts argument as in Exercise 1.2.4.* **Hint.** Follow the computation that starts §1.4.

Consider the behavior for large times. The phases $\phi_\pm(t,x,\xi) = \pm\langle\xi\rangle t + x.\xi$ have gradients

$$\nabla_\xi \phi_\pm(t,x,\xi) \;:=\; \nabla_\xi\Big(\pm\langle\xi\rangle t + x.\xi\Big) \;=\; \frac{\pm t\xi}{\langle\xi\rangle} + x \;=\; t\Big(\frac{\pm\xi}{\langle\xi\rangle} + \frac{x}{t}\Big)\,.$$

At space time points (t,x) so that $t >> 1$ and

$$\frac{\pm\xi}{\langle\xi\rangle} + \frac{x}{t} \ \neq\ 0\,,$$

the phase oscillates rapidly and the contribution to the integral is expected to be small. The contribution to the $a_\pm$ integral from $\xi \sim \underline{\xi}$ is felt predominantly at points where $x/t \sim \mp\underline{\xi}/\langle\underline{\xi}\rangle$. Setting $\tau_\pm(\xi) := \pm\langle\xi\rangle$ one has

$$\frac{\mp\underline{\xi}}{\langle\underline{\xi}\rangle} \ =\ -\nabla_\xi\tau_\pm(\underline{\xi})\,.$$

This agrees with the formula for the group velocity (re)introduced on purely geometric grounds in §2.4.

For $t \to \infty$ the contributions of the plane waves $a_\pm(\xi)e^{i(\tau_\pm(\xi)t+x,\xi)}$ with $\xi \sim \underline{\xi}$ are expected to be felt at points with $x/t \sim -\nabla_\xi\tau_\pm(\underline{\xi})$. A precise version is proved using the method of nonstationary phase.

Proposition 1.3.1. *Suppose that $a_\pm(\xi) \in \mathcal{S}(\mathbb{R}^d)$ and define*

$$\mathbf{V} \ :=\ \big\{\mathbf{v}\ :\ \mathbf{v} = -\nabla_\xi\tau_\pm(\xi)\ \ \text{for some}\ \xi\in \operatorname{supp} \mathrm{a}_\pm\big\}$$

as the closed set of group velocities that appear in the plane wave decomposition of u. For $\mu > 0$ let $\mathbf{K}_\mu \subset \mathbb{R}^d$ denote the set of points at distance $\geq \mu$ from $\mathbf{V}$. Denote by Γ_μ the cone

$$\Gamma_\mu \ :=\ \Big\{(t,x)\ :\ t>0,\ \ \text{and}\ \ x/t\in\mathbf{K}_\mu\Big\}\,.$$

Then for all $N > 0$ and α,

$$(1+t+|x|)^N\ \partial^\alpha_{t,x}u(t,x) \in L^\infty(\Gamma_\mu)\,.$$

Proof. The solution u is smooth with values in $\mathcal{S}$ so one need only consider $\{t \geq 1\}$. We estimate the $+$ summand. The $-$ summand can be treated similarly.

Introduce the first order differential operator

$$\ell(t,x,\partial)\ :=\ \frac{1}{i|\nabla_\xi\phi|^2}\sum_j \frac{\partial\phi}{\partial\xi_j}\,\frac{\partial}{\partial\xi_j}\,,\qquad \text{so},\qquad \ell(t,x,\partial_\xi)\,e^{i\phi}\ =\ e^{i\phi}\,.$$

The coefficients are smooth functions on a neighborhood of Γ_μ, and are homogeneous of degree minus one in (t,x) and satisfy

$$\frac{1}{|\nabla_\xi\phi|^2}\,\Big|\frac{\partial\phi}{\partial\xi_j}\Big|\ \leq\ C(t+|x|)^{-1}\quad \text{for}\quad (t,x,\xi)\in\Gamma_\mu\times\ \operatorname{supp} a_+\,.$$

The identity $\ell\,e^{i\phi} = e^{i\phi}$ implies,

$$\int a_+(\xi)\ e^{i\phi}\ d\xi = \int a_+(\xi)\ \ell^N e^{i\phi}\ d\xi\,.$$

Denote by $\ell^\dagger$ the transpose of ℓ and integrate by parts to find

$$\int a_+(\xi)\ e^{i\phi}\ d\xi = \int [(\ell^\dagger)^N a_+(\xi)]\ e^{i\phi}\ d\xi\,.$$

The operator

$$(\ell^\dagger)^N \ = \ \sum_{|\alpha|\le N} c_\alpha(t,x,\xi)\,\partial_\xi^\alpha$$

with coefficients c_α smooth on a neighborhood of Γ_μ, homogeneous of degree $-N$ in t,x, with

$$|c_\alpha(t,x)| \ \le\ C(\alpha)(t+|x|)^{-N} \quad \text{for} \quad (t,x,\xi)\in\Gamma_\mu\times\ \operatorname{supp} a_+\,.$$

It follows that

$$\Big|\int a_+(\xi)\ e^{i\phi}\ d\xi\Big| \ \le\ C\,(t+|x|)^{-N}\,.$$

Since the t,x derivatives of this integral are again integrals of the same form, this suffices to prove the proposition. ∎

Introduce for $\mu << 1$, $\tilde{\mathbf{V}}_\mu := \mathbb{R}^d\setminus\mathbf{K}_\mu$ an open set slightly larger than $\mathbf{V}$. For $t\to\infty$ virtually all the energy of a solution is contained in the cone $\big\{(t,x)\ :\ x/t\in\tilde{\mathbf{V}}\big\}$. This is particularly interesting when $a_\pm$ are supported in a small neighborhood of a fixed $\underline{\xi}$. For large times virtually all the energy is localized in a small conic neighborhood of the pair of lines $x=-t\,\nabla_\xi\tau_\pm(\underline{\xi})$ that travel with the group velocities associated to $\underline{\xi}$.

The integration by parts method introduced in this proof is very important. The next estimate for nonstationary oscillatory integrals is a straight forward application. The fact that the estimate is uniform in the phases is useful.

Lemma of Nonstationary Phase 1.3.2. *Suppose that Ω is a bounded open subset of $\mathbb{R}^d$ and that $C_1>1$. Then there is a constant $C_2>0$ so that for all $\forall f\in C_0^m(\Omega)$, and $\phi\in C^m(\Omega\,;\mathbb{R})$ satisfying*

$$\forall|\alpha|\le m\,,\quad \|\partial^\alpha\phi\|_{L^\infty}\le C_1\,,\quad \text{and,}\quad \forall x\in\Omega\,,\quad C_1^{-1}\ \le|\nabla_x\phi|\ \le\ C_1\,,$$

one has the estimate,

$$\Big|\int\ e^{i\phi/\epsilon}\ f(x)\ dx\Big| \ \le\ C_2\,\epsilon^m \sum_{|\alpha|\le m}\|\partial^\alpha f\|_{L^1}\,.$$

Exercise 1.3.2. *Prove the Lemma.* **Hint.** Use

$$\ell(x,\partial_x) \ :=\ \frac{\nabla_x\phi}{i\,|\nabla_x\phi|^2}\,.\,\partial_x\,.$$

Example. A special case are the phases $\phi=x.\xi$ with ξ belonging to a compact subset of $\mathbb{R}^d\setminus 0$. The Lemma is then equivalent to the rapid decay of the Fourier transform of smooth compactly supported functions. That decay is proved by integration by parts. The general result can be reduced to the special case of the Fourier transform. Since the gradient of ϕ does not vanish, for each $\underline{x}\in\operatorname{supp} f$ there is a neighborhood and a nonlinear change of coordinates so that in the new coordinates ϕ is equal to x_1. Using a partition of unity, one can suppose that f is the sum of a finite number of functions each supported in one of the neighborhoods. For each such function, a change of coordinates yields an integral of the form

$$\int e^{ix_1/\epsilon}\, g(x)\, dx \;=\; c\,\hat{g}(1/\epsilon, 0, \ldots, 0)\,,$$

which is rapidly decaying since it is the transform of an element of $C_0^\infty(\mathbb{R}^d)$.

Exercise 1.3.3. *Suppose that $f \in H^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ and that u is the unique solution of the Klein-Gordon equation with initial data*

$$u(0,x) = f(x)\,, \qquad u_t(0,x) = g(x)\,. \tag{1.3.2}$$

Prove that for any $\epsilon > 0$ and $R \geq 0$, there is a $\delta > 0$ so that

$$\limsup_{t\to\infty} \int_{|x|>(1-\delta)t-R} u_t^2 + u_x^2 + u^2\, dx \;<\; \epsilon\,. \tag{1.3.3}$$

Hint. Replace $\hat{f}, \hat{g}$ by compactly support smooth functions making an error at most $\epsilon/2$ in energy. Then use the above proposition noting that the group velocities are uniformly smaller than one on the supports of $a_\pm$.

Discussion. Note that as $\xi \to \infty$ the group velocities approach ± 1. High frequencies will propagate at speeds nearly equal to one. In particular they travel at the same speed. High frequency signals stay together better than low frequency signals. Since singularities of solutions are made of only the high frequencies (modifying the data by an element of $\mathcal{S}$ modifies the solution by such an element and therefore by a smooth term) one expects singularities to propagate at speeds ± 1 which is exactly what is true for the fundamental solution. Once known for the fundamental solution it follows for all. The simple proof is an exercise in my book [Rauch 1992, pg. 164-165].

The analysis of Exercise 1.3.3 does not apply to the fundamental solution since the latter does not have finite energy. However it belongs to $C^j(\mathbf{R} : H^{s-j}(\mathbb{R}))$ for all $j \in \mathbb{N}$ and $s < 1/2$. Thus the next result provides a good replacement of (1.3.3).

Exercise 1.3.4. *Suppose that u is the fundamental solution of the Klein-Gordon equation (1.1.6) and that $s < 1/2$. If $0 \leq \chi \in C^\infty(\mathbb{R})$ is a plateau cutoff supported on the positive half line, that is*

$$\chi(x) = 0 \quad \text{for} \quad x \leq 0 \qquad \text{and} \quad \chi(x) = 1 \quad \text{for} \quad x \geq 1\,,$$

then for all $R \geq 0$,

$$\lim_{t\to\infty} \|\, \chi(R + |x| - t)\, u(t,x)\, \|_{H^s(\mathbb{R}_x)} = 0\,. \tag{1.3.4}$$

Hint. Prove that

$$\|\, \chi\, u(t)\, \|_{H^s(\mathbb{R})} \leq C\Big(\, \|u(0)\|_{H^s(\mathbb{R})} + \|u_t(0)\|_{H^{s-1}(\mathbb{R})}\Big)$$

with C independent of t and the initial data. Conclude that it suffices to prove (3.2.4) with initial data $u(0), u_t(0)$ dense in $H^s \times H^{s-1}$. Take the dense set to be data with Fourier Transform in $C_0^\infty(\mathbb{R})$.

These examples illustrate the important observation that the propagation of singularities in solutions and the propagation of the majority of the energy may be governed by different rules. For the Klein Gordon equation at least, both answers can be determined from considerations of group velocities.

§1.4. Fourier synthesis and rectilinear propagation.

For equations with constant coefficients, solutions of the initial value problem are expressed as Fourier integrals. Injecting short wavelength initial data and performing an asymptotic analysis yields the approximations of geometric optics. This is how such approximations were first justified in the nineteenth century. It is also the motivating example for the more general theory. The short wavelength approximations explain the *rectilinear propagation of waves* in homogeneous media. This is the first of the three basic physical laws of geometric optics. It explains, among other things, the formation of shadows. The short wavelength solutions are also the building blocks in the analysis of the laws of reflection and refraction.

Consider the initial value problem

$$\Box u \;:=\; u_{tt} - \Delta\, u \;:=\; \frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} \;=\; 0\,, \qquad u(0,x) = f\,, \quad u_t(0,x) = g\,. \tag{1.4.1}$$

Fourier transformation with respect to the x variables yields

$$\partial_t^2 \hat u(t,\xi) \;+\; |\xi|^2\, \hat u(t,\xi) \;=\; 0\,, \qquad \hat u(0,\xi) = \hat f\,, \quad \partial_t \hat u(0,\xi) = \hat g\,.$$

Solve the ordinary differential equations in t to find

$$\hat u(t,\xi) \;=\; \hat f(\xi)\, \cos t|\xi| + \hat g(\xi)\, \frac{\sin t|\xi|}{|\xi|}\,.$$

Write

$$\cos t|\xi| \;=\; \frac{e^{it|\xi|} + e^{-t|\xi|}}{2}, \qquad \sin t|\xi| \;=\; \frac{e^{it|\xi|} - e^{-t|\xi|}}{2i},$$

to find

$$\hat u(t,\xi) \;=\; a_+(\xi)\, e^{i(x\xi - t|\xi|)} \;-\; a_-(\xi)\, e^{i(x\xi + t|\xi|)}\,, \tag{1.4.2}$$

with,

$$2\, a_+ \;:=\; \hat f \;+\; \frac{\hat g}{i|\xi|}\,, \qquad 2\, a_- \;:=\; \hat f \;-\; \frac{\hat g}{i|\xi|}\,. \tag{1.4.3}$$

The right hand side of (1.4.2) is an expression in terms of the plane waves $e^{i(x\xi \mp t|\xi|)}$ with amplitudes $a_\pm(\xi)$ and dispersion relations $\tau = \mp|\xi|$. The group velocities associated to $a_\pm$ are

$$\mathbf{v} \;=\; -\nabla_\xi \tau \;=\; -\nabla_\xi(\mp|\xi|) \;=\; \pm\frac{\xi}{|\xi|}\,.$$

The solution is the sum of two terms,

$$u_\pm(t,x) \;:=\; \frac{1}{(2\pi)^{d/2}} \int a_\pm(\xi)\; e^{i(x\xi \mp t|\xi|)}\, d\xi\,.$$

Using $\mathcal{F}\big(\partial u/\partial x_j\big) = i\xi_j \hat u$, and Parseval's Theorem shows that the conserved energy for the wave equation is equal to

$$\frac12 \int |u_t(t,x)|^2 \;+\; |\nabla_x u(t,x)|^2\; dx \;=\; \int |\xi|^2 \big(|a_+(\xi)|^2 \;+\; |a_-(\xi)|^2\big)\; d\xi\,.$$

There are conservations of all orders. Each of the following quantities is independent of time,

$$\frac{1}{2}\,\|\nabla_{t,x}u(t)\|^2_{H^s(\mathbb{R}^d)} \;=\; \int \langle\xi\rangle^{2s}\ |\xi|^2\ \big(|a_+(\xi)|^2\ +\ |a_-(\xi)|^2\big)\ d\xi\,.$$

Consider initial data a wave packet with wavelength of order ϵ and phase equal to x_1/ϵ,

$$u^\epsilon(0,x) \;=\; \gamma(x)\,e^{ix_1/\epsilon}\,, \qquad u^\epsilon_t(0,x) \;=\; 0\,, \qquad \gamma\ \in\ \cap_s H^s(\mathbb{R}^d)\,. \tag{1.4.4}$$

The choice $u_t = 0$ postpones dealing with the factor $1/|\xi|$ in (1.4.3). The initial value is an envelope or profile γ multiplied by a rapidly oscillating exponential.
Applying (1.4.3) with $g = 0$ and with

$$\hat f(\xi) \;=\; \hat u(0,\xi) \;=\; \mathcal{F}\big(\gamma(x)\,e^{ix_1/\epsilon}\big) \;=\; \hat\gamma(\xi - \mathbf{e}_1/\epsilon)\,,$$

yields $u = u_+ + u_-$ with,

$$u^\epsilon_\pm(t,x) \;:=\; \frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\xi - \mathbf{e}_1/\epsilon)\ e^{i(x\xi\mp t|\xi|)}\,d\xi\,.$$

Analyse u^ϵ_+. The other term is analogous. For ease of reading, the subscript plus is omitted. Introduce

$$\zeta\ :=\ \xi\ -\ \mathbf{e}_1/\epsilon, \qquad \xi\ =\ \frac{\mathbf{e}_1\ +\ \epsilon\zeta}{\epsilon}\,,$$

to find,

$$u^\epsilon(t,x) \;=\; \frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\ e^{ix(\mathbf{e}_1+\epsilon\zeta)/\epsilon}\ e^{-it|\mathbf{e}_1+\epsilon\zeta|/\epsilon}\ d\zeta\,. \tag{1.4.5}$$

The approximation of geometric optics comes from injecting the first order Taylor approximation,

$$\big|\mathbf{e}_1 + \epsilon\zeta\big| \;\approx\; 1 + \epsilon\zeta_1\,,$$

yielding,

$$u^\epsilon_{\rm approx} \;:=\; \frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\ e^{ix(\mathbf{e}_1+\epsilon\zeta)/\epsilon}\ e^{-it(1+\epsilon\zeta_1)/\epsilon}\ d\zeta\,.$$

Collecting the rapidly oscillating terms $e^{i(x_1-t)/\epsilon}$ which do not depend on ζ gives,

$$u_{\rm approx} \;=\; e^{i(x_1-t)/\epsilon}\,a(t,x), \qquad a(t,x)\;:=\;\frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\,e^{i(x\zeta - t\zeta_1)}\,d\zeta\,. \tag{1.4.6}$$

Write $x - t\zeta_1 = (x - t\mathbf{e}_1).\zeta$ to find,

$$a(t,x) \;=\; \frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\,e^{i(x-t\mathbf{e}_1)\zeta}\,d\zeta \;=\; \frac{1}{2}\,\gamma(x - t\mathbf{e}_1)\,.$$

The approximation is a wave translating rigidly with velocity equal to $\mathbf{e}_1$. The waveform γ is arbitrary. The approximate solution resembles the collumnated light from a flashlight. If the support of γ is small the approximate solution resembles a light ray.

The amplitude a satisfies the transport equation

$$\frac{\partial a}{\partial t} + \frac{\partial a}{\partial x_1} \ = \ 0$$

so is constant on the **rays** $x = \underline{x} + t\mathbf{e}_1$. The construction of a family of short wavelength approximate solutions of D'Alembert's wave equations requires only the solutions of a simple transport equation.

The dispersion relation of the family of plane waves,

$$e^{i(x.\xi+\tau t)} \ = \ e^{i(x.\xi-|\xi|t)},$$

is $\tau = -|\xi|$. The velocity of transport, $\mathbf{v} = (1, 0, \ldots, 0)$, is the group velocity $\mathbf{v} = -\nabla_\xi \tau(\underline{\xi}) = \underline{\xi}/|\underline{\xi}|$ at $\underline{\xi} = (1, 0, \ldots, 0)$. For the opposite choice of sign the dispersion relation is $\tau = |\xi|$, the group velocity is $-\mathbf{e}_1$, and the rays are the lines $x = \underline{x} - t\mathbf{e}_1$.

Had we taken data with oscillatory factor $e^{ix.\xi/\epsilon}$ then the propagation would be at the group velocity $\pm\xi/|\xi|$. The approximate solution would be

$$\frac{1}{2}\left(e^{i(x.\xi - t|\xi|)/\epsilon}\,\gamma\Big(x - t\frac{\xi}{|\xi|}\Big) \ + \ e^{i(x.\xi+t|\xi|)/\epsilon}\,\gamma\Big(x + t\frac{\xi}{|\xi|}\Big)\right).$$

The approximate solution (1.4.6) is a function $H(x - t\mathbf{e}_1)$ with $H(x) = e^{ix_1/\epsilon}\,h(x)$. When h has compact support or more generally tends to zero as $|x| \to \infty$ the approximate solution is localized and has velocity equal to $\mathbf{e}_1$. The next result shows that when $d > 1$, no exact solution can have this form. In particular the distribution $\delta(x - \mathbf{e}_1 t)$ which is the most intuitive notion of a light ray is **not** a solution of the wave equation or Maxwell's equation.

Proposition 1.4.1. *If $d > 1$, $s \in \mathbb{R}$, $K \in H^s(\mathbb{R}^d)$ and $u = K(x - \mathbf{e}_1 t)$ satisfies $\Box u = 0$, then $K = 0$.*

Exercise 1.4.1. *Prove Proposition 1.4.1.* **Hint.** Prove and use a Lemma. **Lemma.** *If $k \le d$, $s \in \mathbb{R}$, and, $w \in H^s(\mathbb{R}^d)$ satisfies $0 = \sum_k^d \partial^2 w/\partial^2 x_j$, then $w = 0$.*

Next, analyse the error in (1.4.6). The first step is to extract the rapidly oscillating factor in (1.4.5) to define an exact amplitude $a^\epsilon_{\rm exact}$,

$$u^\epsilon(t,x) = e^{i(x_1 - t)/\epsilon}\ a_{\rm exact}(\epsilon, t, x)\,,$$

$$a_{\rm exact}(\epsilon, t, x) \ := \ \frac{1}{(2\pi)^{d/2} 2} \int \hat\gamma(\zeta)\ e^{ix.\zeta}\ e^{-it(|\mathbf{e}_1 + \epsilon\zeta| - 1)/\epsilon}\ d\zeta\,. \tag{1.4.7}$$

Proposition 1.4.2. *The exact and approximate solutions of $\Box u^\epsilon = 0$ with Cauchy data (1.4.4) are given by*

$$u^\epsilon = \sum_\pm e^{i(x_1 \mp t)/\epsilon}\ a^\pm_{\rm exact}(\epsilon, t, x)\,, \qquad u^\epsilon_{\rm approx} = \sum_\pm e^{i(x_1 \mp t)/\epsilon}\ \frac{\gamma(x \mp \mathbf{e}_1 t)}{2}\,,$$

as in (1.4.7) and (1.4.6). The error is $O(\epsilon)$ on bounded time intervals. Precisely, there is a constant $C > 0$ so that for all s, ϵ, t,

$$\Big\| a^\pm_{\rm exact}(\epsilon, t, x) \ - \ \frac{\gamma(x \mp \mathbf{e}_1 t)}{2} \Big\|_{H^s(\mathbb{R}^N)} \ \le \ C\,\epsilon\,|t|\,\big\|\gamma\big\|_{H^{s+2}(\mathbb{R}^d)}\,.$$

Proof. It suffices to estimate the error with the plus sign. The definitions yield

$$a^+_{\rm exact}(\epsilon,t,x) \ - \ \gamma(x-\mathbf{e}_1 t)/2 \ = \ C\int \hat\gamma(\zeta)\ e^{ix.\zeta}\ \big(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon)} - e^{-it\zeta_1}\big)\ d\zeta\,.$$

The definition of the $H^s(\mathbb{R}^d)$ norm yields

$$\Big\|a^+_{\rm exact}(\epsilon,t,x) \ - \ \gamma(x-\mathbf{e}_1 t)/2\Big\|_{H^s(\mathbb{R}^N)} \ = \ \Big\|\langle\zeta\rangle^s\ \hat\gamma(\zeta)\ \big(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} - e^{-it\zeta_1}\big)\Big\|_{L^2(\mathbb{R}^N)}.$$

Taylor expansion yields for $|\beta|\le 1/2$,

$$|\,\mathbf{e}_1+\beta\,| \ = \ 1+\beta_1+r(\beta)\,, \qquad |r(\beta)|\le C\,|\beta|^2\,.$$

Increasing C if needed, the same inequality is true for $|\beta|\ge 1/2$ as well.
Applied to $\beta=\epsilon\zeta$ this yields,

$$\Big|t\big(\big|\mathbf{e}_1+\epsilon\zeta\big|-1\big)/\epsilon - \zeta_1 x_1\Big| \ \le \ C\,\epsilon\,|t|\,|\zeta|^2\,,$$

so

$$\Big|e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} \ - \ e^{-it\zeta_1}\Big| \ \le \ C\,\epsilon\,|t|\,|\zeta|^2\,.$$

Therefore

$$\Big\|\langle\zeta\rangle^s\ \hat\gamma(\zeta)\ \big(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} - e^{-it\zeta_1}\big)\Big\|_{L^2(\mathbb{R}^d)} \ \le \ C\,\epsilon\,|t|\,\Big\|\langle\zeta\rangle^s|\zeta|^2\ \hat\gamma(\zeta)\Big\|_{L^2}. \tag{1.4.8}$$

Combining (1.4.7-1.4.8) yields the estimate of the Proposition. ∎

The approximation retains some accuracy so long as $t=o(1/\epsilon)$.

The approximation has the following geometric interpretation. One has a superposition of plane waves $e^{i(x\xi+t|\xi|)}$ with $\xi=(1/\epsilon,0,\dots,0)+O(1)$. Replacing ξ by $(1/\epsilon,0,\dots,0)$ and $|\xi|$ by $1/\epsilon$ in the plane waves yields the approximation (1.4.6).

The wave vectors, ξ, make an angle $O(\epsilon)$ with $\mathbf{e}_1$. The corresponding rays have velocities which differ by $O(\epsilon)$ so the rays remain close for times small compared with $1/\epsilon$. For longer times the fact that the group velocities are not parallel is important. The wave begins to spread out. Parallel group velocities is a reasonable approximation for times $t=o(1/\epsilon)$.

The example reveals several scales of time. For times $t<<\epsilon$, u and its gradient are well approximated by their initial values. For times $\epsilon<<t<<1$ $u\approx e^{i(x-t)/\epsilon}a(0,x)$. The solution begins to oscillate in time. For $t=O(1)$ the approximation $u\approx a(t,x)\,e^{i(x-t)/\epsilon}$ is appropriate. For times $t=O(1/\epsilon)$ the approximation ceases to be accurate. The more refined approximations valid on this longer time scale are called *diffractive geometric optics*. The reader is referred to [Donnat, Joly Métiver, and Rauch] for an introduction in the spirit of Chapters 7-8.

It is typical of the approximations of geometric optics, that

$$\Box\big(u_{\rm approx}-u_{\rm exact}\big) \ = \ \Box u_{\rm approx}=O(1)\,,$$

is not small. The error $u_{\rm approx}-u_{\rm exact}=O(\epsilon)$ is smaller by a factor of ϵ. The residual $\Box u_{\rm approx}$ is rapidly oscillatory, so applying $\Box^{-1}$ gains the factor ϵ.

The analysis just performed can be carried out without fundamental change for initial oscillations with nonlinear phase. A nice description including the phase shift on crossing a focal point can be found in [Hörmander 1983, §12.2].

Next the approximation is pushed to higher accuracy with the result that the residuals can be reduced to $O(\epsilon^N)$ for any N. Taylor expansion to higher order yields,

$$|\mathbf{e}_1 + \eta| \;=\; 1 + \eta_1 + \sum_{|\alpha|\geq 2} c_\alpha \eta^\alpha\,, \qquad |\eta| < 1, \tag{1.4.9}$$

so

$$\big(|\mathbf{e}_1 + \epsilon\zeta| - 1\big)/\epsilon \sim \zeta_1 + \sum_{|\alpha|\geq 2} \epsilon^{|\alpha|-1}\, c_\alpha\, \zeta^\alpha,$$

$$e^{it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} \;\sim\; e^{it\zeta_1}\, e^{\sum_{|\alpha|\geq 2} it\epsilon^{|\alpha|-1}\, c_\alpha \zeta^\alpha} \;\sim\; e^{it\zeta_1}\Big(1 + \sum_{j\geq 1} \epsilon^j\, h_j(t,\zeta)\Big)\,.$$

Here, $h_j(t,\zeta)$ is a polynomial in t,ζ. Injecting in the formula for $a_{\rm exact}(\epsilon,t,x)$ yields an expansion

$$a_{\rm exact}(\epsilon,t,x) \;\sim\; a_0(t,x) + \epsilon\, a_1(t,x) + \epsilon^2 a_2(t,x) + \cdots\,, \qquad a_0(t,x) = \gamma(x - \mathbf{e}_1 t)/2, \tag{1.4.10}$$

$$a_j \;=\; \frac{1}{(2\pi)^{-d/2}\, 2}\int \hat\gamma(\zeta)\; e^{i(x\zeta - t\zeta_1)}\;\; h_j(t,\zeta)\; d\zeta \;=\; \frac{1}{2}\,\big(h_j(t,\partial/i)\gamma\big)(x - \mathbf{e}_1 t)\,. \tag{1.4.11}$$

The series is asymptotic as $\epsilon \to 0$ in the sense of Taylor series. For any s, N, truncating the series after N terms yields an approximate amplitude which differs from $a_{\rm exact}$ by $O(\epsilon^{N+1})$ in L^2 *uniformly on compact time intervals*. The H^s error for $s \geq 0$ is $O(\epsilon^{N+1-s})$.

Exercise 1.4.2. *Compute the precise form of the first corrector* a_1.

Formula (1.4.11) implies that if the Cauchy data are supported in a set $\mathcal{O}$, then the amplitudes a_j are all supported in the tube of rays

$$\mathcal{T} \;:=\; \Big\{(t,x)\;:\; x = \underline{x} + t\mathbf{e}_1, \quad \underline{x} \in \mathcal{O}\Big\}\,. \tag{1.4.12}$$

Warning. Though the a_j are supported in this tube, it is not true that $a^\epsilon_{\rm exact}$ is supported in the tube. The map $\epsilon \mapsto a_{\rm exact}(\epsilon,t,x)$ is not analytic. If it were, the Taylor series would converge to the exact solution which would then have support in the tube. When $d \geq 2$, the function $u = 0$ is the only solution of D'Alembert's equation with support in a tube of rays with cross section of finite d dimensional Lebesgue measure. This follows from the fact that for finite energy solutions, the energy in the tube tends to zero. †

To analyse the oscillatory initial value problem with $u(0) = 0$, $u_t(0) = \beta(x)\, e^{ix_1/\epsilon}$ requires one more idea to handle the contributions from $\xi \approx 0$ in the expression

$$u(t,x) \;=\; (2\pi)^{-d/2}\int \frac{\sin t|\xi|}{|\xi|}\;\hat\beta\Big(\xi - \frac{\mathbf{e}_1}{\epsilon}\Big)\; e^{ix\xi}\; d\xi\,.$$

† This is proved by approximation by regular solutions. For Cauchy data in $C_0^\infty(\mathbb{R}^d)$, the energy in the tube is $O(t^{(1-d)})$. This can be proved using the fundamental solution. Alternatively, if the Fourier transform of the Cauchy data belongs to $C_0^\infty(\mathbb{R}^d_\xi \setminus 0)$ one has the same estimate using the inequality of stationary phase from Appendix 3.II (see Lemma 3.4.2).

Choose $\chi \in C_0^\infty(\mathbb{R}^d_\xi)$ with $\chi = 1$ on a neighborhood of $\xi = 0$. The cutoff integrand is equal to

$$\chi(\xi)\ \frac{\sin t|\xi|}{|\xi|}\ \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s}\ k_s(\xi - \mathbf{e}_1/\epsilon)\ e^{ix\xi}\,, \qquad k_s(\xi)\ :=\ \langle\xi\rangle^s\ \hat\beta(\xi)\ \in\ L^2(\mathbb{R}^d_\xi)\,.$$

A simple upper bound is,

$$\Big\|\chi(\xi)\ \frac{\sin t|\xi|}{|\xi|}\ \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s}\Big\|_{L^\infty(\mathbb{R}^d)}\ \le\ C_s\,|t|\,\epsilon^s\,, \qquad 0<\epsilon\le 1\,.$$

It follows that

$$\Big\|\chi(\xi)\ \frac{\sin t|\xi|}{|\xi|}\ \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s}\ k_s(\xi - \mathbf{e}_1/\epsilon)\ \Big\|_{L^2(\mathbb{R}^d)}\ \le\ C_s\,|t|\,\epsilon^s\big\|\beta\big\|_{H^s(\mathbb{R}^d)}\,.$$

The small frequency contribution is negligable in the limit $\epsilon \to 0$. It is removed with a cutoff as above and then the analysis away from $\xi = 0$ proceeds by decomposition into plane wave as in the case with $u_t(0) = 0$. It yields left and right moving waves with the same phases as before.

Exercise 1.4.3. *Solve the Cauchy problem for the anisotropic wave equation, $u_{tt} = u_{xx} + 4u_{yy}$ with initial data given by*

$$u^\epsilon(0,x)\ =\ \gamma(x)\,e^{ix.\xi/\epsilon}\,, \qquad u^\epsilon_t(0,x)\ =\ 0\,, \qquad \gamma\ \in\ \cap_s H^s(\mathbb{R}^d)\,.$$

Find the leading term in the approximate solution to u_+. In particular, find the velocity of propagation as a function of ξ. **Discussion.** The velocity is equal to the group velocity from §1.3.

§1.5. A cautionary example in geometric optics.

A typical science text discussion of a mathematics problem involves simplifying the underlying equations. The usual criterion applied is to ignore terms which are small compared to other terms in the equation. It is striking that in many of the problems treated under the rubric of geometric optics, such an approach can lead to completely inaccurate results. It is an example of an area where more careful mathematical consideration is not only useful but necessary.

Consider the initial value problems

$$\partial_t u^\epsilon + \partial_x u^\epsilon + u^\epsilon\ =\ 0\,, \qquad u^\epsilon\big|_{t=0} = a(x)\cos(x/\epsilon)\,,$$

in the limit $\epsilon \to 0$. The function a is assumed to be smooth and to vanish rapidly as $|x| \to \infty$ so the initial value has the form of wave packet. The initial value problem is uniquely solvable and the solution depends continuously on the data. The exact solution of the general problem

$$\partial_t u + \partial_x u + u\ =\ 0\,, \qquad u\big|_{t=0} = f(x)\,,$$

is $u(t,x) = e^{-t}\,f(x-t)$ so the exact solution u^ϵ is

$$u^\epsilon(t,x)\ =\ e^{-t}\,a(x-t)\ \cos((x-t)/\epsilon)\,.$$

In the limit as $\epsilon \to 0$ one finds that both $\partial_t u^\epsilon$ and $\partial_x u^\epsilon$ are $O(1/\epsilon)$ while $u^\epsilon = O(1)$ is negligibly small in comparison. Dropping this small term leads to the simplified equation for an approximation v^ϵ,

$$\partial_t v^\epsilon + \partial_x v^\epsilon\ =\ 0\,, \qquad v^\epsilon\big|_{t=0} = a(x)\cos(x/\epsilon)\,.$$

The exact solution is

$$v^\epsilon(t,x) \;=\; a(x-t)\cos\big((x-t)/\epsilon\big)\,,$$

which misses the exponential decay. It is **not** a good approximation. The two large terms compensate so that the small term is not negligible compared to their sum.

§1.6. The law of reflection.

Consider the wave equation $\Box u = 0$ in the half space $\mathbb{R}^d_- := \{x_1 \le 0\}$. At $\{x_1 = 0\}$ a boundary condition is required. The condition encodes the physics of the interaction with the boundary.

Since the differential equation is of second order one might guess that two boundary conditions are needed as for the Cauchy problem. An analogy with the Dirichlet problem for the Laplace equation suggests that one condition is required.

A more revealing analysis concerns the case of dimension $d = 1$. D'Alembert's formula shows that at all points of space time the solution consists of the sum of two waves one moving toward the boundary and the other toward the interior. The waves approaching the boundary will propagate to the edge of the domain. At the boundary one does not know what values to give to the waves which move into the domain. The boundary condition must give the value of the incoming wave in terms of the outgoing wave. That is one boundary condition.

Factoring

$$\partial_t^2 - \partial_x^2 \;=\; (\partial_t - \partial_x)(\partial_t + \partial_x) \;=\; (\partial_t + \partial_x)(\partial_t - \partial_x),$$

shows that $(\partial_t - \partial_x)(u_t + u_x) = 0$ so $u_t + u_x$ is transported to the left. Similarly, $u_t - u_x$ moves to the right. Thus from the initial conditions, $u_t - u_x$ is determined everywhere in $x \le 0$ including the boundary $x = 0$. The boundary condition at $\{x = 0\}$ must determine $u_t + u_x$. The conclusion is that half of the information needed to find all the first derivatives is already available and one needs only one boundary condition.

For the Dirichlet condition,

$$u(t,x)\big|_{x_1=0} = 0\,. \tag{1.6.1}$$

Differentiating (1.6.1) with respect to t shows that $u_t(t,0) = 0$, so at $t = 0$ $(u_t + u_x) = -(u_t - u_x)$ showing that at the boundary, the incoming wave is equal to -1 times the outgoing wave.

In the case $d \ge 1$ consider the Cauchy data,

$$u(0,x) = f\,, \qquad u_t(0,x) = g\,, \quad \text{for} \quad x_1 \le 0\,. \tag{1.6.2}$$

If the data are supported in a compact subset of $\mathbb{R}^d_-$ then, for small time the support of the solution does not meet the boundary. When waves hit the boundary they are reflected. The goal of this section is to describe this reflection process.

Uniqueness of solutions and finite speed of propagation for (1.6.1)-(1.6.2) are both consequences of a local energy identity. A function is a solution if and only if the real and imaginary parts are solutions. Thus it suffices to treat the real case for which

$$u_t\,\Box u \;=\; \partial_t e - \sum_{j\ge 1} \partial_j(u_t \partial_j u)\,, \qquad e := \frac{u_t^2 + |\nabla_x u|^2}{2}\,.$$

Denote by Γ a backward light cone

$$\Gamma \;:=\; \Big\{(t,x) \;:\; |x - \underline{x}|^2 < \underline{t} - t\Big\}$$

and by $\tilde\Gamma$ the part in $\{x_1 < 0\}$,

$$\tilde\Gamma \ := \ \Gamma \ \cap \ \big\{x_1 < 0\big\}\,.$$

For any $0 \le s < \underline{t}$ the section at time s is denoted

$$\tilde\Gamma(s) \ := \ \tilde\Gamma \ \cap \ \big\{t = s\big\}\,.$$

Both uniqueness and finite speed are consequences of the following estimate.

Proposition 1.6.1. *If u is a smooth solution of (1.6.1)-(1.6.2), then for $0 < t < \underline{t}$,*

$$\phi(t) \ := \ \int_{\tilde\Gamma(t)} e(t,x)\ dx$$

is a nonincreasing function of t.

Proof. Translating the time if necessary it suffices to show that for $s > 0$, $\phi(s) \le \phi(0)$.
In the identity

$$0 \ = \ \int_{\tilde\Gamma\cap\{0\le t\le s\}} u_t\ \Box u\ dt\,dx\,.$$

Integrate by parts to find integrals over four distinct parts of the boundary. The tops and bottoms contribute $\phi(t)$ and $-\phi(0)$ respectively. The intersection of $\tilde\Gamma(s)$ with $x_1 = 0$ yields

$$\int_{\tilde\Gamma(s)\cap\{x_1=0\}} u_t\ \partial_1 u\ dt\,dx_2\ \dots\ dx_d\,.$$

The Dirichlet condition implies that $u_t = 0$ on this boundary so the integral vanishes.
The contribution of the sides $|x - \underline{x}| = \underline{t} - t$ yield an integral of

$$n_0\, e \ + \ \sum_{j=1}^{d} n_j\, u_t\, \partial_j u\,,$$

where $(n_0, n_1, n_2, \dots, n_d)$ is the outward unit normal. Then

$$n_0 \ = \ \Big(\sum_{j=1}^{d} n_j^2\Big)^{1/2} \ = \ \frac{1}{\sqrt2}\,, \qquad \Big|\sum_{j=1}^{d} n_j\, u_t\, \partial_j u\Big| \ \le \ \frac{1}{\sqrt2}\ |u_t||\nabla_x u| \ \le \ \frac{1}{\sqrt2}\, e\,.$$

Thus the integrand from the contributions of sides is nonnegative, so the integral over the sides is nonnegative.
Combining yields

$$0 \ = \ \int_{\tilde\Gamma\cap\{0\le t\le s\}} u_t\ \Box u\ dt\,dx \ \ge \ \phi(t) - \phi(0)\,,$$

and the estimate follows. ∎

§1.6.1. The method of images.

Introduce the notations,

$$x = (x_1, x'), \quad x' := (x_2, \dots, x_d), \qquad \xi = (\xi_1, \xi'), \quad \xi' := (\xi_2, \dots, \xi_d).$$

Definitions. *A function f on $\mathbb{R}^{1+d}$ is* **even** *(resp.* **odd***) in x_1 when*

$$f(t, x_1, x') \;=\; f(t, -x_1, x') \qquad \text{resp.} \quad f(t, -x_1, x') \;=\; -f(t, x_1, x').$$

Define the **reflection operator** R *by*

$$(Rf)(t, x_1, x') \;:=\; f(t, -x_1, x')\,.$$

The even (resp. odd) parts of a function f are defined by

$$\frac{f + Rf}{2}, \qquad \text{resp.} \qquad \frac{f - Rf}{2}.$$

Proposition 1.6.2. i. *If $u \in C^\infty(\mathbb{R}^{1+d})$ is a solution of $\Box u = 0$ that is odd in x_1, then its restiction to $\{x_1 \le 0\}$ is a smooth solution of $\Box u = 0$ satisfying the Dirichlet boundary condition (1.6.1) .*
ii. *Conversely, if $u \in C^\infty(\{x_1 \le 0\})$ is a smooth solution of $\Box u = 0$ satisfying (1.6.1) then the odd extension of u to $\mathbb{R}^{1+d}$ is a smooth odd solution of $\Box u = 0$.*

Proof. i. Setting $x_1 = 0$ in the identity $u(t, x_1, x') = -u(t, -x_1, x')$ shows that (1.6.1) is satisfies.
ii. First prove by induction on n that

$$\forall n \ge 0, \qquad \frac{\partial^{2n} u}{\partial^{2n} x_1}\bigg|_{x_1=0} \;=\; 0\,. \tag{1.6.3}$$

The case $n = 0$ is (1.6.1).
Since the derivatives ∂_t and ∂_j for $j > 1$ are parallel to the boundary along which $u = 0$, it follows that u_{tt} and $\partial_j^2 u$ with $j > 1$ vanish at $x_1 = 0$. The equation $\Box u = 0$ implies

$$\frac{\partial^2 u}{\partial x_1^2} \;=\; \frac{\partial^2 u}{\partial t^2} \;-\; \sum_{j=2}^{d} \frac{\partial^2 u}{\partial x_j^2}.$$

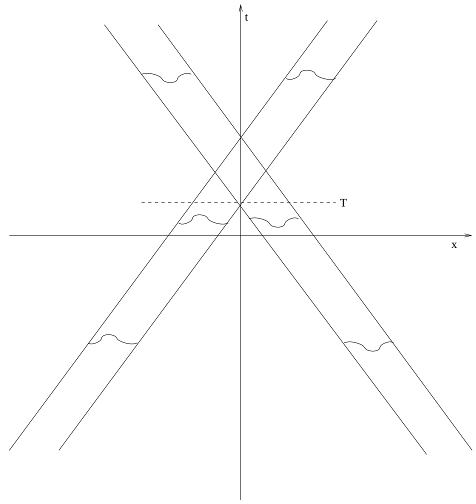
The right hand side vanishes on $\{x_1 = 0\}$ proving the case $n = 1$.
If the case $k \ge 1$ is known, apply the case k to the odd solution $\partial_1^2 u$ to prove the case $k + 1$. This completes the proof of (1.6.3).
Denote by $\tilde u$, the odd extension of u. It is not hard to prove using Taylor's theorem that (1.6.3) is a necessary and sufficient condition for $\tilde u \in C^\infty(\mathbb{R}^{1+d})$. The equation $\Box \tilde u = 0$ for $x_1 \ge 0$ follows from the equation in $x_1 \le 0$ since $\Box \tilde u$ is odd. ∎

Example. Suppose that $d = 1$ and that $f \in C_0^\infty(]-\infty, 0[)$ so that $u = f(x - t)$ is a solution of (1.6.1), (1.6.2) representing a wave which approaches the boundary $\{x = 0\}$ from the left. To describe the reflection use images as follows. The solution in $\{x < 0\}$ is the restriction to $x < 0$ of an odd solution of the wave equation. For $x < 0$ that solution is equal to the given function in $x < 0$ and to minus its reflection in $\{x > 0\}$,

$$u \;=\; f(x - t) \;-\; f(-x - t)\,.$$

The formula on the right is an odd solution of the wave equation which is equal to u in $t < 0$ so is therefore the solution for all time. The solution u is the restriction to $x < 0$.

An example is sketched in the figure. In $\mathbb{R}^{1+1}$ one has an odd solution of the wave equation.

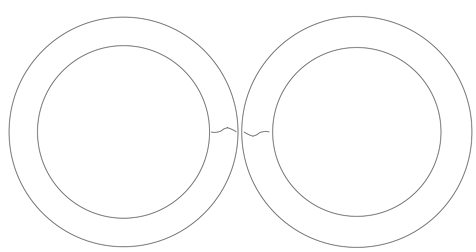


Reflection in dimension $d = 1$

There is a righward moving wave with postitve profile and a leftward moving wave with negative profile equal to -1 times the reflection of the first.

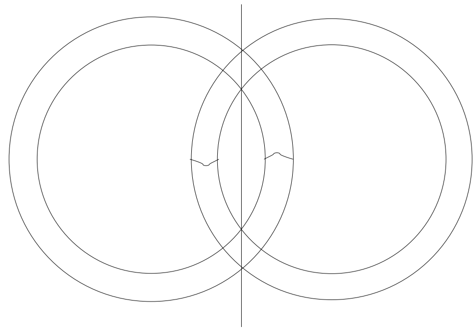
Viewed from $x < 0$, there is a wave with positive profile which arrives at the boundary at time T. At that time a leftward moving wave seems to emerge from the boundary. It is the reflection of the wave arriving at the boundary. If the wave arrives at the boundary with amplitude a on an incoming ray, the reflected wave on the reflected ray has amplitude $-a$. The *coefficient of reflection* is equal to -1. This is the same result found in the first paragraphs of §1.6.

Example. Suppose that $d = 3$ and in $t < 0$ one has a spherically symmetric wave approaching the boundary. Until it reaches the boundary the boundary condition does not play a role. The reflection is computed by extending the incoming wave to an odd solution consisting of the given solution and its negative in mirror image. The moment when the original wave reaches the boundary from the left, its image arrives from the right.



Spherical wave arrives at the boundary

In the figure the wave on the left has positive profile and that on the right a negative profile.



Spherical wave with reflection

In the figure above the middle line represents the boundary. Viewed from $x<0$, the wave on the left disappears into the boundary and a reflected spherical wave emerges with profile flipped. The profiles of spherical waves in three space preserve their shape but decrease in amplitude as they spread.

§1.6.2. The plane wave derivation.

In many texts you will find a derivation which goes as follows. Begin with the plane wave solutions

$$e^{i(x.\xi+t\tau)}, \qquad \xi\in\mathbb{R}^d, \quad \tau=\mp|\xi|\,.$$

Since u is everywhere of modulus one, no solution of this sort can satisfy the Dirichlet boundary condition.

Seek a solution of the initial boundary value problem which is a sum of two plane waves,

$$e^{i(x.\xi-t|\xi|)} \;+\; A\,e^{i(x.\eta+t\sigma)}, \qquad A\in\mathbb{C}\,.$$

In order that the solutions satisfy the wave equation one must have $\sigma^2=|\eta|^2$. In order that the plane waves sum to zero at $x_1=0$ it is necessary and sufficient that $\eta'=\xi'$, $\sigma=-|\xi|$, and $A=-1$. Since $\sigma^2=|\eta|^2$ it follows that $|\eta|=|\xi|$ so

$$\eta=(\pm\xi_1,\xi_2,\dots,\xi_d)\,.$$

The sign + yields the solution $u=0$. Denote

$$\tilde x \;:=\; (-x_1,x_2,\dots,x_d), \qquad \tilde\xi \;:=\; (-\xi_1,\xi_2,\dots,\xi_d).$$

The sign minus yields the interesting solution.

$$e^{i(x.\xi-t|\xi|)} \;-\; e^{i(x.\tilde\xi-t|\tilde\xi|)}$$

which is twice the odd part of $e^{i(x.\xi-t|\xi|)}$.

The textbook interpretation of the solution with $\tau=-|\xi|$ and $\xi_1>0$ is that $e^{i(x.\xi-t|\xi|)}$ is a plane wave approaching the boundary $x_1=0$, and $e^{i(x.\tilde\xi-t|\tilde\xi|)}$ moves away from the boundary. The first is an incident wave and the second is a reflected wave. The factor $A=-1$ is the reflection coefficient. The direction of motions are given group velocity computed from the dispersion relation.

Both waves are of infinite extent and of modulus one everywhere in space time. They have finite energy density but infinite energy. They both meet the boundary at all times. It is questionable to think of either one as incoming or reflected. The next subsection shows that there are localized waves which are clearly incoming and reflected waves with the property that when they interact with the boundary the local behavior resembles the plane waves.

For more general mixed initial boundary value problems, there are other wave forms which need to be included. The key is that solutions of the form $e^{i(x.\xi+t\tau)}$ are acceptable in $x_1 < 0$ for ξ', τ real and $\mathrm{Im}\,\xi_1 \le 0$. When $\mathrm{Im}\,\xi_1 < 0$ the associated waves are localized near the boundary. The Rayleigh waves in elasticity are a classic example. They carry the devastating energy of earth quakes. Waves of this sort which do not propagate are needed to analyse total reflection which is described at the end of §1.7. The reader is referred to [Benzoni-Gavage - Serre], [Chazarain-Piriou], [Taylor 1981], [Hormander 1982 v.II], [Sakamoto], for more information.

§1.6.3. Reflected high frequency wave packets.

Consider solutions which for small time are equal to high frequency solutions from §1.3,

$$u^\epsilon \;=\; e^{i(x.\xi - t|\xi|)/\epsilon}\, a(\epsilon,t,x)\,, \qquad a(\epsilon,t,x) \;\sim\; a_0(t,x) + \epsilon\, a_1(t,x) + \cdots\ , \tag{1.6.5}$$

with

$$\xi = (\xi_1,\xi_2,\dots,\xi_d)\,, \qquad \xi_1 > 0\,.$$

Then $a_0(t,x) \;=\; h(x - t\xi/|\xi|)$ is constant on the rays $\underline{x} + t\xi/|\xi|$. If the Cauchy data are supported in a set $\mathcal{O} \subset\subset \{x_1 < 0\}$ then the amplitudes a_j are supported in the tube of rays

$$\mathcal{T} \;:=\; \Big\{(t,x) \;:\; x = \underline{x} + t\xi/|\xi|, \quad \underline{x} \in \mathcal{O}\Big\}\,, \tag{1.6.6}$$

Finite speed shows that the wave as well as the geometric optics approximation stays strictly to the left of the boundary for small $t > 0$.

The method of images computes the reflection. Define v^ϵ to be the reversed mirror image solution,

$$v^\epsilon(t,x_1,x_2,\dots,x_d) \;:=\; -u^\epsilon(t,-x_1,x_2,\dots,x_d)\,.$$

The solution of the Dirichlet problem is then equal to the restriction of $u^\epsilon + v^\epsilon$ to $\{x_1 \le 0\}$.

Then

$$\tilde v^\epsilon \;=\; -\,e^{i(\tilde x.\xi - t)/\epsilon}\, h(\tilde x - t\xi) \;+\; \text{h.o.t} \;=\; -\,e^{i(\tilde x.\xi - t)/\epsilon}\, \tilde h(x - t\tilde\xi) \;+\; \text{h.o.t}\,.$$

To leading order, $u^\epsilon + v^\epsilon$ is equal to

$$e^{i(x.\xi - t)/\epsilon}\, h(x - t\xi) \;-\; e^{i(\tilde x.\xi - t)/\epsilon}\, \tilde h(x - t\tilde\xi)\,. \tag{1.6.7}$$

The wave represented by u^ϵ has leading term which moves with velocity $\xi/|\xi|$. The wave corresponding to v^ϵ has leading term with velocity $\tilde\xi/|\tilde\xi|$. which comes from $\xi/|\xi|$ by reversing the first component. At the boundary $x_1 = 0$, the tangential components of $\xi/|\xi|$ and $\tilde\xi/|\tilde\xi|$ are equal and their normal components are opposite. The directions are related by the standard law that the angle of incidence equals the angle of reflection. The amplitude of the reflected wave v^ϵ on the reflected ray is equal to -1 time the amplitude of the incoming wave u^ϵ on the incoming wave. This is summarized by the statement that the reflection coefficient is equal to -1.

Suppose that $\underline{t}, \underline{x}$ is a point on the boundary and $\mathcal{O}$ in a neighborhood of size large compared to the wavelength ϵ and small compared to the scale on which h varies. Then, on $\mathcal{O}$, the solution is approximately equal to

$$e^{i(x.\xi - t)/\epsilon}\, h(\underline{x} - \underline{t}\xi/|\xi|) \;-\; e^{i(\tilde x.\xi - t)/\epsilon}\, \tilde h(\underline{x} - \underline{t}\tilde\xi/|\tilde\xi|)\,.$$

This recovers the reflected plane waves of §1.6.2. An observer on such an intermediate scale sees the structure of the plane waves. Thus, even though the plane waves are completely nonlocal, the asymptotic solutions of geometric optics shows that they predict the local behavior at points of reflection.

The method of images also solves the Neumann boundary value problem in a half space using *even* mirror reflection in $x_1 = 0$. It shows that for the Neumann condition, the reflection coefficient is equal to 1.

Proposition 1.6.2. i. *If $u \in C^\infty(\mathbb{R}^{1+d})$ is an even solution of $\Box u = 0$, then its restiction to $\{x_1 \le 0\}$ is a smooth solution of $\Box u = 0$ satisfying the Neumann boundary condition*

$$\partial_1 u|_{x_1=0} = 0\,, \tag{1.6.8}$$

ii. *Conversely, if $u \in C^\infty(\{x_1 \le 0\})$ is a smooth solution of $\Box u = 0$ satisfying (1.6.8) then the even extension of u to $\mathbb{R}^{1+d}$ is a smooth odd solution of $\Box u = 0$.*

The analogue of (1.6.3) in this case is

$$\forall n \ge 0, \qquad \frac{\partial^{2n+1} u}{\partial x_1^{2n+1}}\bigg|_{x_1=0} \;=\; 0\,. \tag{1.6.9}$$

Exercise 1.6.1. *Prove the Proposition.*

Exercise 1.6.2. *Prove uniqueness of solutions by the energy method.* **Hint.** Use the local energy identity.

Exercise 1.6.3 *Verify the assertion concerning the reflection coefficient by following the examples above. That is, consider the case of dimension $d = 1$, the case of spherical waves with $d = 3$ and the behavior in the future of a solution which near $t = 0$ is a high frequency asymptotic solution approaching the boundary.*

§1.7. Snell's law of refraction.

Refraction is the bending of waves as they pass through media whose propagation speeds vary from point to point. The simplest situation is when media with different speeds occupy half spaces, for example $x_1 < 0$ and $x_1 > 0$. The classical physical situations are when light passes from air to water or from air to glass. It is observed that the angles of incidence and refraction are so that for fixed materials the ratio $\sin\theta_i/\sin\theta_r$ is independent of the incidence angle. Fermat observed that this would hold if the speed of light were different in the two media and light light path was a path of least time. In that case, the quotient of sines equal to the ratio of the speeds, c_i/c_r. In this section we derive this behavior for a model problem quite close to the natural Maxwell equations.

The simplified model with the same geometry is,

$$u_{tt} - \Delta\, u = 0 \quad \text{in} \quad x_1 < 0\,, \qquad u_{tt} - c^2\,\Delta\, u = 0 \quad \text{in} \quad x_1 > 0\,, \qquad 0 < c < 1\,. \tag{1.7.1}$$

In $x_1 < 0$ the speed is equal to 1 which is greater than the speed c in $x > 0$. To see that c is the speed of the latter equation one can factor the one dimensional operator $\partial_t^2 - c^2\partial_x^2 = (\partial_t = c\partial_x)(\partial_t - c\partial_x)$ or use the formula for group velocity with dispersion relation $\tau^2 = |\xi|^2$.

A transmission condition is required at $x_1 = 0$ to encode the interaction of waves with the interface. In the one dimesional case, there are waves which approach the boundary from both sides. The waves which move from the boundary into the interior must be determined from the waves which arrive from the interior. There are two arriving waves and two departing waves. One needs two boundary conditions.

We analyse the transmission condition that imposes continuity of u and $\partial_1 u$ across $\{x_1 = 0\}$. Seek solutions of (1.7.1) satsifying the transmission condition,

$$u(t,0^-,x') \;=\; u(t,0^+,x')\,, \qquad \partial_1 u(t,0^-,x') \;=\; \partial_1 u(t,0^+,x')\,. \tag{1.7.2}$$

Denote by square brackets the jump

$$\big[u\big](t,x') \;:=\; u(t,0^+,x') \;-\; u(t,0^-,x')\,.$$

The transmission condition is then

$$\big[u\big] \;=\; 0\,, \qquad \big[\partial_1 u\big] \;=\; 0\,.$$

For solutions which are smooth on both sides of the boundary $\{x_1 = 0\}$, the transmission condition (1.7.2) and be differentiated in t or $x_2, \dots, x_d$ to find

$$\Big[\partial^\beta_{t,x'} u\Big] \;=\; 0\,, \qquad \Big[\partial^\beta_{t,x'}\partial_1 u\Big] \;=\; 0\,. \tag{1.7.3}$$

The partial differential equations then imply that in $x_1 < 0$ and $x_1 > 0$ respectively one has

$$\frac{\partial^2 u}{\partial x_1^2} \;=\; \frac{\partial^2 u}{\partial t^2} \;-\; \sum_{j=2}^{d} \frac{\partial^2 u}{\partial x_j^2}\,, \qquad \frac{\partial^2 u}{\partial x_1^2} \;=\; \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2} \;-\; \sum_{j=2}^{d} \frac{\partial^2 u}{\partial x_j^2},$$

Therefore at the boundary

$$\left[\frac{\partial^2 u}{\partial x_1^2}\right] \;=\; \left(1 - \frac{1}{c^2}\right)\frac{\partial^2 u}{\partial t^2}\,.$$

The second derivative $\partial_1^2 u$ is expected to be discontinuous at $\{x_1 = 0\}$.

The physical conditions for Maxwell's Equations at an air-water or air-glass interface can be analysed in the same way. In that case, the dielectric constant is discontinuous at the interface.

Define

$$\gamma(x) \;:=\; \begin{cases} 1 & \text{when} \quad x_1 > 0 \\ c^{-2} & \text{when} \quad x_1 < 0\,, \end{cases} \qquad e(t,x) \;:=\; \frac{\gamma\, u_t^2 + |\nabla_x u|^2}{2}\,,$$

From (1.7.1) it follows that solutions suitably small at infinity satisfy

$$\partial_t \int_{x_1<0} e\;dx \;=\; \int u_t(t,0^-,x')\;\partial_1 u(t,0^+,x')\;dx'\,,$$
$$\partial_t \int_{x_1>0} e\;dx \;=\; -\int u_t(t,0^+,x')\;\partial_1 u(t,0^+,x')\;dx'\,.$$

The transmission condition guarantees that the terms on the right compensate exactly so

$$\partial_t \int_{\mathbb{R}^3} e\;dx \;=\; 0\,.$$

This suffices to prove uniqueness of solutions. A localized argument as in §1.6.1, shows that signals travel at most at speed one.

Exercise 1.7.1. *Prove this finite speed result.*

A function $u(t,x)$ is called **piecewise smooth** if its restriction to $x_1 < 0$ (resp. $x_1 > 0$) has a C^∞ extension to $x_1 \le 0$ (resp. $x_1 \ge 0$). The Cauchy data of piecewise smooth solutions must be piecewise smooth (with the analogous definition for functions of x only). They must, in addition, satisfy conditions analogous to (1.6.3).

Propostion 1.7.1. *If u is a piecewise smooth solutions u of the transmission problem, then the partial derivatives satisfy the sequence of compatibility conditions, for all $j \ge 0$,*

$$\Delta^j \{u, u_t\}(t, 0^-, x_2, x_3) \;=\; (c^2\Delta)^j \{u, u_t\}(t, 0^+, x_2, x_3)\,,$$

$$\Delta^j \partial_1 \{u, , u_t\}(t, 0^-, x_2, x_3) \;=\; (c^2\Delta)^j \partial_1 \{u, u_t\}(t, 0^+, x_2, x_3)\,.$$

ii. *Conversely, if the piecewise smooth f, g satisfy for all $j \ge 0$,*

$$\Delta^j \{f, g\}(0^-, x_2, x_3) \;=\; (c^2\Delta)^j \{f, g\}(0^+, x_2, x_3)\,, \tag{1.7.4}$$

$$\Delta^j \partial_1 \{f, g\}(0^-, x_2, x_3) \;=\; (c^2\Delta)^j \partial_1 \{f, g\}(0^+, x_2, x_3)\,, \tag{1.7.5}$$

then there is a piecewise smooth solution with these Cauchy data.

Proof. **i.** If u is a piecewise smooth solution then so is $\partial_t^j u$ for any j. Use (1.7.2) for pure time derivatives,

$$\big[\partial_t^j u\big] \;=\; 0\,, \qquad \big[\partial_t^j \partial_1 u\big] \;=\; 0\,. \tag{1.7.6}$$

The case $j = 1$ yields the necessary condition

$$\big[g\big] \;=\; 0\,, \qquad \big[\partial_1 g\big] \;=\; 0\,.$$

For the higher orders, compute with $k \ge 1$,

$$\partial_t^{2k} u\big|_{t=0} \;=\; \begin{cases} \Delta^k u & \text{when} \quad x_1 < 0 \\[2ex] (c^2\Delta)^k u & \text{when} \quad x_1 > 0, \end{cases}$$

$$\partial_t^{2k-1} u\big|_{t=0} \;=\; \begin{cases} \Delta^k u & \text{when} \quad x_1 < 0 \\[2ex] (c^2\Delta)^k u & \text{when} \quad x_1 > 0. \end{cases}$$

Thus, the transmission conditions (1.7.6) proves **i.**

The proof of **ii.** is technical, interesting, and omitted. One can construct solutions using finite differences almost as in §2.2. The shortest existence proof to state uses the spectral theorem for self adjoint operators.* The general regularity theory for such transmission problems can be obtained

* For those with sufficient background, the Hilbert space is $\mathcal{H} := L^2(\mathbb{R}^d\,;\,\gamma\,dx)$.

$$D(\mathcal{A}) \;:=\; \Big\{ w \in H^2(\mathbb{R}^d_+) \cap H^2(\mathbb{R}^d_-) \;:\; [w] = [\partial_1 w] = 0 \Big\}\,,$$

by folding them to a boundary value problem and using the results of [Rauch-Massey, Sakamoto]. ▮

Next consider the mathematical problem whose solution explains Snell's law. The idea is to send a wave in $x_1 < 0$ toward the boundary and ask how it behaves in the future. Suppose

$$\xi \in \mathbb{R}^d, \qquad |\xi| = 1\,, \qquad \xi_1 > 0\,,$$

and consider a short wavelength asymptotic solution in $\{x_1 < 0\}$ as in §1.6.3,

$$I^\epsilon \ \sim\ e^{i(x.\xi - t)/\epsilon}\, a(\epsilon, t, x)\,, \qquad a(\epsilon, t, x)\ \sim\ a_0(t,x) + \epsilon\, a_1(t,x) + \cdots\,, \tag{1.7.5}$$

where for $t < 0$ the support of the a_j is contained in a tube of rays with compact cross section and moving with speed ξ. One can take a to vanish outside the tube. Since the incoming waves are smooth and initially vanish identically on a neighborhood of the interface $\{x_1 = 0\}$, the compatibilities are satisfied and there is a family of piecewise smooth solutions u^ϵ defined on $\mathbb{R}^{1+d}$. The tools prepared yield an infinitely accurate description of the family of solutions u^ϵ.

To solve the problem, seek an asymptotic solution which at $\{t = 0\}$ is equal to this incoming wave. A first idea is to find a transmitted wave which continues the incoming wave into $\{x_1\} > 0$.
Seek the transmitted wave in $x_1 > 0$ in the form

$$T^\epsilon \ \sim\ e^{i(x.\eta + t\tau)/\epsilon}\ d(\epsilon, t, x)\,, \qquad d(\epsilon, t, x)\ \sim\ d_0(t,x) + \epsilon\, d_1(t,x) + \cdots\,,$$

In order that this be an approximate solution moving away from the interface one must have

$$\tau^2 = c^2|\eta|^2, \qquad |\eta| = 1/c\,.$$

The incoming wave, when restricted to the interface $x_1 = 0$ oscillates with phase $(x'.\xi' - t)/\epsilon$. At the interface, the proposed transmitted wave oscillates with phase $(x'.\eta' - t\tau)/\epsilon$. In order that there be any chance at all of satisfying the transmission condtions one must take

$$\eta' = \xi', \qquad \tau = -1,$$

so that the two expressions oscillate together.

$$\mathcal{A}w := \Delta w \quad \text{in} \quad x_1 < 0, \qquad \mathcal{A}w := c^2\Delta \quad \text{in} \quad x_1 > 0\,.$$

Then,

$$(\mathcal{A}u, v)_{\mathcal{H}} \ =\ (u, \mathcal{A}v)_{\mathcal{H}} \ =\ -\int \nabla u.\nabla v\ dx\,,$$

so $-\mathcal{A} \geq 0$. The elliptic regularity theorem implies that $\mathcal{A}$ is self adjoint. The regularity theorem is proved, for example, by the methods in [Rauch 1992, Chapter 10]. The solution of the initial value problem is

$$u \ =\ \cos t\sqrt{-\mathcal{A}}\ f \ +\ \frac{\sin t\sqrt{-\mathcal{A}}}{\sqrt{-\mathcal{A}}}\ g\,.$$

For piecwise H^∞ data, the sequence of compatibilities is equivalent to the data belonging to $\cap_j D(\mathcal{A}^j)$.

The equation $\tau^2 = c^2|\eta|^2$ implies

$$\eta_1^2 \;=\; \frac{\tau^2}{c^2} - |\eta'|^2 \;=\; \frac{1}{c^2} - |\xi'|^2\,.$$

Impose $\eta_1 > 0$ so the transmitted wave moves into the region $x_1 > 0$ to find

$$\eta_1 = \Big(\frac{1}{c^2} - |\xi'|^2\Big)^{1/2} \;>\; \xi_1.$$

Thus,

$$T^\epsilon \;\sim\; e^{i(x.\eta - t)/\epsilon}\, d(\epsilon, t, x)\,, \qquad \eta = \left(\Big(\frac{1}{c^2} - |\xi'|^2\Big)^{1/2}\,,\ \xi'\right). \tag{1.7.6}$$

From section 1.6.3 we know that the leading amplitude d_0 must be constant on the rays $t \mapsto (t, \underline{x} + c\,t\,\eta/|\eta|)$. To determine d_0 it suffices to know the values $d_0(t, 0^+, x')$ at the interface. One could choose d_0 to guarantee the continuity of u or of $\partial_1 u$, but not both. One cannot construct an good approximated solution consisting of just an incident and transmitted wave.

Add to the recipe a reflected wave. Seek a reflected wave in $x_1 \geq 0$ in the form

$$R^\epsilon \;\sim\; e^{i(x.\zeta + t\sigma)/\epsilon}\, b(\epsilon, t, x)\,, \qquad b(\epsilon, t, x) \;\sim\; b_0(t, x) + \epsilon\, b_1(t, x) + \cdots\,.$$

In order that the reflected wave oscillate with the same phase as the incident wave in the boundary $x_1 = 0$, one must have $\zeta' = \xi'$ and $\sigma = -1$. To satisfy the wave equation in $x_1 < 0$ requires $\sigma^2 = |\zeta|^2$. Together these imply $\zeta_1^2 = \xi_1^2$. To have propagation away from the boundary requires $\zeta_1 = -\xi_1$ so $\zeta = \tilde{\xi}$. Therefore,

$$R^\epsilon \;\sim\; e^{i(x.\tilde{\xi} - t)/\epsilon}\, b(\epsilon, t, x)\,, \qquad b(\epsilon, t, x) \;\sim\; b_0(t, x) + \epsilon\, b_1(t, x) + \cdots\,. \tag{1.7.7}$$

Summarizing seek

$$v^\epsilon \;=\; \begin{cases} I^\epsilon + R^\epsilon & \text{in} \quad x_1 < 0 \\ \quad T^\epsilon & \text{in} \quad x_1 > 0 \end{cases}.$$

The continuity required at $x_1 = 0$ forces

$$e^{i(x'.\xi' - t)/\epsilon}\, \big(a(\epsilon, t, 0, x') + b(\epsilon, t, 0, x')\big) \;=\; e^{i(x'.\xi' - t)/\epsilon}\, d(\epsilon, t, 0, x')\,. \tag{1.7.8}$$

The continuity of u and $\partial_1 u$ hold if and only if at $x_1 = 0$ one has

$$a \;+\; b \;=\; d\,, \qquad \text{and,} \qquad \frac{i\xi_1}{\epsilon} a + \partial_1 a \;-\; \frac{i\xi_1}{\epsilon} b + \partial_1 b = \frac{i\eta_1}{\epsilon} d + \partial_1 d\,. \tag{1.7.9}$$

The first of these relations yields

$$\Big(a_j \;+\; b_j \;-\; d_j\Big)_{x_1 = 0} = 0, \qquad j = 0, 1, 2, \ldots\,, \tag{1.7.10}$$

The second relation in (1.7.9) is expanded in powers of ϵ. The coefficients of ϵ^j must match for all all $j \geq -1$. The leading order is ϵ^{-1} and yields

$$\big(a_0 - b_0 - (\eta_1/\xi_1) d_0\big)_{x_1 = 0} = 0\,. \tag{1.7.11}$$

Since a_0 is known, the $j = 0$ equation from (1.7.10) together with (1.7.11) yield a system of two linear equations for the two unknown b_0, d_0

$$\begin{pmatrix} -1 & 1 \\ 1 & \eta_1/\xi_1 \end{pmatrix} \begin{pmatrix} b_0 \\ d_0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_0 \end{pmatrix} .$$

Since the matrix is invertible, this determines the values of b_0 and d_0 at $x_1 = 0$.

The amplitude b_0 (resp. d_0) is constant on rays with velocity $\tilde{\xi}$ (resp. $c\eta/|\eta|$). Thus the leading amplitudes are determined throughout the half spaces on which they are defined.

Once these leading terms are known the ϵ^0 term from the second equation in (1.7.9) shows that on $x_1 = 0$,

$$a_1 - b_1 - d_1 = \text{known} .$$

Note that a_1 is also known so that together with the case $j = 2$ from (1.7.10) this suffices to determine b_1, d_1 on $x_1 = 0$. Each satisfies a transport equation along rays which is the analogue of (1.4.12). Thus from the initial values just computed on $x_1 = 0$ they are determined everywhere. The higher order correctors are detemined analogously.

Once the b_j, d_j are determined, one can choose b, c as functions of ϵ with the known Taylor expansions at $x = 0$. They can be chosen to have supports in the appropriate tubes of rays and to satisfy the transmission conditions (1.7.9) exactly.

The function u^ϵ is then an infinitely accurate approximate solution in the sense that it satisfies the transmission and initial conditions exactly while the residuals

$$v^\epsilon_{tt} - \Delta v^\epsilon := r^\epsilon \quad \text{in} \quad x_1 < 0 , \qquad v^\epsilon_{tt} - c^2 \Delta v^\epsilon := \rho^\epsilon,$$

satisfy for all N, s, T there is a C so that

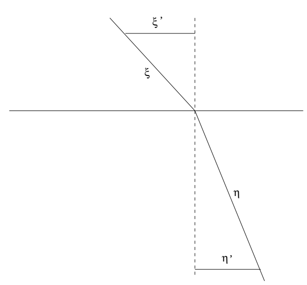
$$\big\| r^\epsilon \big\|_{H^s([-T,T]\times\{x_1<0\})} + \big\| \rho^\epsilon \big\|_{H^s([-T,T]\times\{x_1>0\})} \le C \epsilon^N .$$

From the analysis of the transmission problem it follows that with new constants,

$$\big\| u^\epsilon - v^\epsilon \big\|_{H^s([-T,T]\times\{x_1>0\})} \le C \epsilon^N .$$

The proposed problem of describing the family of solutions u^ϵ is solved.

The angles of incidence and refraction, θ_i and θ_r, given by the directions of propagation of the incident and transmitted waves. From the figure



one finds,

$$\sin\theta_i = \frac{|\xi'|}{|\xi|}, \qquad \text{and,} \qquad \sin\theta_r = \frac{|\eta'|}{|\eta|} = \frac{|\xi'|}{|\xi|/c} .$$

Therefore

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{1}{c},$$

is independent of θ_i. The high frequency asymptotic solutions explain Snell's law. This is the last of the three basic laws of geometric optics. The law depends only on the phases. The phases are determined by the requirement that the restriction of the phases to $x_1 = 0$ equal the restriction of the incoming phase. They do not depend on the transmission condition that we chose. It is for this reason that the conclusion is the same for the correct transmission problem for Maxwell's equations.

On a neighborhood $(\underline{t}, \underline{x}) \in \{x_1 = 0\}$ which is small compared to the scale on which a, b, c vary and large compared to ϵ, the solution resembles three interacting plane waves. In science texts one usually computes for which such triples the transmission condition is satisfied in order to find Snell's law. The asymmptotic solutions of geometric optics show how to overcome the criticism that the plane waves have modulus independent of (t, x) so cannot reasonably be viewed as either incoming or outgoing.

For a more complete discussion of reflection and refraction see [Taylor 1981, Benzoni-Gavage and Serre]. In particular these treat the phenomenon of *total reflection* which can anticipated as follows. From Snell's law one sees that $\sin\theta_r < 1/c$ and approaches that value as θ_i approaches $\pi/2$. The refracted rays lie in the cone $\theta_r < \arcsin(1/c)$. Reversing time shows that light rays from below approaching the surface at angles smaller than this critical angle traverse the surface tracing backward the old incident rays. For angles larger than $\arcsin(1/c)$ there is no continuation as a ray above the surface possible. One can show by constructing infinitely accurate approximate solutions that there is total reflection. Below the surface there is a reflected ray with the usual law of reflection. The role of a third wave is played by a boundary layer of thickness $\sim \epsilon$ above which the solution is $O(\epsilon^\infty)$.

Chapter 2. The linear Cauchy problem

Hyperbolic initial value problems with constant coefficients are efficiently analysed using the Fourier transform. Such equations describe problems where the medium is identical at all points. When the physical properties of the medium vary from point to point, the corresponding models have variable coefficients. The initial value problem in such situations is usually not explicitly solvable by Fourier transform. Constant coefficient systems also arise as the linearization at constant solutions of translation invariant nonlinear operators.

The key idea of stability or well posedness, emphasized by Hadamard, is that for a model to be reasonable, one must know that small changes in the data, for example the initial data, can only result in small changes in predictions. For a linear equation, this is equivalent to showing that small data yields small solutions. In the linear case, normed vector spaces are often appropriate settings to describe this continuity. In this case, continuity is equivalent to showing that the norm of solutions is bounded by at most a constant times a norm of the data.

For hyperbolic problems, for example Maxwell's equations or D'Alembert's wave equation, L^2 norms and associated L^2 Sobolev spaces yield better estimates (for example without loss of derivatives) than L^p or Hölder spaces. Initial data in Sobolev spaces H^s yield solutions with values in H^s, while the analogous statement for C^α or $W^{s,p}$, $p \neq 2$ is false in space dimension greater than one.

The analysis of constant coefficient hyperbolic systems is summarized in Appendix 2A.

§2.1. Energy estimates for symmetric hyperbolic systems.

§2.1.2. The constant coefficient case.

Three classic examples of hyperbolic equations are D'Alembert's equation of vibrating strings, Maxwell's equations of electrodynamics and Euler's equations of inviscid compressible fluid flow. The first two have constant coefficients. Linearizing the the third at a constant state also yields a constant coefficient system.

Maxwell's equations describe electric and magnetic field strengths $E(t,x), B(t,x)$ which are vector fields defined on $\mathbb{R}^{1+3}$. There are two dynamic equations

$$E_t = c\,\mathrm{curl}\,B - 4\pi\mathbf{j}\,, \qquad B_t = -c\,\mathrm{curl}\,E\,, \qquad c = 3\times 10^{10}\ \text{cm./sec.} \tag{2.1.1}$$

The vector field $\mathbf{j}(t,x)$ is the current density measuring the flow of charge. It is a source term which is assumed to be given. These equations determine E, B from their initial data once $\mathbf{j}$ is known. Not all initial data and sources are physically relevant. The physical solutions are a subset of the dynamics defined by the additional Maxwell equations

$$\mathrm{div}\,E = 4\pi\rho, \quad \text{and} \quad \mathrm{div}\,B = 0\,, \tag{2.1.2}$$

where $\rho(t,x)$ is the charge density.

Taking the divergence of the first equation in (2.1.1) and the time derivative of the first equation in (2.1.2) shows that the *continuity equation*,

$$\partial_t \rho = -\mathrm{div}\,\mathbf{j}\,, \tag{2.1.3}$$

follows from the Maxwell system. This equation expresses the *conservation of charge* and must be satisfied by the given source terms ρ and $\mathbf{j}$.

Taking the divergence of (2.1.1) and using (2.1.3) yields

$$\partial_t \operatorname{div} B = \partial_t \operatorname{div} (E - 4\pi\rho) = 0\,.$$

Thus, when the continuity equation is satisfied, the constraint equations (2.1.2) hold for all time as soon as they are satisfied at time $t = 0$. In summary the sources must satisfy (2.1.3) and the solutions of interest are those which satisfy (2.1.2). †

The system (2.1.1) is a symmetric hyperbolic system in the following sense. Introduce the $\mathbb{R}^6$ valued unknown $u := \big(E\,,\, B\big)$. Then equation (2.1.1) has the form

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{3} A_j\,\frac{\partial u}{\partial x_j} = f\,, \qquad f := \big(\mathbf{j}\,,\, 0\big)\,, \tag{2.1.4}$$

with constant 6×6 real matrices A_j.

Exercise 2.1.1. *Compute the matrices A_j. In particular, verify that they are symmetric.*

Definition. *A constant coefficient operator*

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} A_j\,\frac{\partial}{\partial x_j}$$

on $\mathbb{R}^d$ is **symmetric hyperbolic** *when the coefficient matrices are hermitian symmetric matrices.*

The importance of symmetry is that it leads to simple L^2 and more generally H^s estimates.

Symmetric systems with constant coefficients are efficiently analyzed using the Fourier transform. The transform and its inverse are given by,

$$u(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix.\xi}\,\hat{u}(\xi)\;d\xi\,,$$

where

$$\hat{u}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix.\xi}\,u(\xi)\;d\xi\,.$$

The Fourier transform is also denoted by $\mathcal{F}$.

Consider the case $f = 0$ and take the Fourier transform in x to find

$$\partial_t \hat{u}(t,\xi) \;+\; \sum i\,A_j\,\xi_j\,\hat{u}(t,\xi) \;=\; 0\,.$$

† This argument is historically very important. Experiments with charges and currents yielded the equations,

$$0 \;=\; c\,\operatorname{curl} B - 4\pi\mathbf{j}\,, \qquad B_t \;=\; -c\,\operatorname{curl} E\,,$$

together with (2.1.1) and (2.1.2). Taking the divergence of the first and using (2.1.2) yields $\rho_t = 0$ showing that the equations are incomplete in the case of nonstatic ρ. Modifying the first by inserting an unknown term F on the left of the first equation one finds that the equations are coherent exactly when $\operatorname{div} F = \operatorname{div} E_t$. This together with the symmetry of the equations in the pair E, B lead Maxwell to propose the equations with E_t on the left of the first equation.

For ξ fixed, integrate the ordinary differential equation in time to find,

$$\hat{u}(t,\xi) \;=\; e^{-it\sum A_j\xi_j}\,\hat{u}(0,\xi)\,.$$

The symmetry implies that $\exp(-it\sum A_j\xi_j)$ is a unitary matrix-valued function of t,ξ. Thus for all t,ξ

$$\|\hat{u}(t,\xi)\|^2 = \|\hat{u}(0,\xi)\|^2\,.$$

For Maxwell's equations this asserts that for all ξ,

$$|\hat{E}(t,\xi)|^2 + |\hat{B}(t,\xi)|^2 = \text{independent of time}$$

which expresses the conservation of energy at every frequency. Integrating $d\xi$ and using the Plancherel theorem implies that the L^2 norm is conserved, that is for all t

$$\|u(t)\|_{L^2(\mathbb{R}^d)} = \|u(0)\|_{L^2(\mathbb{R}^d)}\,.$$

For the fields this asserts that

$$\int_{\mathbb{R}^d} |E|^2 + |B|^2\; dx = \text{ independent of time}$$

which is the physical law of *conservation of energy.* More generally, the Sobolev H^s norms defined for $s\in\mathbb{R}$ by

$$\|v\|^2_{H^s(\mathbb{R}^d)} \;:=\; \int_{\mathbb{R}^d} (1+|\xi|^2)^s\,|\hat{v}(\xi)|^2\; d\xi$$

are conserved. When $s\neq 0$, these norms have no natural physical interpretation. They are important in the mathematical analysis. The next result summarizes the conclusions.

Proposition 1.2.1. *Suppose that $L=\partial_t+\sum A_j\partial_j$ is a constant coefficient symmetric hyperbolic operator. For $g\in\cap_s H^s(\mathbb{R}^d) \;:=\; H^\infty(\mathbb{R}^d)$ there is one and only one solution*

$$u\in \cap_k\, C^k((\mathbb{R}\;;\;H^k(\mathbb{R}^d)) \;:=\; C^\infty(\mathbb{R}\;;\;H^\infty(\mathbb{R}^d))$$

of the initial value problem,

$$L\,u \;=\; 0, \qquad u\big|_{t=0} \;=\; g\,.$$

The solution is given by $\hat{u}(t)=e^{-it\sum A_j\xi_j}\,g$. For all t,s, $\|u(t)\|_{H^s(\mathbb{R}^d)}=\|u(0)\|_{H^s(\mathbb{R}^d)}$.

A proof of the last identities of the Propostion without using the Fourier Transform is based on local conservation laws. Denote with brackets, $\langle\;,\;\rangle$ the scalar product in $\mathbb{C}^N$, and by ∂ either ∂_t or ∂_j for some j. When A is hermitian symmetric,

$$\partial\big\langle Au\,,\,u\big\rangle = \big\langle A\partial u\,,\,u\big\rangle \;+\; \big\langle Au\,,\,\partial u\big\rangle = \big\langle A\partial u\,,\,u\big\rangle \;+\; \big\langle u\,,\,A\partial u\big\rangle = 2\mathrm{Re}\,\langle A\partial u\,,\,u\rangle,$$

where the symmetry is used at the second step. Summing shows that if L is the differential operator on the left in (2.1.4), then for $u\in C^1$,

$$\partial_t\big\langle u\,,\,u\big\rangle \;+\; \sum_j \partial_j\big\langle A_j u\,,\,u\big\rangle \;=\; 2\mathrm{Re}\,\big\langle Lu\,,\,u\big\rangle\,.$$

The same argument proves the equality in $L^1_{\rm loc}(\mathbb{R}^{1+d})$ for functions u so that

$$u \in L^\infty_{\rm loc}(\mathbb{R}\ ;\ H^1_{\rm loc}(\mathbb{R}^d)), \qquad u_t \in L^1_{\rm loc}(\mathbb{R}\ ;\ L^2_{\rm loc}(\mathbb{R}))$$

in which case $\langle Lu, u\rangle \in L^1_{\rm loc}(\mathbb{R}^{1+d})$. When $Lu = 0$, integrating this identity over $[0,t]\times\mathbb{R}^d$ proves the conservation of the L^2 norm. To prove this one must justify the identities

$$\partial_t \|u(t)\|^2 \ =\ 2\,\mathrm{Re}\int \overline{u}_t\, u_t\ dx, \qquad \text{and,} \qquad \int \partial_j \langle A_j u\,,\, u\rangle\ dx\ =\ 0\,.$$

The sides of each equality are continuous on the space of u considered and vanish on the dense subset of $u \in C^\infty_0(\mathbb{R}^{1+d}$.

Similarly if $Lu = 0$ then also $Lw = 0$ where $w := \partial^\alpha_x u$. The L^2 conservation for these solutions with $|\alpha| \le s$ yields an H^s conservation law when s is an integer. For the noninteger case the L^2 conservation for $w := (1-\Delta_x)^{s/2}u$ gives the desired identity.

§2.1.3. The variable coefficient case.

Analogous results are valid for variable coefficient operators (for example, Maxwell's equations in a nonhomogeneous dielectric) satisfying a symmetry hypothesis. The introduction of this class of operators and the observation that it is ubiquitous in mathematical physics is due to K.O. Friedrichs.

Definition. *In $\mathbb{R}^{1+d}$ introduce coordinates $y = y_0, y_1, \cdots, y_d := t, x_1, \cdots, x_d$. A partial differential operator*

$$L(y,\partial) = \sum_{\mu=0}^{d} A_\mu(y)\,\frac{\partial}{\partial y_\mu} + B(y) \tag{2.1.5}$$

is called **symmetric hyperbolic** *if and only if*

i. *the coefficient matrices A_μ, and B have uniformly bounded derivatives,*

$$\sup_{y\in\mathbb{R}^{1+d}} \left\| \partial^\alpha_y \Big(A_\mu(y), B(y)\Big)\right\| < \infty\,, \tag{2.1.6}$$

ii. *the A_μ are hermitian symmetric valued, and*

iii. *A_0 is strictly positive in the sense that there is a $c > 0$ so that for all y,*

$$A_0(y)\ \ge\ c\,I\,. \tag{2.1.7}$$

The L^2 estimate has a generalization to such problems. *For a first version suppose that $A_0 = I$,* and denote by

$$G(t)\ :=\ \sum_j A_j(t,x)\,\partial_j\ +\ B(t,x)\,.$$

Denote the $L^2(\mathbb{R}^d)$ scalar product by

$$(f,h)\ :=\ \int_{\mathbb{R}^d} f(x)\,\overline{h(x)}\ dx\,.$$

The adjoint differential operator

$$G^*(t) \;:=\; -\sum_j A_j^*(t,x)\,\partial_j \;+\; B(t,x)^* \;-\; \sum_j(\partial_j\,A_j^*)$$

is defined so that for ϕ and ψ belonging to $C_0^\infty(\mathbb{R}^d)$,

$$(G(t)\,\phi,\psi) \;=\; (\phi, G^*(t)\psi)\,.$$

Then,

$$G(t)+G(t)^* \;=\; B(y)+B^*(y)-\sum_{j=1}^{3}(\partial_j A_j(y))$$

is multiplication by a uniformly bounded matrix. Thus there is a C so that

$$\|\,G(t)+G(t)^*\| \;\le\; 2C\,. \tag{2.1.8}$$

The operator G is nearly antiselfadjoint. If

$$u\in C(\mathbb{R}\,;\,H^1(\mathbb{R}^d)),\qquad u_t\in C^1(\mathbb{R}\,;\,L^2(\mathbb{R}^d)),\qquad \text{and}\qquad f\;:=\;Lu\,,$$

then,

$$u'(t)+G(t)\,u(t) \;=\; f(t). \tag{2.1.9}$$

By hypothesis $\|u(t)\|^2\in C^1(\mathbb{R})$ and,

$$\frac{d}{dt}\,\|u(t)\|^2 \;=\; (u,u')+(u',u)\,.$$

Using (2.1.9) yields,

$$\frac{d}{dt}\,\|u(t)\|^2 \;=\; (u,-Gu)+(-Gu,u)+2\mathrm{Re}\,(u,f)\,. \tag{2.1.10}$$

The Cauchy-Schwartz inequality shows that

$$2\mathrm{Re}\,(u,f)\le 2\,\|u(t)\|\,\|f(t)\|\,, \tag{2.1.11}$$

and the near antisymmetry implies that

$$(u,-Gu)+(-Gu,u) \;=\; -\big(u,(G+G^*)u\big)\le 2C\|u\|^2\,. \tag{2.1.12}$$

The left hand side of (2.1.10) is $\frac{d}{dt}\,\|u(t)\|^2 \;=\; 2\,\|u(t)\|\,\frac{d}{dt}\,\|u(t)\|$, so,

$$2\,\|u(t)\|\,\frac{d}{dt}\,\|u(t)\| \;\le\; 2C\,\|u(t)\|^2 \;+\; 2\,\|u(t)\|\,\frac{d}{dt}\,\|f(t)\|\,.$$

Where $u(t)\ne 0$, dividing (2.1.10) by $\|u(t)\|$ yields,

$$\frac{d\|u(t)\|}{dt}\;\le C\|u(t)\|+\|f(t)\|\,,\quad \text{equivalently}\quad \frac{d\big(e^{-Ct}\,\|u(t)\|\big)}{dt}\;\le e^{-Ct}\,\|f(t)\|\,. \tag{2.1.13}$$

Since $\|u(t)\|$ is C^1 in t, (2.1.13) extends by continuity to the closure of $\{t : u(t) \neq 0\}$. The complement of that closure is an open set on which $u(t) = 0$ so (2.1.13) is valid there too.

Integrating (2.1.13) yields the fundamental estimate,

$$\|u(t)\| \leq e^{Ct}\, \|u(0)\| + \int_0^t e^{C(t-\sigma)}\, \|f(\sigma)\|\, d\sigma\,. \tag{2.1.14}$$

We next describe three methods for extending this argument to the case $A_0 \neq I$. The first is by reduction to that case Let $v = A_0^{1/2} u$. The equation

$$\tilde{L}\, v \;:=\; A_0^{-1/2}\, Lu\, A_0^{-1/2} v \;=\; A_0^{-1/2} f \;:=\; \tilde{f}.$$

is equivalent to the original equation $Lu = f$. The operator $\tilde{L}$ is symmetric hyperbolic since its coeffient matrices are $A_0^{-1/2}\, A_\mu\, A_0^{-1/2}$. The coefficient of ∂_t comes from $\mu = 0$ so is I. In this way the general case can be reduced to the case $A_0 = I$.

Next revisit the proof using integration by parts. Equation (2.1.10) shows that the estimate (2.1.14) is proved by taking the real part of the scalar product $(u(t)\,,\,(Lu)(t))$. This argument generalizes to the case of an equation of the form

$$A_0(t)\, \frac{du}{dt} \;+\; G\, u = f\,, \tag{2.1.15}$$

where A_0 is strictly positive with $\|dA_0/dt\| \leq C'$. The starting point is either

$$\frac{d}{dt}\, \big(u(t), A_0(t)\, u(t)\big)\,, \tag{2.1.16}$$

or equivalently

$$0 = \mathrm{Re}\, \Big(\, u\,,\; A_0(t) \frac{du}{dt} + G\, u - f\, \Big)\,.$$

One finds that

$$\|u(t)\| \leq C\, e^{Ct} \Big(\, \|u(0)\| + \int_0^t e^{-C(t-\sigma)}\, \|f(\sigma)\|\; d\sigma \Big)\,. \tag{2.1.17}$$

Exercise 2.1.2. *Carry out the last two derivations of the estimate (2.1.17).*

Proposition 2.1.1 *For every $s \in \mathbb{R}$ there is a constant $C(s, L)$ so that for all $t \geq 0$, for all $u \in C^1(\mathbb{R}\,;\, H^s(\mathbb{R}^d)) \cap C(\mathbb{R}\,;\, H^{s+1}(\mathbb{R}^d))$,*

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \leq C\, e^{Ct} \|u(0)\|_{H^s(\mathbb{R}^d)} + \int_0^t C\, e^{C(t-\sigma)} \|(Lu)(\sigma)\|_{H^s(\mathbb{R}^d)}\; d\sigma\,. \tag{2.1.18}$$

Proof. The procedure of the paragraph after (2.1.14), reduces to the case $A_0 = I$. The main step is to prove (2.1.18) for integer $s \geq 0$ when $A_0 = I$. The case $s = 0$ is (2.1.17).

For any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq s$, the basic L^2 estimate (2.1.14) implies that

$$\|\partial_x^\alpha\, u(t)\| \leq C\, e^{Ct}\, \|\partial_x^\alpha\, u(0)\| + \int_0^t C\, e^{C(t-\sigma)}\, \|L\, \partial_x^\alpha\, u\,(\sigma)\|\, d\sigma\,. \tag{2.1.19}$$

Using the product rule for differentiation and the fact that $A_0 = I$ one finds that,

$$L\,\partial_x^\alpha\, u = \partial_x^\alpha\, L\, u + \sum_{|\beta|\le s} C_{\alpha,\beta}(y)\,\partial_x^\beta\, u$$

with smooth bounded matrix valued functions $C_{\alpha,\beta}$. Equivalently

Definition. *For integer $\sigma \ge 0$, $\mathrm{Op}(\sigma, \partial_x)$ denotes the family of partial differential operators in x of degree σ whose coefficients have derivatives bounded on $\mathbb{R}^{1+d}$. $\mathrm{Op}(\sigma, \partial_{t,x})$ is defined similarly.*

Then,

$$[L, \partial^\alpha] = [G, \partial_x^\alpha] \in \mathrm{Op}(|\alpha|, \partial_x)\,, \quad \text{more generally,} \quad \big[\mathrm{Op}(m, \partial_x)\,,\, \partial_x^\alpha\big] \;\in\; \mathrm{Op}(m + |\alpha| - 1, \partial_x).$$

Define

$$\psi(t) \;:=\; \sum_{|\alpha|\le s} \big\|\partial_x^\alpha u(t)\big\|_{L^2(\mathbb{R}^d)}\,.$$

Summing (2.1.19) over all $|\alpha| \le s$ yields,

$$\psi(t) \;\le\; C\,e^{Ct}\,\psi(0) \;+\; \int_0^t C\,e^{C(t-\sigma)}\,\psi(\sigma)\;d\sigma \;+\; \int_0^t C\,e^{C(t-\sigma)}\,\|f(\sigma)\|_{H^s(\mathbb{R}^d)}\;d\sigma\,.$$

If follows from Gronwall's Lemma below that one has the same estimate with a larger constant C' and without the middle term on the right hand side.

Gronwall's Lemma 2.1.3. *If $0 \le g, \psi \in L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$, $0 \le h \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$ and*

$$\psi(t) \;\le\; g(t) \;+\; \int_0^t h(\sigma)\,\psi(\sigma)\;d\sigma\,, \quad \text{a.e. } \; t > 0\,, \tag{2.1.20}$$

then, with $H(t) := \int_0^t h(\sigma)\,d\sigma$,

$$\psi(t) \;\le\; g(t) + e^{H(t)} \int_0^t e^{-H(\sigma)}\,h(\sigma)\,g(\sigma)\;d\sigma\,, \quad \text{a.e. } \; t > 0\,.\,.$$

Proof of Gronwall's Lemma. Denote by γ the absolutely continuous function

$$\gamma(t) \;:=\; \int_0^t h(\sigma)\,\psi(\sigma)\;d\sigma\,.$$

Then

$$\gamma'(t) \;=\; h(t)\psi(t) \;\le\; h(t)g(t) + h(t)\gamma(t)\,,$$

where the integral inequality is used in the last step.
Therefore

$$\big(e^{-H(t)}\gamma(t)\big)' \;=\; e^{-H(t)}\big(\gamma' - h\,\gamma\big) \;\le\; e^{-H(t)}\,h(t)\,g(t)\,.$$

Since $\gamma(0) = 0$, integrating this inequality from $t = 0$ to t yields

$$e^{-H(t)}\,\gamma(t) \;\le\; \int_0^t e^{-H(\sigma)}\,h(\sigma)\,g(\sigma)\,d\sigma\,.$$

This together with (2.1.21) which asserts that $\psi \le g + \gamma$ completes the proof. ∎

Exercise 2.1.3. *Show how Gronwall's Lemma suffices to erase the middle term in the estimate at the cost of increasing the constant C.* **Hint.** The source term involving f is part of g.

Following [Lax, 1955], we prove (2.1.9) for $0 > s \in \mathbb{Z}$. With integer $s < 0$, introduce $\sigma := |s| > 0$ and $u := (1-\Delta)^\sigma v$, so

$$v \;=\; (1-\Delta)^{-\sigma} u, \qquad \|u(t)\|_s \;=\; \|u(t)\|_{-\sigma} \;=\; \|(1-\Delta)^{-\sigma} u\|_\sigma \;=\; \|v(t)\|_\sigma\,.$$

Use the estimate for the positive integer value σ on v. Toward that end, we need to estimate $\|Lv(t)\|_\sigma$.

Since $(1-\Delta)^\sigma$ is a scalar differential operator of degree 2σ,

$$\Big[L\,,\,(1-\Delta)^\sigma\Big] \;\in\; \mathrm{Op}(2\sigma, \partial_x)\,.$$

In particular, $\big[L\,,\,(1-\Delta)^\sigma\big]$ is for each t a contintuous map $H^\sigma \to H^{-\sigma}$ with bound independent of t.

Compute

$$Lu = L\big((1-\Delta)^\sigma v\big) = (1-\Delta)^\sigma Lv + [L,(1-\Delta)^\sigma]v = (1-\Delta)^\sigma Lv + \mathrm{Op}(2\sigma,\partial_x)v\,.$$

Therefore

$$Lv \;=\; (1-\Delta)^{-\sigma}\Big(Lu \;+\; \mathrm{Op}(2\sigma,\partial_x)v\Big)\,,$$

so with $\|\ \|_s$ denoting the $H^s(\mathbb{R}^d)$ norm,

$$\|Lv(t)\|_\sigma \le \|Lu(t)\|_{-\sigma} + C\|v(t)\|_\sigma\,.$$

Insert this in the inequality

$$\|v(t)\|_\sigma \;\le\; C\Big(\|v(0)\|_\sigma + \int_0^t \|Lv(r)\|_\sigma \; dr\Big)$$

to find

$$\|u(t)\|_{-\sigma} \;\le\; C\bigg(\|u(0)\|_{-\sigma} + \int_0^t \|u(r)\|_{-\sigma} + \|Lu(r)\|_{-\sigma}\; dr\bigg)\,.$$

Gronwall's lemma yields the desired estimate (2.1.19) for the case $s = -\sigma$.

This completes the proof of (2.1.9) for integer values of s. The estimate for s not equal to an integer follows by interpolation. ∎

The *a priori* estimates show that solutions of $Lu = 0$ grow at most exponentially in time. The simple example $u_t = u$ shows that such growth occurs. The derivation of estimates for the derivatives shows that derivatives grow at most exponentially but perhaps with at a faster rate. The next example shows that derivatives may grow more rapidly.

Examples. 1. Let $a(x) := -\arctan(x)$, and $L := \partial_t + a(x)\partial_x$. Solutions of $Lu = 0$ are constant on the characteristic curves which converge exponentially rapidly to $x = 0$ as $t \to +\infty$. The

$L^\infty(\mathbb{R})$ norm of solutions is independent of time and the $L^p(\mathbb{R})$ norm of solutions tends to zero for all $1 \le p < \infty$. For compactly supported initial data and such p,

$$\int |u(t,x)|^p \ dx \ \sim \ e^{-t}, \qquad \text{as} \qquad t \to +\infty\,.$$

For $u(0,\cdot) \in C_0^\infty(\mathbb{R}) \setminus 0$ and t large, the solution is compressed into an interval of width $\sim e^{-t}$ so the derivatives are $\sim e^t$ and one finds

$$\int |\partial_x u(t,x))|^p \ dx \ \sim \ e^{-t}\, e^{pt}, \qquad \text{as} \qquad t \to +\infty\,.$$

For $1 \le p < \infty$ the rate of growth of the derivatives is different than that of the solutions.

2. For $L := \partial_t + a(x)\partial_x + a'(x)/2$, $G(x,\partial_x) = a\partial_x + a'/2$ is antiselfadjoint and the time evolution conserves the $L^2(\mathbb{R})$ norm. The compression occurs as in the preceding example but the amplitudes grow so that the L^2 norm of the solution between any pair of characteristic curves is conserved. The amplitudes of the derivatives grow even faster by a factor e^t. For this modification, the L^2 norm is constant and the L^2 norm of first derivatives grow as e^t The L^2 norm of derivatives of order s grow as e^{st}.

§2.2. Existence theorems for symmetric hyperbolic systems.

As is the case with many good estimates, the corresponding existence theorem lingers not far behind.

Friedrichs' Theorem 2.2.1. *If $g \in H^s(\mathbb{R}^d)$ and $f \in L^1_{\rm loc}(\mathbb{R}; H^s(\mathbb{R}^d))$ for some $s \in \mathbb{R}$, then there is one and only one solution $u \in C(\mathbb{R}\,;\, H^s(\mathbb{R}^d))$ to the initial value problem*

$$Lu \ = \ f\,, \qquad u|_{t=0} \ = \ g\,. \tag{2.2.1}$$

In addition, there is a constant $C = C(L,s)$ independent of f, g so that for all $t > 0$,

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \le C\, e^{Ct} \|u(0)\|_{H^s(\mathbb{R}^d)} + \int_0^t C\, e^{C(t-\sigma)} \|f(\sigma)\|_{H^s(\mathbb{R}^d)}\ d\sigma\,, \tag{2.2.2}$$

with a similar estimate for $t < 0$.

Theorem 2.2.2 gives additional regularity in time assuming that f is smoother in t.

The solution u is constructed as the limit of approximate solutions u^h. The u^h are solutions of a differential-difference equation obtained by replacing x derivatives by centered difference quotients. This replaces the generator of the dynamics by a bounded linear operator so the existence of the approximate solution follows from existence for ordinary differential equations. The important step is to prove uniform bounds for the u^h as $h \to 0$.

As a warm up consider the simple initial value problem

$$\partial_t u + \partial_x u = 0\,, \qquad u(0,x) = g(x)\,,$$

with $x \in \mathbb{R}^1$. Define the centered difference operator by

$$\delta^h \phi(x) = \frac{\phi(x+h) - \phi(x-h)}{2h}\,.$$

Approximate solutions are defined as solutions of

$$\partial_t u^h + \delta^h u^h = 0\,, \qquad u^h(0,x) = g(x)\,.$$

Note that as operators on H^s, the norms of the generators diverge to infinity as $h \to 0$. This corresponds to the fact that the difference operators δ^h converge to the unbounded operator ∂_x.

Exercise 2.2.1. *Show that for any $s \in \mathbb{R}$ and $g \in H^s(\mathbb{R})$ this recipe determines a sequence of approximate solutions which as $h \to 0$, converge in $C(]-\infty,\infty[\,;\,H^s(\mathbb{R}))$ to the exact solution.*
Hint (Von Neumann). Use the Fourier Transform in x.

One does not have similar good behavior for the finite difference approximations

$$\partial_t u^h + i\,\delta^h u^h = 0\,, \qquad u^h(0,x) = g(x)\,,$$

to the nonhyperbolic initial value problem

$$\partial_t u + i\,\partial_x u = 0\,, \qquad u(0,x) = g(x)\,.$$

For this initial value problem and generic g there is nonexistence (see chapter 3 of my book).

Exercise 2.2.2. *For the approximations to the nonhyperbolic initial value problem prove that there is a $c > 0$ so that*

$$\liminf_{h\to 0}\ \|u^h(t)\|^2_{L^2(\mathbb{R})}\ \geq\ c\int_{\mathbb{R}} e^{2t\xi}\,|\hat g(\xi)|^2\,d\xi\,.$$

In particular, if the right hand side is infinite, $u^h(t)$ does not converge in $L^2(\mathbb{R})$ as h tends to zero. The right hand side is infinite for generic $g \in C_0^\infty(\mathbb{R})$. In the same way, prove that for generic smooth g, $u^h(t)$ is unbounded in $H^s(\mathbb{R})$ for all $t \neq 0$ and $s < 0$.

Proof of Friedrich's Theorem. It suffices to consider the case $A_0 = I$.
Step 1. Existence for $g \in \cap_s H^s(\mathbb{R}^n)$ and $f \in \cap_s C^s(\mathbb{R}\,;\,H^s(\mathbb{R}^d))$.
For such f, g, define approximate solutions u^h as solutions of

$$L^h u^h = f\,, \qquad u^h(0) = g\,,$$

where for $h > 0$, L^h comes from L upon replacing the unbounded antiselfadjoint operators ∂_j with $j \geq 1$ by the bounded antiselfadjoint finite difference operators δ_j^h defined by

$$\delta_j^h \phi\ :=\ \frac{\phi(x_1,\cdots,x_j+h,\cdots,x_d) - \phi(x_1,\cdots,x_j-h,\cdots x_d)}{2h}\,.$$

Then $L^h = \partial_t + G^h(t)$ and for every s, $t \mapsto G^h(t)$ is a smooth function with values in $\mathcal{L}(H^s(\mathbb{R}^d))$. The norm of G^h is $O(1/h)$. It follows that the u^h are uniquely determined as solutions of ordinary differential equations in time and satisfy the crude estimate $\|u^h(t)\|_{H^s} \leq C(s)\,e^{C(s)/h}\|u(0)\|_{H^s(\mathbb{R}^d)}$. This would be true for any system of partial differential operators L even those with ill posed Cauchy problems. For the symmetric hyperbolic systems, the operators L^h satisfy estimates like (2.1.19) with constants independent of h. That is, for each $s \in \mathbb{R}$ there is a constant $C(s,L)$, so that for all $0 < h < 1$ and all $u \in C^1(\mathbb{R}\,;\,H^s(\mathbb{R}^d))$, and all $t > 0$

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \leq e^{Ct}\|u(0)\|_{H^s(\mathbb{R}^d)} + \int_0^t e^{C(t-\sigma)}\|(L^h u)(\sigma)\|_{H^s(\mathbb{R}^d)}\,d\sigma\,. \qquad (2.2.3)$$

Note that for $u \in C^1(\mathbb{R}\,;\,H^s)$ it follows that $L^h u \in C(\mathbb{R}\,;\,H^s)$.

The proof of (2.2.3) for $s = 0$ mimics the proof for L. The key ingredient is almost antiselfadjointness expressed by the bound,

$$\|(A_j\delta_j^h \;+\; (A_j\delta_j^h)^*)w\|_{L^2(\mathbb{R}^d)} \;\le\; \big(\sup_{\mathbb{R}^{1+d}} |\nabla_x A_j|\big)\; \|w\|_{L^2(\mathbb{R}^d)}.$$

The right is independent of h. To prove this bound, use the fact that δ_j^j is antiselfadjoint to find

$$\big(A_j\delta_j^h\big)^* \;=\; (\delta_j^h)^* A_j^* \;=\; -\delta_j^h\, A_j \;=\; -A_j\delta_j^h - [A_j, \delta_j^h]\,.$$

Denoting by $\mathbf{e}_j$ the unit vector in the j direction, and suppressing the j's for ease of reading, $[A_j, \delta_j^h]\, w$ is equal to,

$$A(y)\,\frac{w(y+h\mathbf{e}) - w(y-h\mathbf{e})}{2h} \;-\; \frac{A(y+h\mathbf{e})\,w(y+h\mathbf{e}) - A(y-h\mathbf{e})w(y-h\mathbf{e})}{2h}\,.$$

Regrouping yields

$$[A_j, \delta_j^h]\, w \;=\; \left[\frac{A(y) - A(y+h\mathbf{e})}{2h}\right] w(y+h\mathbf{e}) \;+\; \left[\frac{A(y-h\mathbf{e}) - A(y)}{2h}\right] w(y-h\mathbf{e})\,.$$

The bound follows. Writing $L^h = \partial_t + G^h$, (2.2.3) for $s = 0$ follows from $\|G^h + (G^h)^*\| \le 2C$ with C independent of h.

The proof of (2.2.3) for $s \ge 0$ integer, uses the $s = 0$ result to estimate $\partial_x^\alpha u^h$ for $|\alpha| \le s$. Use the equation,

$$L^h \partial_x^\alpha u^h \;=\; \partial_x^\alpha L^h u^h \;+\; [L^h, \partial_x^\alpha] u^h\,.$$

For $|\alpha| = 1$, direct computation of the commutator yields

$$[A_j\delta_j^h, \partial_x] \;=\; (\partial_x A_j)\delta_j^h\,, \qquad [B, \partial_x] \;=\; (\partial_x B)\,.$$

Therefore

$$[L^h, \partial_x] \;=\; \mathrm{Op}(0) \;+\; \mathrm{Op}(0)\delta_x^h\,.$$

where this means a finite sum of terms of the type described. The general case is

$$[L^h, \partial_x^\alpha] \;=\; \mathrm{Op}(|\alpha|-1, \partial_x) \;+\; \mathrm{Op}(|\alpha|-1, \partial_x)\delta_x^h\,. \tag{2.2.4}$$

Exercises 2.2.3. i. *Prove (2.2.4) by induction on s.*

ii. *Prove (2.2.3) for integer $s \ge 0$ using (2.2.4).*

iii. *Prove for negative integer s by Lax's method as in the proof of the estimate for L. The general case follows by interpolation.*

Apply (2.2.3) to u^h to find that for all $n, \alpha, s \ge 1$, the family

$$\big\{\partial_x^\alpha u^h\big\} \quad \text{is bounded in} \quad C\big([-T,T]\,;\,H^s(\mathbb{R}^d)\big) \subset L^\infty([-T,T]\,:\,H^s(\mathbb{R}^d))\,.$$

Since H^s is Hilbert so a dual space, latter space is a dual. We use the weak star compactness of bounded sets in duals.

The differential equations together with the regularity of f then shows that

$$\{\partial_{t,x}^\beta u^h\} \quad \text{is bounded in} \quad L^\infty([-T,T]\,:\,H^s(\mathbb{R}^d))\,.$$

The Cantor diagonal process allows one to extract a subsequence $u^{h(k)}$ with $h(k)\to 0$ as $k\to\infty$ so that for all T and all α, as $k\to\infty$,

$$\partial_{t,x}^\beta u^{h(k)} \quad\to\quad \partial_{t,x}^\beta u \qquad \text{in} \quad L^\infty([-T,T]\,:\,H^s(\mathbb{R}^d))\,.$$

in the weak star topology. In particular, $u^{h(k)}$ converges in $H^s([-T,T]\times\mathbb{R}^d)$ so $u^{k(h)}|_{t=0}$ converges to $u|_{t=0}$ in all $H^{s-1/2}(\mathbb{R}^d)$. So $u(0)=g$.

Passing to the limit in the equation satisfied by $u^{h(k)}$ shows that u satisfies $Lu=f$. This completes the proof of Step 1.

Step 2. Existence in the general case.

Choose

$$g_n\in C_0^\infty(\mathbb{R}^d)\,,\qquad g_n\to g \ \text{ in } \ H^s(\mathbb{R}^d)\,,$$

$$f_n\in C_0^\infty(\mathbb{R}^{1+d})\,,\qquad f_n\to f \ \text{ in } \ L^1_{\rm loc}(\mathbb{R}\,;\,H^s(\mathbb{R}^d)\,.$$

Let u^n be the solution from the first step with data f_n, g_n.

Estimate (2.2.3) implies that for all T and as $n,m\to\infty$,

$$u^n-u^m \ \to\ 0 \quad\text{in}\quad C\big([-T,T]\,;\,H^s(\mathbb{R}^d)\big) \tag{2.2.5}$$

By completeness there is a $u\in C(\mathbb{R}\,;\,H^s(\mathbb{R}^d))$ which is the limit uniformly on compact time intervals of the u^n. It follows that $Lu=f$ and $u|_{t=0}=g$.

Step 3. Uniqueness.

Suppose that $u\in C(\mathbb{R}\,;\,H^s(\mathbb{R}^d))$ satisfies $Lu=0$ and $u|_{t=0}=0$. The differential equation implies that $u_t\in C(\mathbb{R}\,;\,H^{s-1}(\mathbb{R}^d))$. so $u\in C^1(\mathbb{R}\,;\,H^{s-1}(\mathbb{R}^d))$. This is sufficient to apply the case $s-1$ of Proposition 2.1.1 for $s-1$. ∎

Using difference approximations to prove existence goes back at least to the work of Cauchy (1840) and most notably Peano on ordinary differential equations. In the context of partial differential equations, note the seminal paper of Courant, Friedrichs and Lewy (1928). The method has the advantages of being constructive and wide applicability. In Appendix 2B, we present an alternate functional analytic method for passing from the *apriori* estimates to existence. It is elegant and strictly limited to linear problems.

The next result shows that when the source term f is differentiable in time, then so is u.

Corollary 2.2.2. *If $m\ge 1$ and $\partial_t^k f\in C(\mathbb{R}\,;\,H^{s-k}(\mathbb{R}^d))$ for $k=0,1,\cdots,m-1$, then*

$$u\in C^k\big(\mathbb{R}\,;\,H^{s-k}(\mathbb{R}^d)\big)\qquad\text{for}\quad k=0,1,\cdots,m\,.$$

Proof. It suffices to consider $A_0=I$. For $m=1$ write,

$$u_t \ =\ -\sum A_j\,\partial_j u - Bu + f.$$

The hypothesis together with Theorem 2.2.1 shows that the right hand side is continuous with values in H^{s-1}. For $m = 2$,

$$u_{tt} \;=\; \partial_t\Big(- \sum A_j\, \partial_j u - Bu + f\Big).$$

From the case $m = 1$, the first terms are continuous with values in H^{s-2}. The hypothesis on f treats the last term. An induction completes the proof. ∎

We next prove a formula which expresses solutions of the inhomogeneous equation $Lu = f$ in terms of solutions of the Cauchy problem for $Lu = 0$. The formula is motivated as follows. Let $h(t)$ denote heaviside's function. For f supported in $t \ge 0$ seek u also supported in the future with $Lu = f$. The source term f is the sum on σ of the singular sources $f\delta(t-\sigma)$ with $\sigma \ge 0$. The solution of $Lv = f\delta(t-\sigma)$ with a response supported in $t \ge \sigma$ is equal to $v = h(t-\sigma)w$ where w is the solution ot the Cauchy problem,

$$Lw \;=\; 0, \qquad w(\sigma) \;=\; A_0^{-1} f(\sigma)\,.$$

Summing the solutions v yields the formula of the next Proposition.

Define the operator $S(t,\sigma)$ from $H^{-\infty}(\mathbb{R}^d) := \cup_s H^s(\mathbb{R}^d)$ to itself by $S(t,\sigma)g := u(t)$ where u is the solution of

$$Lu \;=\; 0\,, \qquad u(\sigma) \;=\; g\,.$$

The operator $S(t,\sigma)$ is the operator that marches from time σ to time t. Corollary 2.2.2 implies that for $g \in H^s(\mathbb{R}^d)$, $S(t,\sigma)g \in C^k\big(\mathbb{R}\,;\, H^{s-k}(\mathbb{R}^d)\big)$ and, and by definition, $L(t,x,\partial_{t,x})S = 0$. For any R, $\{S(t,\sigma)\,:\, |t,\sigma| \le R\}$ is bounded $\mathrm{Hom}(H^s, H^{s-k})$.

Duhamel's Proposition 2.2.3. *If $f \in L^1_{\mathrm{loc}}([0,\infty[\;;\, H^s(\mathbb{R}^d))$ then the solution of the initial value problem,*

$$Lu \;=\; f \quad \text{on} \quad [0,\infty[\times\mathbb{R}^d, \qquad u(0) \;=\; 0, \tag{2.2.6}$$

is given by

$$u(t) \;=\; \int_0^t S(t,\sigma)\; A_0^{-1}\, f(\sigma)\; d\sigma\,. \tag{2.2.7}$$

Proof. It suffices to prove (2.2.7) for $f \in C([0,\infty[\,;\, H^s(\mathbb{R}^d))$ since the general result then follows by approximation. For such f define u by (2.2.7). On compact sets of t, σ one has

$$\|S(t,\sigma)\|_{H^s \to H^s} \;+\; \|S_t(t,\sigma)\|_{H^s \to H^{s-1}} \;\le\; C\,. \tag{2.2.8}$$

It follows that $u \in C([0,\infty[\,;\, H^s(\mathbb{R}^d))$ and differentiating under the integral sign,

$$u_t \;=\; S(t,t)(A_0^{-1} f(t)) \;+\; \int_0^t S_t(t,\sigma)\; f(\sigma)\; d\sigma \;=\; A_0^{-1} f(t) \;+\; \int_0^t S_t(t,\sigma)\; f(\sigma)\; d\sigma\,,$$

and,

$$\partial_j u \;=\; \int_0^t \partial_j S(t,\sigma)\; f(\sigma)\; d\sigma\,.$$

It follows from $LS = 0$, that $Lu = f$. Since $u(0) = 0$ the result follows from uniqueness of solutions to (2.2.6) ∎

§2.3. Finite speed of propagation.

The proof of Theorem 2.2.1 works as well for the operator $L + i\Delta_x$ as for L since the additional operator is exactly antisymmetric and commutes with ∂_y. However the resulting evolution equations do not the finite speed of propagation. An important aspect of the L^2 estimates which form the basis of the results in §2.2 is that the local form of the energy law implies finite speed. That is not the case for the energy law for $L + i\Delta$.

One of the goals of the theory of partial differential equations is to be able to derive precise qualitative properties of solutions from properties of the symbol. The precise speed estimates proved in §2.5 is a striking success.

§2.3.1. The method of characteristics.

Corolary 2.1.3 already addressed finite speed using the method of characteristics. Here we use this method in the simplest case of one dimensional homogeneous constant coefficient systems,

$$\partial_t u + A\partial_x u \;=\; 0\,, \qquad A = A^*\,.$$

The change of variable $w = Wv$ with unitary W yields $\partial_t Wv + \partial_x AWv = 0$. Multiplying by W^* yields

$$\partial_t w \;+\; W^* A W\, \partial_x w \;=\; 0\,.$$

Choose W which diagonalises A to find,

$$\partial_t w \;+\; D\,\partial_x w \;=\; 0\,, \qquad D = \operatorname{diag}\big(\lambda_1, \lambda_2 \;\cdots\; \lambda_d\big)\,, \qquad \lambda_1 \le \lambda_2 \le \ldots \le \lambda_N\,.$$

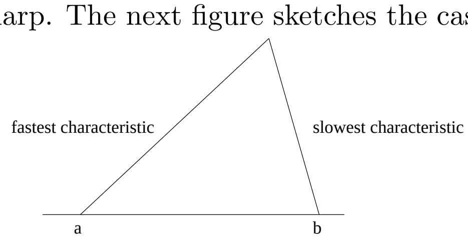
The exact solution is given by undistorted traveling waves,

$$w_j(t,x) \;=\; g_j(x - \lambda_j\, t)\,.$$

If $g = 0$ on an interval $I :=]a, b[$ then for $t \ge 0$, u vanishes on the triangle

$$\Big\{(t,x)\;:\; t \ge 0\,,\;\; \text{and}\;\; a + \lambda_N\, t \;<\; x \;<\; b - \lambda_1\, t\Big\}\,.$$

A degenerate case is when $\lambda_1 = \lambda_N$ in which case the triangle becomes a strip bounded by parallel lines. The triangle is called a *domain of determination* of the interval because the initial values of u on I determine the values of u on the triangle. Considering traveling waves with speeds λ_N and λ_1 shows that this result is sharp. The next figure sketches the case $\lambda_1 < 0 < \lambda_N$.



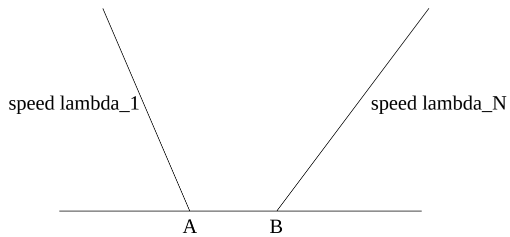
Domain of determination of I

The left hand boundary moves at largest velocity and the right hand boundary at the smallest.

Viewed another way, the values of f on an interval $J = [A, B]$ influence the future values of the solution u only on the set

$$\Big\{(t,x)\;:\; t \ge 0\,,\;\; \text{and}\;\; A + \lambda_1 t \;\le\; x \;\le\; B + \lambda_N t\Big\}\,.$$

It is called a *domain of influence* of J and is sketched below in the case $\lambda_1 < 0 < \lambda_N$.



Domain of influence of J

§2.3.2. Speed estimates uniform in space

The estimates in this section concern propagation estimates which are uniform in space time. In subsequent sections precise bounds taking into account the variation of speed with y will be given. The uniform results take into account the variation of speed with direction. The results in this section are as sharp for operators with constant coefficient principal part. That case is the backbone of the precise results of §2.5.

Usually one thinks of speed in terms of distance traveled divided by time. However, for a general symmetric hyperbolic operator, there is no natural metric to measure distance. Introducing an artificial metric, for example the Euclidean metric, often leads to imprecision in the results. The problem we analyse in this section depends only on affine geometry. It is to find the smallest space time half space which contains the support of solutions whose Cauchy data have support in a half space in $\{t = 0\}$.

Define

$$\Lambda \ := \ \inf \Big\{ \ell \ : \ \forall y\,, \quad A_0(y)^{-1/2}\, A_1(y)\, A_0(y)^{-1/2} \ \le \ \ell\, I \Big\}\,. \tag{2.3.1}$$

Λ is the smallest upper bound for the eigenvalues of $A_0(y)^{-1/2}\, A_1(y)\, A_0(y)^{-1/2}$.

Lemma 2.3.1. *Suppose that L is a symmetric hyperbolic operator, $s \in \mathbb{R}$, Λ is defined by (2.3.1),*

$$u \in C([0,\infty[\,;\, H^s(\mathbb{R}^d)) \quad \text{satisfies} \quad Lu = 0\,,$$

and

$$\operatorname{supp} u(0,x) \ \subset \ \big\{ x_1 \ \le \ 0 \big\}\,.$$

Then, for $0 \le t$,

$$\operatorname{supp} u(t,x) \ \subset \ \big\{ x_1 \ \le \ \Lambda\, t \big\}\,.$$

Example. If $A_0^{-1/2} A_1 A_0^{-1/2}$ is independent of y, one obtains the same estimate for propagation in the x_1 direction that was obtained in the preceding section.

Proof. Choose $f_n \in C_0^\infty(\mathbb{R}^d)$ supported in $\{x_1 \le 0\}$ with $f_n \to f$ in $H^s(\mathbb{R}^d)$. Denote by u_n the solution of $L\, u_n = 0$ with $u_n(0,x) = f_n(x)$. For all $T > 0$, $u_n \to u$ in $C\big([0,T] \ : \ H^s(\mathbb{R}^d)\big)$. Therefore, it suffices to show that u_n is supported in $\{x_1 \le \Lambda\, t\}$. Thus, it suffices to consider the case when the initial data of u belongs to $C_0^\infty(\mathbb{R}^d)$.

Use a local version of the basic energy law. When $A_\mu = A_\mu^*$, and $\langle\ ,\ \rangle$ is the scalar product in $\mathbb{C}^N$,

$$\big\langle A_\mu \partial_\mu u\,,\, u \big\rangle \ + \ \big\langle u\,,\, A_\mu \partial_\mu u \big\rangle \ = \ \partial_\mu \big\langle A_\mu u\,,\, u \big\rangle \ - \ \big\langle (\partial_\mu A_\mu) u\,,\, u \big\rangle\,.$$

Denote $L_1(y,\partial) := \sum_\mu A_\mu \partial_\mu$. Summing on μ yields

$$2\mathrm{Re}\,\big\langle L_1(y,\partial)u\,,\,u\big\rangle \;=\; \sum_\mu \partial_\mu \big\langle A_\mu u\,,\,u\big\rangle \;+\; \big\langle \big(\sum_\mu \partial_\mu A_\mu\big)u\,,\,u\big\rangle\,.$$

Adding the lower order terms yields the energy balance law

$$\begin{aligned}\partial_t \left\langle A_0(t,x)\,u(t,x), u(t,x)\right\rangle + \sum_{j=1}^{d} \partial_j \left\langle A_j(t,x)\,u(t,x), u(t,x)\right\rangle = \\ \big\langle Z(t,x)\,u(t,x), u(t,x)\big\rangle + 2\,\mathrm{Re}\,\big\langle (Lu)(t,x), u(t,x)\big\rangle\,,\end{aligned} \tag{2.3.2}$$

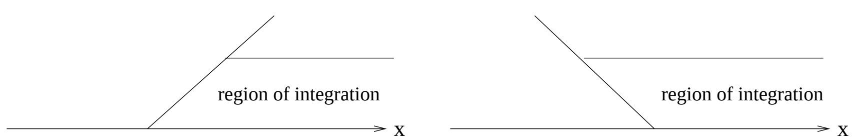
where Z is the smooth matrix valued function

$$Z(y) \;:=\; -B(y) - B^*(y) + \sum_{\mu=0}^{d} \frac{\partial A_\mu(y)}{\partial y_\mu}\,. \tag{2.3.3}$$

Integrate this identity over the region where we want to prove that $u = 0$,

$$\Omega(\underline{t}) \;:=\; \Big\{(t, x_1, x_2, \dots, x_d) \;:\; 0 \le t \le \underline{t} \quad \text{and} \quad x_1 \ge t\Lambda\Big\}\,. \tag{2.3.4}$$

The case $\Lambda > 0$ (resp. $\Lambda < 0$) are sketched on the left (resp. right) in the figure below.



Define

$$\Phi(t) \;:=\; \Big(\int_{x_1 \ge \Lambda t} \big\langle A_0(t,x)\,u(t,x)\,,\,u(t,x)\big\rangle\,dx\Big)^{1/2}\,,$$

so $\Phi(t)$ is equivalent to the L^2 norm of $u(t)$ on $\{x_1 \ge \Lambda\,t\}$. Integrate (2.3.2) over $\Omega(\underline{t})$. Denote the lateral boundary of $\Omega(\underline{t})$ by

$$\mathcal{B}(\underline{t}) \;:=\; \Big\{(t,x) \;:\; 0 \le t \le \underline{t}\,, \quad \text{and} \quad x_1 = \Lambda\,t\Big\}\,.$$

Integrate by parts[†] to find

$$\Phi(\underline{t})^2 \;=\; \Phi(0)^2 \;-\; \int_{\Omega(\underline{t})} \langle Z\,u\,,\,u\rangle\;dx\,dt \;-\; \int_{\mathcal{B}(\underline{t})} \big\langle L_1(y,\nu)\,u\,,\,u\big\rangle\;d\sigma\,,$$

where ν is a the unit outward normal and $d\sigma$ is the surface area element.
The function $\phi(t,x) := \Lambda\,t - x_1$ is negative in the interior of the region and positive in the interior of the complement. The outward pointing normals, ν, to the lateral boundary are positive multiples of

$$d\phi \;=\; \Big(\frac{\partial\phi}{\partial t}\,,\,\frac{\partial\phi}{\partial x_1}\,,\,0,\ldots,0\Big) \;=\; \Big(\Lambda\,,\,-1\,,\,0\,,\,\ldots\,,0\Big)\,.$$

The boundary matrix $L_1(y,\nu) = \sum_\mu A_\mu(y)\nu_\mu$ is a positive multiple of

$$\Lambda A_0 - A_1 \;=\; A_0^{1/2}\Big(\Lambda I - A_0^{-1/2}\,A_1\,A_0^{-1/2}\Big)A_0^{1/2} \;\geq\; 0$$

from the definition of Λ. Therefore integral over $\mathcal{B}(\underline{t})$ is nonnegative.
By hypothesis, $\Phi(0) = 0$. The volume integral satisfies,

$$\Big|\int_{\Omega(\underline{t})} \langle Z\,u\,,\,u\rangle\;dx\,dt\Big| \;\leq\; C\int_0^{\underline{t}} \Phi(t)^2\;dt\,.$$

Combining yields

$$\Phi(\underline{t})^2 \;\leq\; C\int_0^{\underline{t}} \Phi(\sigma)^2\;d\sigma\,.$$

Gronwall's lemma implies that $\Phi \equiv 0$ and the proof is complete. ∎

Remark. The same proof fails for the operator $L + i\Delta_x$ because there is no choice of Λ which guarantees that the boundary terms from the $i\Delta$ will be nonnegative.

[†] This integration by parts involves smooth functions u all of whose derivatives are square integrable. This is sufficient. For example we show that

$$\int\int_{\Omega(\underline{t})} \partial_1\,g\;dx\,dt \;=\; -\int_{\mathcal{B}(\underline{t})} g\,d\sigma\,, \qquad \text{with} \qquad g \;=\; \langle A_1 u\,,\,u\rangle,$$

Denote by B the Banach space of functions g so that $\{g, \partial_1 g\} \subset L^1(\Omega(\underline{t}))$. $C^\infty_{(0)}(\Omega(\underline{t}) \subset B$ is dense. For the trace at $\mathcal{B}$ for elements of the dense set,

$$\int_{\mathcal{B}} |g|\;dt\,dx_2\,\ldots\,dx_d = \int_0^t \int_{\mathbb{R}^{d-1}} \Big|\int_{\Lambda t}^\infty \frac{\partial g}{\partial x_1}\;dx_1\Big|dx'\;dt \leq \|\partial_1 g\|_{L^1(\Omega(\underline{t}))}\,.$$

Therefore the trace $g \mapsto g|_{\mathcal{B}}$ extends as a continuous map $B \to L^1(\mathcal{B})$. And the linear map

$$g \mapsto \int\int_\Omega \partial_1 g\,dx\,dt + \int_{\mathcal{B}} g\,d\sigma$$

is a continuous and vanishes on a dense subset.

Theorem 2.3.2. *Suppose that L is a symmetric hyperbolic operator, $\xi \in \mathbb{R}^d \setminus 0$, and*

$$\Lambda(\xi) \ := \ \inf\Big\{\ell \ : \ \forall y\,, \ \ A_0(y)^{-1/2}\Big(\sum_j A_j(y)\xi_j\Big)\,A_0(y)^{-1/2} \ \le \ \ell\, I\Big\}\,. \tag{2.3.5}$$

If $s \in \mathbb{R}$ and $u \in C([0,\infty[\,;\,H^s(\mathbb{R}^d))$ satisfies $Lu = 0$ and

$$\operatorname{supp} u(0,x) \ \subset \ \big\{x.\xi \ \le \ 0\big\}\,,$$

then for $0 \le t$,

$$\operatorname{supp} u(t,x) \ \subset \ \big\{x.\xi \ \le \ \Lambda(\xi)\,t\big\}\,.$$

Proof. Choose linear spatial coordinates

$$\tilde{x} \ := \ M\,x\,, \qquad \tilde{x}_k \ = \ \sum M_{kj}x_j\,,$$

so that $\sum \xi_j dx_j = d\tilde{x}_1$. Since

$$d\tilde{x}_1 \ = \ \sum \frac{\partial \tilde{x}_1}{\partial x_k}\,dx_k \ = \ \sum M_{1k}\,dx_k$$

this is equivalent to $M_{1k} = \xi_k$.
Since

$$\frac{\partial}{\partial x_j} \ = \ \sum_k \frac{\partial \tilde{x}_k}{\partial x_j}\,\frac{\partial}{\partial \tilde{x}_k} \ = \ \sum_k \ M_{kj}\,\frac{\partial}{\partial \tilde{x}_k}\,,$$

in the new coordinates

$$\sum A_j\,\frac{\partial}{\partial x_j} \ = \ \sum_{j,k} A_j\,M_{kj}\,\frac{\partial}{\partial \tilde{x}_k}\,,$$

so one still has a symmetric hyperbolic system. The coefficient of $\partial/\partial\tilde{x}_1$ is equal to

$$\sum_j A_j\,M_{1j} \ = \ \sum_j A_j\,\xi_j\,.$$

The result follows upon applying Lemma 2.3.1 in the $\tilde{x}$ coordinates. ∎

Sketch of alternate proof. Modify the proof of Lemma 2.3.1 as follows. Use the energy method in the region

$$\Big\{0 \le t \le \underline{t} \ \ \text{and} \ \ x.\xi \ge t\Lambda(\xi)\Big\}.$$

The lateral boundary has equation

$$\phi(t,x) = 0\,, \qquad \phi(t,x) \ := \ t\Lambda(\xi) \ - \ x.\xi\,.$$

The lateral boundary matrix $L_1(y,\nu)$ is a nonegative multiple of

$$L_1(y,d\phi) = L_1(y,\Lambda,-\xi) = \Lambda\,A_0 - \sum_j A_j\xi_j = A_0^{1/2}\Big(\Lambda I - A_0^{-1/2}\big(\sum_j A_j\xi_j\big)A_0^{-1/2}\Big)A_0^{1/2}\,.$$

This is nonnegative by definition of Λ. ∎

Example. The fundamental solution u is the solution of $Lu = 0$ with $u|_{t=0} = \delta(x)$. Then for $t \geq 0$,

$$\operatorname{supp} u \quad \subset \quad \Big\{(t,x) \ : \ \forall\, \xi, \ \ x.\xi \ \leq \ t\Lambda(\xi)\Big\}. \tag{2.3.6}$$

The set on the right is a convex cone in $t \geq 0$ whose section at $t = 1$ is compact. In the $d = 1$ case, $\operatorname{supp} u \subset \big\{(t,x) \ : \ t\lambda_1 \leq x \leq t\lambda_N\big\}$.

Exercise 2.3.1. *Prove the estimate measuring speed using euclidean distance. Define*

$$c_{max} \ := \ \max_{|\xi|=1} \ \Lambda(\xi). \qquad (n.b. \text{ the euclidean } |\xi|.)$$

i. *If $Lu = 0$ and $u(0)$ is supported in $\{|x| \leq r\}$ then $u(t)$ is supported in $\{|x| \leq r + c_{max}|t|\}$.*
ii. *For the solution u in Theorem 2.2.1, let*

$$K \ := \ \operatorname{supp} u(0) \ \cup \ \big(\operatorname{supp} f \cap \{t \geq 0\}\big).$$

Prove that in $\{t \geq 0\}$,

$$\operatorname{supp} u \ \subset \ \Big\{(t,x) \ : \ \exists(\underline{t}, \underline{x}) \in K, \quad |x - \underline{x}| \leq t - \underline{t}\Big\}.$$

Hint. Use Duhamel's formula.

There are distributional right hand sides which are not covered by the sources $f \in L^1_{\rm loc}(\mathbb{R}\,;\, H^s(\mathbb{R}^d))$. An interesting example is $f = \delta(t,x)$. The next Theorem covers general distribution sources. It includes sources with no decay as $x \to \infty$. For those, finite speed is used.

Theorem 2.3.3. *For any distribution $f \in \mathcal{D}'(\mathbb{R}^{1+d})$ supported in $\{t \geq 0\}$ there is one and only one distribution $u \in \mathcal{D}'(\mathbb{R}^{1+d})$ supported in $\{t \geq 0\}$ so that $Lu = f$.*

Proof. Since f has support in $\{t \geq 0\}$, the linear functional

$$C_0^\infty(\mathbb{R}^{1+d}) \ \ni \ v \quad \mapsto \quad \big\langle f, v \big\rangle$$

extends uniquely to a sequentially continuous functional on

$$\Big\{v \in C^\infty(\mathbb{R}^{1+d}) \ : \ \operatorname{supp} v \ \cap \ \{t \geq -1\} \text{ is compact}\Big\}. \tag{2.3.7}$$

Here sequential convergence $v_j \to v$ means that there is compact set K independent of j with $\operatorname{supp} v_j \ \cap \ \{t \geq -1\} \ \subset \ K$, and, v_j and each of its partial derivatives converge uniformly on compacts. Since $C_0^\infty(\mathbb{R}^{1+d})$ is sequentially dense in (2.3.7) there can be at most one extension.

An extension is constructed by choosing $\chi \in C^\infty(\mathbb{R})$ with $\chi = 1$ for $t \geq -1/3$ and $\chi = 0$ for $t \leq -2/3$. Since f is supported in $\{t \geq 0\}$, $\langle f, v\rangle = \langle f, \chi v\rangle$ for all $v \in C_0^\infty$. The right hand side defines the desired extension. Similary the the formula $\langle u, \chi\, v\rangle$ extends the functional $\langle u, v\rangle$ to (2.3.7).

It follows that if u is a solution, then the identity

$$\big\langle u, L^\dagger v\big\rangle \ = \ \big\langle f, v\big\rangle, \tag{2.3.8}$$

extends to v belonging to (2.3.7).

For a test function $\psi \in C_0^\infty(\mathbb{R}^{1+d})$, let v be the solution of,

$$L^\dagger v \;=\; \psi\,, \qquad v = 0 \quad \text{for} \quad t >> 1\,.$$

Exercise 2.3.1**ii.** shows that v belongs to (2.3.7), so formula (2.3.8) implies that

$$\big\langle u, \psi \big\rangle \;=\; \big\langle f, v \big\rangle\,. \tag{2.3.9}$$

This determines the value of any solution u on ψ proving uniqueness of solutions.

Exercise 2.3.2. *Show that the recipe given by formula (2.3.9) defines a solution u, proving existence.* ∎

§2.3.3. Time like and propagation cones.

The results of the preceding section have a geometric interpretation in terms of the affine geometry of the characteristic variety of L.

Definition. *The* **principal symbol** $L_1(y,\eta)$ *of* $L(y,\partial)$ *is the function defined by*

$$L_1(y,\eta) \;:=\; \sum_\mu A_\mu(y)\,\eta_\mu \;=\; A_0(y)\,\tau \;+\; \sum_j A_j(y)\,\xi_j\,.$$

The prinicipal symbol arises by dropping the zero order term and replacing ∂_μ by η_μ. It is an $N \times N$ matrix valued function of (y,η). An alternative definition replaces ∂_μ by $i\,\eta_\mu$ so differs by a factor i from the above. The choice we take is natural if one expresses differential operators using the partial derivatives ∂_μ rather than $\frac{1}{i}\,\partial_\mu$. The advantage of the latter is that it is the Fourier multiplier by η_μ.

The principal symbol is invariantly defined on the cotangent bundle of $\mathbb{R}^{1+d}$. In fact, if ϕ is a smooth real valued function with $d\phi(y) = \eta$ and $v \in \mathbb{R}^N$, then as $\sigma \to \infty$,

$$\partial_\mu e^{i\sigma\phi} = i\sigma\,\frac{\partial\phi}{\partial y_\mu}\,e^{i\sigma\phi} + O(1), \quad \text{so,} \quad L_1(y,\eta)\,v \;=\; \lim_{\sigma\to\infty}\;\frac{1}{i\sigma}\,e^{-i\sigma\phi}\,L(y,\partial)\big(e^{i\sigma\phi}\,v\big)\,.$$

The right hand side is independent of coordinates and $(y, d\phi(y))$ is a well defined element of the cotangent bundle.

Definitions. *The* **characteristic polynomial** *of* $L(y,\partial)$ *is the polynomial* $p(y,\eta)$ *defined by*

$$p(y,\eta) \;:=\; \det L_1(y,\eta)\,.$$

The **characteristic variety** *of* L*, denoted* $\mathrm{Char}\,L$*, is the set of pairs* $(y,\eta) \in \mathbb{R}^{1+d} \times \mathbb{R}^{1+d} \setminus 0$ *such that* $p(y,\eta) = 0$*. Points in the complement of* $\mathrm{Char}\,L$ *are called* **noncharacteristic**.

The characteristic variety is a well defined subset of the cotangent bundle. It is **conic** in the sense that

$$r \in]0,\infty[\quad \text{and} \quad (y,\eta) \in \mathrm{Char}\,L \qquad \Longrightarrow \qquad (y, r\,\eta) \in \mathrm{Char}\,L\,.$$

Definition. *For a symmetric hyperbolic $L(y, \partial)$, the cone of* **forward time like codirections at** y *is,*

$$\mathcal{T}(y) \ := \ \Big\{(\tau, \xi) \ : \ L_1(y, \tau, \xi) \ > \ 0\Big\}.$$

The next result is an immediate consequence of the definition and the fact that $L_1(y, \eta)$ is linear in η.

Proposition 2.3.4. *$\mathcal{T}(y)$ is an open convex cone that contains $(1, 0, \ldots, 0)$. It is equal to the connected component of $(1, 0, \ldots, 0)$ in the noncharacteristic points over y.*

Exercise 2.3.3. *Prove the proposition.*

The linear form $\tau t + \xi.x$ is called time like because if one changes to new coordinates with $t' = \tau t + \xi.x$ then in the new coordinates the cooefficient of $\partial/\partial t'$ is equal to $L_1(y, \tau, \xi)$. It is positive precisely when τ, ξ is time like. In that case, the system in the new coordinates will be symmetric hyperbolic. The new time variable is OK. More generally a proposed nonlinear change with $t' = t'(t, x)$ leads to a system with coefficient $L_1(y, dt'(t, x))$ in front of $\partial/\partial t'$. Reasonable time functions are those whose differential, dt' belongs to the forward time like cone.

Examples 1. *$L = \partial_t + \partial_1$ with $\mathcal{T} = \{\tau + \xi_1 > 0\}$ shows that $\mathcal{T}$ need not be a subset of $\{\tau > 0\}$.*

2. *In the one dimensional constant coefficient case of §2.3.1, the characteristic variety is a finite union of lines given by*

$$\mathrm{Char}\, L \ = \ \cup_j \Big\{(\tau, \xi) \ : \ \tau + \lambda_j \xi = 0\Big\},$$

where the λ_j are the eigenvalues of $A_0^{-1/2} A_1 A_0^{-1/2}$ in nondecreasing order.

For the second example, the rays $x = \lambda_j t + \mathrm{const.}$ describe the propagation of traveling waves. The velocity vectors $(1, \lambda_j)$ are orthogonal to the lines

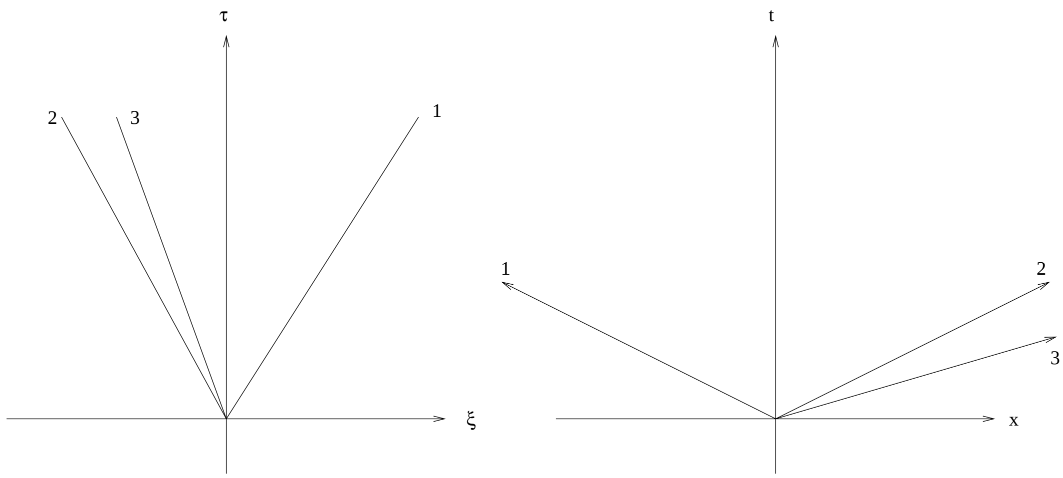
$$\Big\{(\tau, \xi) \ : \ \psi_j(\tau, \xi) \ := \ \tau + \lambda_j \xi = 0\Big\},$$

which belong to the characteristic variety. The conormal directions are scalar multiples of the differential

$$d\psi_j \ = \ \left(\frac{\partial \psi_j}{\partial \tau}, \frac{\partial \psi_j}{\partial \xi}\right) \ = \ \Big(1, \lambda_j\Big).$$

The lines of the characteristic variety are in the dual space $\mathbb{R}^2_{\tau,\xi}$. The normals to such lines define directions in the space time $\mathbb{R}^2_{t,x}$.

These relations are illustrated in the figure below where there are two distinct positive eigenvalues, $\lambda_1 < 0 < \lambda_2 < \lambda_3 = -\lambda_1$.



The rays of propagation in the figure on the right are orthogonal
to the lines in the characteristic variety on the left.

The timelike cone $\mathcal{T}$ is the wedge between the lines labeled 1 and 3. These bounding lines are the steepest lines in the variety on either side of the time like codirection $(1, 0)$. Any line traveling to the right faster than line 3 on the right, has conormal which is time like.

Exercise 2.3.4. *For a three speed system with only positive speeds* $0 < \lambda_1 < \lambda_2 < \lambda_3$*, sketch the graphs of the characteristic variety, the rays* $x = t\lambda_j$*, and the forward time like cone.*

Example. *For Maxwell's equations, (2.1.1),* $L = L_1$ *and*

$$\det L_1(\tau, \xi) = \tau^2 \big(\tau^2 - c^2 |\xi|^2\big)^2 .$$

The characteristic variety is the union of the horizontal hyperplane $\{\tau = 0\}$ *and the light cone* $\tau^2 = c^2 |\xi|^2$*. The forward time like cone is* $\{\tau > c\,|\xi|\}$*.*

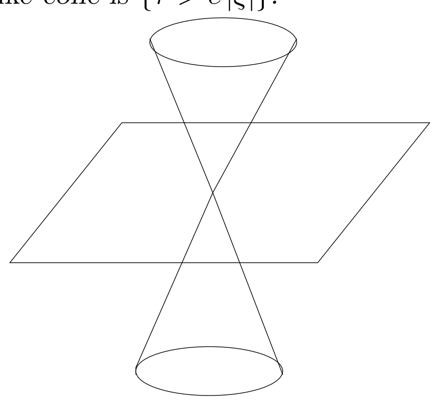


Figure 2.2. Maxwell equation characteristic variety

Return to the general development and in the characteristic polynomial $p(y, \tau, \xi)$.

Definition. *For* $\xi \in \mathbb{R}^d \setminus 0$,

$$\tau_{\max}(y, \xi) \;:=\; \max\ \big\{\tau \in \mathbb{R}\ :\ p(y, \tau, \xi) = 0\big\} . \tag{2.3.10}$$

$\tau_{\max}(y,\xi)$ is the largest eigenvalue of the symmetric matrix $-A_0^{-1/2}(\sum A_j\,\xi_j)A_0^{-1/2}$. Therefore it is uniformly lipschitzean in y,ξ. As a function of ξ it is positively homogeneous of degree one and convex. The time like cone and its closure have equations,

$$\mathcal{T}(y) \;=\; \big\{(\tau,\xi)\;:\;\tau>\tau_{\max}(y,\xi)\big\},\qquad \overline{\mathcal{T}}(y)\;=\;\big\{(\tau,\xi)\;:\;\tau\geq\tau_{\max}(y,\xi)\big\}\,.$$

Definitions. *The closed forward propagation cone is the dual of $\overline{\mathcal{T}}(y)$ defined by*

$$\Gamma^+(y)\;:=\;\Big\{(T,X)\in\mathbb{R}^{1+d}\;:\;\forall(\tau,\xi)\in\overline{\mathcal{T}}(y),\;\;T\tau+X.\xi\;\geq\;0\Big\}\,. \tag{2.3.11}$$

The section at $T=1$ is denoted,

$$\Gamma_1^+(y)\;:=\;\Gamma^+\cap\{T=1\}\,.$$

The definition states that $\Gamma^+(y)$ is the set of all points which lie in the future of the origin $(0,0)\in\mathbb{R}^{1+d}$ with respect to each time like $\tau t+\xi.x$, $(\tau,\xi)\in\mathcal{T}(y)$. For points which are not in $\Gamma^+(y)$ there is a time function so that the point is in the past. This suggests that if $(T,X)\notin\Gamma^+(y)$ and $0<\epsilon<<1$ then the point $y+\epsilon(T,X)$ should not be influenced by waves at y. This is verified in the following examples.

Examples. 1. For $\partial_t+c\partial_x$, $\Gamma^+=\{X=cT\}$, $\Gamma_1^+=\{c\}$, $\tau_{\max}(\xi)=-c\xi$.
2. For $\partial_t^2-c^2\Delta$, $\Gamma^+=\{|x|\leq ct\}$, $\Gamma_1^+=\{|x|\leq c\}$, $\tau_{\max}(\xi)=c|\xi|$.
3. For Maxwell's equations, $\tau_{\max}$ is the same as for the second example, so Γ^+ is also the same.

Example. *If $L_1(y,\partial)=L(\partial)$ has constant coefficients, then the bound (2.3.6) on the support of the fundamental solution is equivalent to the inclusion for $t\geq0$, $\operatorname{supp}u(t)\subset\Gamma^+$.*

Proof. The bound (2.3.6) is in terms of $\Lambda(\xi)$ which for constant coefficients is given by,

$$\begin{aligned}\Lambda(\xi)\;&=\;\text{largest eigenvalue of }\sum_{j=0}^{d}A_0^{-1/2}\,A_j\xi_j\,A_0^{-1/2}\\&=\;\max\Big\{-\tau(\xi)\;:\;\det\Big(\tau I+A_0^{-1/2}\,A_j\xi_j\,A_0^{-1/2}\Big)=0\Big\}\\&=\;\max\Big\{\tau(\xi)\;:\;\det\Big(\tau I-A_0^{-1/2}\,A_j\xi_j\,A_0^{-1/2}\Big)=0\Big\}\\&=\;\tau_{\max}(-\xi)\,.\end{aligned}$$

The bound (2.3.6) yields the set of (T,X) such that the following conditions, each equivalent to the one that precedes, are satisfied,

$$\begin{array}{ll}\forall\xi, & x.\xi\leq\tau_{\max}(-\xi)t\\ \forall\xi, & 0\leq\tau_{\max}(-\xi)t-x.\xi\\ \forall\xi, & 0\leq\tau_{\max}(\xi)t+x.\xi\\ \forall\tau,\xi\in\overline{\mathcal{T}}, & 0\leq\tau t+x.\xi.\end{array}\qquad\blacksquare$$

Since $\mathcal{T}(y)$ is convex and contains an open cone about $\mathbb{R}_+(1,0,\dots,0)$ it follows that $\Gamma_1^+(y)$ is a compact convex set. The next Proposition gives three more relations between Γ^+ and $\mathcal{T}$.

Proposition 2.3.5. i. *The propagation cone Γ^+ has equation*

$$\Gamma^+(y) \;=\; \Big\{(T,X)\;:\;T\ge 0 \quad \text{and,}\quad \forall \xi,\;\; T\,\tau_{\max}(y,\xi)+X.\xi\ge 0\Big\}. \tag{2.3.12}$$

ii. *The forward time like cone is given by the duality*

$$\overline{\mathcal{T}}(y) \;=\; \Big\{(\tau,\xi)\;:\;\forall (T,X)\in\Gamma^+(y)\setminus 0\,,\;\; \tau T+\xi.X \;\ge\; 0\Big\}. \tag{2.3.13}$$

$\mathcal{T}$ is given by the same formula with $\ge$ replaced by $>$.
iii.

$$\tau_{\max}(\xi) \;=\; \max_{X\in\Gamma_1^+} -X.\xi\,, \tag{2.3.14}$$

Proof. i. Suppress the y dependence. Take $(\tau,\xi)=(1,0)$ in (2.3.11) to show that $T\ge 0$ in Γ^+. Γ^+ is defined by $T\tau+x.\xi\ge 0$ when $\tau\ge\tau_{\max}$. Since $T\ge 0$ this holds if and only if it holds when $\tau=\tau_{\max}$, proving (2.3.12)
ii. From the definition of Γ^+ it follows that

$$\forall\;(T,X)\in\Gamma^+,\;\;\forall\;(\tau,\xi)\in\overline{\mathcal{T}},\qquad T\tau+X\xi\ge 0.$$

Therefore,

$$\overline{\mathcal{T}}\;\subset\;\Big\{(\tau,\xi)\;:\;\forall(\tau,\xi)\in\overline{\mathcal{T}},\quad T\tau+X\xi\ge 0\Big\}. \tag{2.3.15}$$

In fact there is equality. If $(\underline{\tau},\underline{\xi})\notin\overline{\mathcal{T}}$, then $\underline{\tau}<\tau_{\max}(\underline{\xi})$ for some $\underline{\xi}$. The point $(\tau_{\max}(\underline{\xi}),\underline{\xi})$ is a boundary point of the closed convex set $\overline{\mathcal{T}}$, so there is a $(T,X)\neq 0$ so that

$$T\tau_{\max}(\underline{\xi})+X\underline{\xi}=0,\qquad\text{and}\qquad \forall\tau,\xi\in\overline{\mathcal{T}},\quad T\tau+X\xi\ge 0.$$

For $\tau=1$ and ξ in a small neighborhood of the origin, $\tau,\xi\in\mathcal{T}$. It follows that $T\neq 0$. Therefore, $T\underline{\tau}+X.\underline{\xi}<0$ showing that $(\underline{\tau},\underline{\xi})$ is not in the set on the right of (2.3.15).
iii. In (2.3.12) it suffices to consider (T,X) with $T=1$ and $X\in\Gamma_1^+$, so

$$\mathcal{T}\;=\;\Big\{(\tau,\xi)\in\mathbb{R}^{1+d}\;:\;\forall X\in\Gamma_1^+,\;\;\tau+X.\xi>0\Big\}.$$

Thus $\mathcal{T}$ has equation $\tau+\min\{X.\xi\;:\;X\in\Gamma_1^+\}>0$. Comparing with $\tau>\tau_{\max}(\xi)$ yields,

$$\tau_{\max}(\xi)\;=\;-\min_{X\in\Gamma_1^+}X.\xi\;=\;\max_{X\in\Gamma_1^+}-X.\xi\,. \qquad \blacksquare$$

The cones $\mathcal{T}(y)$ and $\Gamma^+(y)$ concern the differential operator at the point y. The results of the preceding section give estimates on propagation which are independent of y. They involve the uniform objects of the next definition.

Definitions.

$$\tau_{\max}^{\text{unif}}(\xi)\;:=\;\sup_{y\in\mathbb{R}^{1+d}}\;\tau_{\max}(y,\xi)\,,$$

$$\mathcal{T}_{\rm unif} \ := \ \Big\{(\tau,\xi)\ :\ \tau\ >\ \tau_{\max}^{\rm unif}(\xi)\Big\}, \qquad \overline{\mathcal{T}}_{\rm unif} \ := \ \Big\{(\tau,\xi)\ :\ \tau\ \ge\ \tau_{\max}^{\rm unif}(\xi)\Big\},$$

$$\Gamma^+_{\rm unif} \ := \ \Big\{(T,X)\ :\ \forall(\tau,\xi)\in\overline{\mathcal{T}}_{\rm unif}\,, \quad T\,\tau + X.\xi\ge 0\Big\}\,.$$

Since $\tau_{\max}(y,\xi)$ is the largest eigenvalue of $-A_0(y)^{-1/2}\big(\sum A_j(y)\xi_j\big)A_0(y)^{-1/2}$ it follows that $\tau_{\max}^{\rm unif}$ is uniformly lispschitzean, positive homogeneous of degree one, and convex.

Exercise 2.3.5. *Show that if the coefficients are constant outside a compact set, then* $\mathcal{T}_{\rm unif} = \cap_y\ \mathcal{T}(y)$. *Show that this need not be the case in general.*

Corollary 2.3.6. *Denote by* u *the solution of* $Lu=0$, $u\big|_{t=0}=\delta(x-\underline{x})$. *Then for* $t\ge 0$,

$$\operatorname{supp} u \ \subset\ (0,\underline{x})\ +\ \Gamma^+_{\rm unif}\,. \tag{2.3.16}$$

For general initial data one has

$$\operatorname{supp} u \ \subset\ \cup_{\underline{x}\in\operatorname{supp} u(0)}\Big(\underline{x}\ +\ \Gamma^+_{\rm unif}\Big)\,. \tag{2.3.17}$$

We will show in the next section that this estimate is quite sharp in case L_1 has constant coefficients. A comparably sharp result for variable coefficients is proved in §2.5.

Proof. The second assertion follows from the first. Translating coordinates, it suffices treat the first with $\underline{x}=0$.

The definition (2.3.5) is equivalent to $\Lambda(\xi)$ being the supremum over y of the eigenvalues of $-A_0(y)^{-1/2}\big(\sum A_j(y)\xi_j\big)A_0(y)^{-1/2}$. This in turn is equal to $\tau_{\max}^{\rm unif}(-\xi)$. Corollary 2.3.3 implies that for $t>0$

$$\operatorname{supp} u \ \subset\ \Big\{(t,x)\ :\ \forall\xi \quad x.\xi\ \le\ t\,\tau_{\max}^{\rm unif}(-\xi)\Big\}\,.$$

Therefore, for all ξ, $t\,\tau_{\max}^{\rm unif}(-\xi)-x.\xi\ge 0$. Thus, replacing ξ by $-\xi$ shows that for all ξ, $t\,\tau_{\max}^{\rm unif}(\xi)+x.\xi\ge 0$. Equation (2.3.12) shows that this is equal to the set $\Gamma^+_{\rm unif}$. ∎

Definition. *If* Ω_0 *is an open subset of* $\{t=0\}$ *and* Ω *is a relatively open subset of* $\{t\ge 0\}$, Ω *is a domain of determinacy of* Ω_0 *when every smooth solution,* u*, of* $Lu=0$ *whose Cauchy data vanish in* Ω_0 *must vanish in* Ω.

The idea is that the Cauchy data in Ω_0 determine the solution on Ω. Any subset of a domain of determination of Ω_0 is also such a domain. The union of a family of domains is one, so there is a largest. The larger is Ω the more information one has, so the goal is to find large ones.

Definition. *If* S_0 *is a closed subset of* $\{t=0\}$ *and* S *is a closed subset of* $\{t\ge 0\}$ *then* S *is a domain of influence of* S_0 *when every smooth solution,* u*, of* $Lu=0$ *whose Cauchy data is supported in* S_0 *must be supported in* S.

The idea is that the Cauchy data in S_0 can influence the solution only in S. It does not assert that the data actually does influence in S. In that sense, the name is confusing.

Any closed set containing a domain of influence is also such a domain. The intersection of a family of domains of influence of S_0 is such a domain, so there is a smallest one. The smaller is the domain the more information one has.

S is a domain of influence of S_0 if and only if $\Omega := \{t \geq 0\} \setminus S$ is a domain of determination of $\Omega_0 := \{t = 0\} \setminus S_0$.

The next result rephrases the preceding Corollary.

Corollary 2.3.7. i. *For any S_0, the set*

$$S \ := \ \cup_{x \in S_0} \left(x + \Gamma^+_{\rm unif}\right)$$

is a domain of influence. **ii.** *For any Ω_0,*

$$\Big\{ y \ : \ (y - \Gamma^+_{\rm unif}) \cap \{t = 0\} \quad \subset \quad \Omega_0 \Big\}$$

is a domain of determination.

§2.4 Plane waves, group velocity, and phase velocities.

Plane wave solutions are the multidimensional analogues of traveling waves $f(x - \lambda t)$ in the $d = 1$ case. They are the backbone of our short wavelength asymptotic expansions. It is not unusual for the analysis of a partial differential equation in science texts to consist only of a calculation of plane wave solutions. Gleaning information this way is part of the tool kit of both scientists and mathematicians. In particular, plane waves will be used to show that Corollary 2.3.6 is precise in the case of operators with constant coefficient principal part.

Functions $f(x - \lambda t)$ generate the general solution of constant coefficient systems without lower order terms when $d = 1$. They are compositions with a linear functions of (t, x). The multidimensional analogue is to seek solutions as compositions with linear functions $t\tau + x.\xi = y.\eta$.

Definiton. Plane waves *are functions which depend only on $y.\eta$ for some $\eta \in \mathbb{R}^{1+d}$. That is, functions of the form*

$$u(y) \ := \ a(y.\eta)\,, \qquad a : \mathbb{R} \to \mathbb{C}^N\,. \tag{2.4.1}$$

When $L = L_1(\partial)$ is homogeneous and has constant coefficients and u is a plane wave,

$$\partial_\mu a(y.\eta) = a'(y.\eta)\, \eta_\mu, \quad \text{so,} \quad L(\partial_y)\, u = L_1(\partial_y)\, a(y.\eta) = L_1(\eta)\, a'(y.\eta)\,. \tag{2.4.2}$$

In this case, u is a solution of $L\, u = 0$ precisely when a' takes values in the kernel of $L_1(\eta)$. In particular, η must belong to the characteristic variety.

Exercise 2.4.1. *Compute all plane wave solutions for the following homogeneous constant coefficient operators.*

1. $\Pi_{j=1}^m \left(\frac{\partial}{\partial t} + \lambda_j \frac{\partial}{\partial x} \right)$ *where x is one dimensional and the λ_j are distinct reals.*

2. $\partial_t + {\rm diag}\,(\lambda_1, \dots, \lambda_d)\, \partial_x$ *with x and λ_j as in* **1**.

3. $\Box := \partial_t^2 - \Delta_x$.

4. *For $c = 1$ and characteristic $\eta = (1, 1, 0, \ \dots \ , 0)$, compute all plane wave solutions of the homogeneous Maxwell's equations that is with $\rho = \mathbf{j} = 0$.* **Partial Answer**. *There is a two dimensional space of standing waves, with $\tau = 0$, which do not satisfy the divergence conditions (2.1.2). There is a four dimensional space of solutions with η belonging to the light cone $\tau^2 = |\xi|^2$. For such solutions E, B, ξ form an orthogonal basis for $\mathbb{R}^3$.*

For problems with lower order terms, the computations are not as clean and there tend to be few plane wave solutions.

Exercise 2.4.2. *Compute all plane wave solutions for the following non homogeneous constant coefficient operators.*

5. *The Klein-Gordon equation* $\Box u + u = 0$.

6. *The dissipative wave equation* $\Box u + 2u_t = 0$.

7. *The telegrapher's equation* $u_{tt} - u_{xx} + 2u_t + u = 0$.

Discussion. The first equation is conservative and the last two dissipative. You should see some indication of that in the solutions.

A plane wave $u = a(\tau t + x.\xi)$ has initial value $u_0(x) = u|_{t=0} = a(x.\xi)$. We suppose that a' is not identically zero. The solution is said to have velocity $\mathbf{v}$ when it satisfies $u(t,x) = u_0(x - \mathbf{v}t)$. Compute

$$u_0(x - \mathbf{v}t) \;=\; a\big((x - \mathbf{v}t).\xi\big) \;=\; a\big(x.\xi - (\mathbf{v}.\xi)t\big)\,.$$

Therefore, the solution has velocity $\mathbf{v}$ if and only if $\mathbf{v}$ satisfies

$$\mathbf{v}.\xi \;=\; -\tau\,. \tag{2.4.3}$$

These velocities are called **phase velocities**. Note that for $d > 1$ there are many solutions $\mathbf{v}$. They differ by vectors which are orthogonal to ξ.

A remark on units is in order. If t has units of time and x has units of length, then since (τ, ξ) belongs to the dual space, τ (resp. ξ) has units 1/time (resp. 1/length). Therefore the solutions $\mathbf{v}$ of (2.3.4) have dimensions length/time of a velocity.

The amplitude of a plane wave solution is constant on hyperplanes $\{y.\eta = \text{const.}\}$. The amplitude seen at $t = 0$ on the hyperplane $x.\xi = 0$ appears at $t = 1$ on the hyperplane $\tau + x.\xi = 0$. For example when $\xi = (1, 0, \dots, 0)$, plane waves are functions of $t + x_1$. The amplitude achieved at $t = 0$ on the hyperplane $x_1 = 0$ are achieved at $t = 1$ on the hyperplane $x_1 = -1$.

Any constant vector which translates one of these planes to the other is a reasonable velocity. It is traditional in the applied mathematics literature to call the special choice $\mathbf{v} = \tau\xi/|\xi|^{\mathbf{2}}$ the *phase velocity*. Note that this choice is always parallel to ξ. This is the unique choice which is orthogonal to the hyperplanes $x.\xi = \text{const.}$ with orthogonality measured by the euclidean metric. The reliance on the euclidean metric shows that this notion is not intrinsic. For the simple equation

$$u_{tt} = u_{x_1 x_1} + 4u_{x_2 x_2}$$

or its system analogue

$$\partial_t \;+\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x_1} \;+\; \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \frac{\partial}{\partial x_2}\,,$$

the velocity computed in this way does not correspond to the correct propagation velocity associated to the plane wave solutions. The correct velocity is the **group velocity** from §1.2 and below.

The fact that the phase velocity is not uniquely defined is sufficiently not well known that we pause to discuss it a little more. The givens are the space time of dimension $d+1$ and two linear functions. The first is t which measures the passage of time. The second is the linear function $y.\eta$ whose level surfaces are the surfaces of constant amplitude. From these givens by considering the propagation from time $t = 0$ to time $t = 1$ one constructs the pair of hyperplanes in $\mathbb{R}^d_x$ with equations $\tau + x.\xi = 0$ and $x.\xi = 0$. From two hyperplanes in $\mathbb{R}^d_x$ with $d > 1$, it is impossible to pick a well defined vector which translates one hyperplane to the other. The striking exception is the case $d = 1$ where one has two points and the translation is uniquely determined. The conclusion is that the traditional *phase velocity* makes sense only in one dimensional space. It was exactly in the case $d = 1$ that the notion of phase velocity was introduced in the late nineteenth century.

Definition. *For $(y, \eta) \in \mathrm{Char}\, L$, let $\pi(y, \eta)$ denote the spectral projection of $L_1(y, \eta)$ onto its kernel. That is*

$$\pi(y, \eta) := \frac{1}{2\pi i} \oint_{|z|=r} \big(zI - L_1(y, \eta) \big)^{-1} \, dz \tag{2.4.4}$$

where r is chosen so small that 0 is the only eigenvalue of $L_1(y, \eta)$ in the disk $|z| \le r$.

If $L = L_1$ has constant coefficients and no lower order terms, then a necessary and sufficient condition for (2.4.1) to define a solution of $Lu = 0$ is that $\pi(\eta)\, a' = a'$. In particular, there are nontrivial solutions if and only if $\eta \in \mathrm{Char}\, L$. Except for an additive constant vector, the equation $\pi\, a' = a'$ is equivalent to

$$\pi(\eta)\, a = a\,. \tag{2.4.5}$$

This **polarization** for a recurs in all of our formulas from geometric optics.

The next result shows that for operators with constant coefficient principal part, the bounds on the support in Corollary 2.3.6 are quite precise.

Proposition 2.4.1. i. *If $L = L_1(\partial)$ has constant coefficients and is homogeneous then the fundamental solution u, $Lu = 0$, $u|_{t=0} = \delta(x)$ satisfies for $t \ge 0$, for $t \ge 0$,*

$$\mathrm{conv}\big(\mathrm{supp}\, u\big) \;\; = \;\; \Gamma^+\,, \tag{2.4.6}$$

where the left hand side denotes the convex hull.

ii. *If $L = L_1(\partial) + B(y)$ has constant coefficient principal part and v is the fundamental solution then Γ^+ is the smallest convex cone containing $(\mathrm{supp}\, v) \cap \{t \ge 0\}$.*

Proof. i. In this case, for each $\xi \in \mathrm{Char}(L)$ there are plane wave solutions $f(x.\xi - t\Lambda(\xi))$. Choosing $f(\sigma)$ vanishing for $s > 0$ and so that $0 \in \mathrm{supp}\, f$ shows that Theorem 2.3.2 is sharp in the sense that the solution does NOT vanish on any larger set $\{x.\xi > t\kappa\}$, $\kappa < \Lambda(\xi)$.

The remainder of the proof consists of 3 exercises.

Excercise 2.4.5. *This implies that the fundamental solution cannot be supported in $\{x.\xi \le t\kappa\}$ for any $\kappa < \Lambda(\xi)$.*

Excercise 2.4.6 *When $L = L_1(\partial)$ is homogeneous with constant coefficients, Γ^+ is the smallest convex cone in $t \ge 0$ which contains the support of the fundamental solution.*

Excercise 2.4.7. *Since $\delta(x)$ is homogeneous of degree $-d$, it follows that when $L = L_1(\partial)$, u is homogeneous of degree $-d$. and (2.4.6) follows.*

ii. Suppose next that $L_1(\partial) + B(y)$ had a fundamental solution v which for $t \ge 0$ is supported in convex cone $\tilde\Gamma \subset \{t > 0\}$. Prove that $\Gamma^+ \subset \tilde\Gamma$ as follows. Continue to denote by u the fundamental solution of L_1. Define for $\epsilon > 0$, $v^\epsilon(y) := \epsilon^d\, v(\epsilon y)$. Then v^ϵ is the unique solution of

$$\Big(L_1(\partial) + \epsilon\, B(\epsilon y)\Big) v^\epsilon \;=\; 0\,. \qquad v^\epsilon\big|_{t=0} \;=\; \delta(x)\,.$$

By hypothesis, v^ϵ is supported in $\tilde\Gamma$.

Excercise 2.4.8. *As $\epsilon \to 0$, $v^\epsilon \to u$ in $C\big(\mathbb{R}\,;\,H^s(\mathbb{R}^d)\big)$ for any $s < -d/2$.*

Therefore $\operatorname{supp} u \subset \tilde\Gamma$. Since $\tilde\Gamma$ is convex, part **i.** implies that $\Gamma^+ \subset \tilde\Gamma$. ∎

There is a well defined velocity associated to each point (y, η) of the characteristic variety with the property that the fiber over y is a smooth hypersurface on a neighborhood of $\eta \in \mathbb{R}^{1+d} \setminus 0$. The description is made for y fixed.

The variety is defined by the polynomial equation (with y dependence suppressed),

$$0 \;=\; \det L_1(\tau,\xi) \;:=\; P(\tau,\xi)$$

which is homogeneous of degree N. The coefficient of τ^N is equal to $\det A_0$ which is real. For ξ real, the remaining coefficients are equal to $\det A_0$ times symmetric functions of the real roots τ. Therefore the polynomial P has real coefficients.

Thus $\operatorname{Char} L$ is a real algebraic variety in $\mathbb{R}^{1+d}$. Since for each $\xi \in \mathbb{R}^d$ the equation has only real roots, the variety has for each such ξ at least one and no more than N roots τ. These roots are equal to the eigenvalues of $-A_0^{-1/2}\big(\sum_j \xi_j A_j\big)A_0^{-1/2}$.

The fundamental stratification theorem of real algebraic varieties implies that the variety has dimension d and except for a subvariety of dimension at most $d-1$, is locally a d dimensional real analytic subvariety of $\mathbb{R}^{1+d}_{\tau,\xi}$ (see [Benedetti-Risler]). Such real analytic points are called **smooth points**.

Propostion 2.4.2. *At smooth points the characteristic variety has conormal vector that is not orthogonal to the time like codirection* $(1, 0, \ldots, 0)$. *Therefore, the variety is locally a graph*

$$\tau = \tau(\xi)\,, \qquad \tau(\cdot) \in C^\omega\,.$$

Proof. Denote by ν a conormal vector to $\operatorname{Char} L$ at a smooth point $\underline{\eta}$. We need to show that

$$\langle \nu\,,\,(1,0,\ldots,0)\rangle \;\neq 0\,. \tag{2.4.7}$$

The proof is by contradiction. If (2.4.7) were not true, changing linear coordinates ξ yields

$$\nu = (0,1,0,\ldots,0)\,.$$

Then near $\underline{\tau}, \underline{\eta}$, $\operatorname{Char} L$ has an equation

$$\xi_1 \;=\; f(\tau,\xi')\,, \qquad \xi' := (\xi_2,\ldots,\xi_n)\,, \tag{2.4.8}$$

with

$$f \in C^\omega\,, \qquad f(\underline{\tau},\underline{\xi}') = \underline{\xi}_1\,, \qquad \partial_{\tau,\xi'} f(\underline{\tau},\underline{\xi}') = 0\,. \tag{2.4.9}$$

For ξ' fixed equal to $\underline{\xi}'$ expand f about $\tau = \underline{\tau}$,

$$f(\tau,\underline{\xi}') = \underline{\xi}_1 + a(\tau-\underline{\tau})^r + \text{higher order terms}\,, \qquad a \in \mathbf{R}\setminus 0\,.$$

The gradient condition in (2.4.9)) implies that the integer $r \ge 2$. Solving (2.4.8) for τ as a function of ξ_1 shows that for ξ_1 near $\underline{\xi}_1$, there are r distinct complex roots $\tau \approx \big[(\xi_1 - \underline{\xi}_1)/a\big]^{1/r}$. Since $r \ge 2$, real values of ξ_1 near $\underline{\xi}_1$ with $(\xi_1 - \underline{\xi}_1)/a < 0$ yield nonreal solutions τ.

The solutions are eigenvalues of a symmetric matrix and therefore real. This contradiction proves (2.4.7). ∎

At a smooth point $(\tau, \xi) \in \operatorname{Char} L$, let H denote the hyperplane in (τ, ξ) space which is tangent to the fiber over y. Proposition 2.4.2 guarantees that H has a normal line $\mathbb{R}(1, \mathbf{v})$. The normal line belongs to the tangent space at y. The vector $\mathbf{v}$ is called the *group velocity* associated to (τ, ξ). If the characteristic variety is given locally by the equation $\tau = \tau(\xi)$, then the normal directions are mulitples of the differential

$$d_{\tau,\xi}(\tau - \tau(\xi)) \;=\; \big(1, -\nabla_\xi \tau(\xi)\big) \qquad \text{therefore,} \qquad \mathbf{v} \;=\; -\nabla_\xi \tau(\xi)\,, \tag{2.4.10}$$

This yields the classic formula for the group velocity recalled in §1.3.

Since $\tau(\xi)$ is homogeneous of degree one in ξ, the Euler homogeneity relation reads $\xi.\nabla_\xi \tau(\xi) = \tau(\xi)$. Thus, the group velocity satisifies $-\xi.\mathbf{v} = \tau$ which is the relation (2.4.3) defining the phase velocities. *For $d \geq 2$, at smooth points of the characteristic variety the group velocity is the correct choice among the infinitely many phase velocities.*

Exercise 2.4.9. *Compute the group velocities for the equation $u_{tt} = u_{x_1x_1} + 4u_{x_2x_2}$ or its system analogue. Note that for most ξ, the velocity is not parallel to ξ.*

The set of all such normal lines is called the *ray cone* at y. If the characteristic variety at y is given by the *irreducible* equation $q(\tau, \xi) = 0$, then the ray cone is the conic real algebraic variety in (T, X) defined the equations

$$q(\tau, \xi) = 0\,, \qquad X\,\frac{\partial q(\tau, \xi)}{\partial \tau} + T\,\nabla_\xi q(\tau, \xi) = 0\,. \tag{2.4.11}$$

In formula (2.4.11), (T, X) are coordinates in the tangent space at y.

General solutions of constant coefficient initial value problems can often be expressed as a Fourier superposition of exponential solutions of the form $e^{i(\tau t + x.\xi)}$ with ξ real and $\tau \in \mathbb{C}$. Since τ may not be real these need not be plane waves. For constant coefficient operators and fixed ξ, the solutions of this form come from the (possibly complex) roots τ of the equation

$$\det L(i\tau, i\xi) = 0\,.$$

The roots $\tau_j(\xi)$ where they are nice functions of ξ define the *dispersion relations* of the equation. Of particular importance is the case of conservative systems for which the roots are automatically real.

Exercise 2.4.10. *Suppose that the constant coefficient $L(\partial_y)$ is symmetric hyperbolic and conservative in the sense that $B = -B^*$. This hold in particular if $B = 0$. Then for solutions of $Lu = 0$, the energy*

$$\int_{\mathbb{R}^d} \big\langle\, A_0\, u(t, x)\,,\, u(t, x)\,\big\rangle\; dx \tag{2.4.11}$$

is independent of time. Prove that the roots τ must be real in this case.

Exercise 2.4.11. *Find the dispersion relations for the wave equation, the Klein-Gordon Equation $\Box u + u = 0$, and the Schrödinger equation $u_t + i\Delta_x u = 0$. You must extend the notion of dispersion relation beyond the first order case to solve these problems.*

For dissipative equations the exponential solutions decay in time which corresponds to roots τ with positive imaginary parts. Hadamard's analysis of well posedness of initial value problems rests on the observation that one does not have continuous dependence on initial conditions if there exist exponential solutions whose imaginary parts tend to $-\infty$. A systematic use of exponential solutions in the study of initial value problems can be found in Chapter 3 of my book *Partial Differential Equations*.

Finally note that the *ellipticity* of a partial differential operator is defined by the absence of plane wave solutions.

Definition. *A first order system of partial differential operators $L(y, \partial_y)$ is* **elliptic at** y *if the constant coefficient homogeneous operator $L_1(\underline{y}, \partial_y)$ has no nonconstant plane wave solutions. This is equivalent to the invertibility of $L_1(\underline{y}, \eta)$ for all real η. The system is elliptic on an open set if this property holds for all points y in the set.*

Exercise 2.4.12. *Verify the ellipticity of your favorite elliptic operators. This should include at least the Laplacian, and the Cauchy-Riemann system. For the Laplacian L_1 must be replaced by L_2 in the definition of ellipticity. In general, ellipticity of an $m^{\rm th}$ order operator is equivalent to the invertibility of $L_m(y, \eta)$ for all real η.*

§2.5. Precise speed estimate

In the last section we proved that for operators with constant coefficient principal part, the forward propagation cone Γ^+ gives a good bound on the propagation of influence. In the variable coefficient case it is reasonable to expect that $\Gamma^+(y)$ describes the propagation at y. The central concept is that of influence curves which are curves whose tangents lie in the local propagation cones.

Definition *A lipschitzean curve $[a, b] \ni t \mapsto (t, \gamma(t))$ is an* **influence curve** *when for almost all t, $(1, \gamma'(t)) \in \Gamma^+(t, \gamma(t))$. The curve $(-t, \gamma(t))$ a backward influence curve when $(-1, \gamma') \in -\Gamma^+(-t, \gamma(t))$ for almost all t.*

The uniform boundedness of the sets Γ_1^+ implies that influence curves are uniformly lipschitzean. Peano's existence proof combining Euler's scheme and Ascoli's Theorem, implies that influence curves exist with arbitrary initial values.

The convexity of the sets Γ_1^+ implies that uniform limits of influence curves are influence curves. Indeed, if $y^n(t) = (t, \gamma_n(t))$ is such a uniformly convergent sequence, then $\gamma_n' \in \Gamma_1^+(t, \gamma_n(t))$. This uniform bound allows us to pass to a subsequence for which

$$\gamma_n' \quad \rightarrow \quad f(t)$$

weak star in $L^\infty([a, b])$. Since the Γ_1^+ are convex one has $f(t) \in \Gamma_1^+(t, \gamma(t))$ for almost all t.

On the other hand, g_n' converges to g' in the sense of distributions. Therefore $g' = f$ and therefore $(t, \gamma(t))$ is an influence curve.

Ascoli's Theorem implies that from any sequence of influence curves defined on $[a, b]$ whose initial points lie in a bounded set, one can extract a uniformly convergent subsequence.

Following Leray's IAS notes, define emissions as follows.

Definition. *If $K \subset [0, T] \times \mathbb{R}^d$ is a closed set, the* **forward emission** *of K denoted $\mathcal{E}^+(K)$ is the union of forward influence curves beginning in K. The* **backward emission**, *defined with backward influence curves, is denoted $\mathcal{E}^-$.*

The emissions are closed subsets of $[0, T] \times \mathbb{R}^d$.

Theorem 2.5.1. *If $u \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ satisfies $L\,u = 0$ then the support of u is contained in $\mathcal{E}^+\big(\mathrm{supp}\,(u|_{t=0})\big)$.*

The proof of this theorem, which is taken from [R, 2005], uses fattened propagation cones. The fattening gives one a little wiggle room, as in [Leray]. It also regularizes the boundary of emissions. We fatten Γ by shrinking $\mathcal{T}$.

Definitions. *For $\epsilon > 0$, define the shrunken time like cone,*

$$\mathcal{T}^\epsilon(y) \ := \ \Big\{(\tau,\xi) \in \mathbb{R}^{1+d} \ : \ \tau \ > \ \tau_{\max}(y,\xi) \ + \ \epsilon|\xi|\Big\}\,.$$

Define the fattened propagation cone, $\Gamma^{+,\epsilon}(y)$, to be the dual cone of $\overline{\mathcal{T}}^\epsilon(y)$. Denote by $\mathcal{E}^\pm_\epsilon$ the emissions defined with the $\Gamma^{\pm,\epsilon}$.

While $\Gamma^+(y)$ can be a lower dimensional cone, $\Gamma^{+,\epsilon}(y)$ has nonempty interior. The fattened cones, $\Gamma^{+,\epsilon}(y)$ are strictly convex, increasing in ϵ and contain $\Gamma^{+,\epsilon/2}(y)\setminus 0$ in their interior. In addition, $\cap_{0<\epsilon<1}\Gamma^{+,\epsilon}(y) = \Gamma^+(y)$.

Lemma 2.5.2. *To prove Theorem 2.5.1, it suffices to show that if $\epsilon > 0$, $\underline{y} \in [0,T]\times\mathbb{R}^d$, and, $\mathcal{E}^-_\epsilon(\underline{y})$ does not meet $\mathrm{supp}(u|_{t=0})$, then u vanishes on $\mathcal{E}^-_\epsilon(\underline{y})$.*

Proof. To prove Theorem 2.5.1, one must show that

$$[0,T]\times\mathbb{R}^d \ni y \notin \mathcal{E}^+(\mathrm{supp}(u(0,\cdot)) \quad \Longrightarrow \quad y \notin \mathrm{supp}(u)\,.$$

If $y \notin \mathcal{E}^+(\mathrm{supp}(u(0,\cdot))$, then points $\underline{y}$ on a neighborhood of y in $[0,T]\times\mathbb{R}^d$ are also not in $\mathcal{E}^+(\mathrm{supp}(u(0,\cdot))$. Therefore it suffices to show that

$$[0,T]\times\mathbb{R}^d \ni \underline{y} \notin \mathcal{E}^+(\mathrm{supp}(u(0,\cdot)) \quad \Longrightarrow \quad u(\underline{y}) = 0\,.$$

From the definitions,

$$\underline{y} \notin \mathcal{E}^+(\mathrm{supp}(u(0,\cdot)) \quad \Longleftrightarrow \quad \mathcal{E}^-(\underline{y}) \ \cap \ \mathrm{supp}(u(0,\cdot)) \ = \ \phi\,.$$

The compact sets $\mathcal{E}^-_\epsilon(\underline{y})$ decrease as ϵ decreases, and,

$$\cap_{0<\epsilon<1}\ \mathcal{E}^-_\epsilon(\underline{y}) \ = \ \mathcal{E}^-(\underline{y})\,.$$

Therefore, for ϵ small, $\mathcal{E}^-_\epsilon(\underline{y})$ does not meet $\mathrm{supp}\,(u|_{t=0})$.
To prove Theorem 2.5.1, it suffices to show that if $\mathcal{E}^-_\epsilon(\underline{y})$ does not meet $\mathrm{supp}(u|_{t=0})$ then $u(\underline{y}) = 0$. This is equivalent to the statement of the lemma. ∎

The next lemma is an accessibility theorem, in the sense of control theory.

Lemma 2.5.3 *If $\underline{y} = (\underline{t},\underline{x}) \in [0,T[\times\mathbb{R}^d$ and $\epsilon \in]0,1[$, then there is a $0 < \delta \le T - \underline{t}$ so that*

$$\mathcal{E}^+_\epsilon(\underline{y}) \ \supset \ \big\{\underline{y} + \Gamma^{+,\epsilon/2}(\underline{y})\big\} \cap \big\{\underline{t} \le t \le t + \delta\big\}\,.$$

δ can be chosen uniformly for $\underline{y} \in [0,T[\times\mathbb{R}^d$. An analogous result holds for backward emissions.

Proof. Continuity of $\Gamma^{+,\epsilon}(y)$ with respect to y implies that there is a δ_0 so that

$$|y - \underline{y}| < \delta_0 \quad \Longrightarrow \quad \Gamma^{+,\epsilon}(y) \supset \Gamma^{+,\epsilon/2}(\underline{y})\,.$$

Therefore, a curve $(t, \gamma(t))$ with $(1, \gamma') \in \Gamma^{+,\epsilon/2}(\underline{y})$ is an influence curve so long as it stays in $\{|y - \underline{y}| < \delta_0\}$. Choose $0 < \delta \le \delta_0$ so that this holds for $t \in [\underline{t}, \underline{t} + \delta]$ on influence curves starting in $\{|y - \underline{y}| < \delta\}$. This completes the proof for y fixed.

That the constants can be chosen uniformly follows from the fact that $\Gamma^{+}(y)$ is uniformly continuous. ∎

Lemma 2.5.4. *For any q and $\epsilon > 0$, the set $\mathcal{E}_\epsilon^-(q)$ has lipschitz boundary. The boundary has a tangent plane at almost all points. At such points, the conormals belong to $\mathcal{T} \cup -\mathcal{T}$.*

Proof. Suppose that $\underline{y} \ne q$ belongs to the boundary of $\mathcal{E}_\epsilon^-(q)$. Then for t close to and greater than $\underline{t}$, $\underline{y} + \Gamma^{+,\epsilon/4}(\underline{y})$ belongs to the complement of $\mathcal{E}_\epsilon^-(q)$. To prove this note that if there were points

$$z = (t, x) \ \in \ \underline{y} + \Gamma^{+,\epsilon/4}(\underline{y}) \ \cap \ \mathcal{E}_\epsilon^-(q) \quad \text{with} \quad \underline{t} < t < \underline{t} + \delta,$$

as in the figure below, then $\mathcal{E}_\epsilon^-(z) \subset \mathcal{E}_\epsilon^-(q)$. Lemma 2.5.? (access) implies that $\mathcal{E}_\epsilon^-(z)$ contains a neighborhood of $\underline{y}$. contradicting the fact that $\underline{y}$ is a boundary point.

On the other hand, since $\underline{y}$ belongs the emission, $\mathcal{E}_\epsilon^-(\underline{y})$ belongs the emission. Lemma 2.5.? (access) implies that for $\underline{t} > t > t - \delta$, the emission from $\underline{y}$ contains $\underline{y} - \Gamma_1^{+,\epsilon/2}(\underline{y})$. The interior of that set is thus a subset of the interior of the emission. Thus the boundary of the emmision near $\underline{y}$ is sandwiched between $\underline{y} + \Gamma^{+,\epsilon/4}(\underline{y})$ and $\underline{y} - \Gamma^{+,\epsilon/4}(\underline{y})$ as in the figure below.

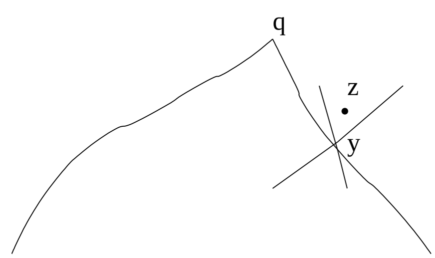


Figure 2.5.1.

At $\underline{y} = q$, it is also true that the boundary of $\mathcal{E}_\epsilon(q)$ is sandwiched between $\underline{y} + \Gamma^{+,\epsilon/4}(\underline{y})$ and $\underline{y} - \Gamma^{+,\epsilon/4}(\underline{y})$.

This shows that at all points, the boundary satisfies a two sided cone condition with cones $\pm\Gamma^{+,\epsilon/2}(\underline{y})$ which are lipschitzean in their dependence on $\underline{y}$. This proves the desired lipschitz regularity of the boundary. The differentiability then follows from Rademacher's theorem asserting the almost everywhere differentiability of lipschitz functions.

At points, $\underline{y}$, of differentiability, the tangent plane locally separates the cones $\underline{y} \pm \Gamma^{+,\epsilon/4}(\underline{y})$. This implies that the plane separates the smaller cones $\underline{y} \pm \Gamma^{+}(\underline{y})$. In §1.1 it was noted that such separating planes are exactly those which have conormals in $\mathcal{T}(\underline{y}) \cup -\mathcal{T}(\underline{y})$. ∎

Proof of Theorem 2.5.1. We verify the criterion of Lemma 2.5.2. For $0 < t < \underline{t}$ define

$$\Omega_t \ := \ \mathcal{E}_\epsilon^-(\underline{y}) \ \cap \ [0, t] \times \mathbb{R}^d, \qquad \mathcal{B}_t \ := \ \partial\mathcal{E}_\epsilon^-(\underline{y}) \cap \,]0, t[\times \mathbb{R}^d\,.$$

The set $\mathcal{B}_t$ is the lateral boundary of Ω_t.

Define

$$\phi(s) \ := \ \int_{\mathcal{E}_\epsilon^-(\underline{y}) \cap \{t=s\}} |u(s,x)|^2 \, dx \, .$$

The energy conservation law (2.3.2) implies that For smooth solutions of $Lu = 0$ and $0 < t < \underline{t}$, one has

$$0 \ = \ \int_{\Omega_t} \partial_t(u,u) \ + \ \sum_j \partial_j(A_j u, u) + (Zu, u) \ dx \, dt \, ,$$

Integrating by parts yields

$$\phi(t) \ - \ \phi(0) \ + \ \int_{\mathcal{B}_t} (\sum_{\mu=0}^{d} \nu_\mu A_\mu u, u) \ d\sigma \ + \ \int_{\Omega_t} (Zu, u) \ dx \, dt \ = \ 0 \, , \qquad (2.5.1)$$

where ν is the unit outward normal to $\mathcal{B}$ and $d\sigma$ is the element of d dimensional surface area in the boundary $\mathcal{B}$. The conormal ν is almost everywhere well defined with respect to surface area. From Lemma 2.5.4, ν belongs to $\mathcal{T} \cup -\mathcal{T}$. Since $\mathcal{E}_\epsilon^-(\underline{y})$ is a backward emission, the outward conormals belong to $\mathcal{T}$. Therefore the matrix $\sum_\mu \nu_\mu A_\mu$ is strictly positive. Using this in (2.5.1) yields

$$\phi(t) \ \le \ \phi(0) + \|Z\|_{L^\infty([0,T]\times\mathbb{R}^d)} \ \int_0^t \phi(s) \, ds \, .$$

Since $\mathcal{E}^-$ does not meet the support of $u|_{t=0}$, $\phi(0) = 0$. Gronwall's Lemma implies that $\phi(t) = 0$ for $0 \le t \le \underline{t}$. This shows that u vanishes in $\mathcal{E}_\epsilon^-(\underline{y})$ so completes the proof. ∎

§2.6. Local Cauchy problems.

Once finite speed is established it is not hard to show that the Cauchy problem has unique solutions for data and operators only defined locally. This is needed in Chapter 5 where some operators are defined only where locally defined phases exist.

Assumption. *Suppose that $0 < T < \infty$ and $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^{1+d}$ lying on one side of its compact boundary. Let $\Omega = \mathcal{O} \cap \{0 < t < T\}$. Assume that $\overline{\Omega}$ is a domain of determinacy for $L(y, \partial)$ in the sense that,*

$$\forall \, y \in \overline{\Omega}, \qquad \mathcal{E}^-(y) \cap \{0 \le t \le T\} \ \subset \ \overline{\Omega} \qquad (2.6.1)$$

Denote by $\overline{\Omega}_\sigma := \{(t,x) \in \overline{\Omega} \ : \ t = \sigma\}$ the section at time σ.

Theorem 2.6.1. *Suppose that L is a symmetric hyperbolic operator with coefficients defined only on $\overline{\Omega}$ with partial derivatives of all orders bounded. If $g \in C^\infty(\overline{\Omega}_0)$ and $f \in C^\infty(\overline{\Omega})$, then there is one and only one solution $u \in C^\infty(\overline{\Omega})$ of the initial value problem*

$$Lu = f \quad \text{on} \quad \overline{\Omega} \, , \qquad u(0) = g \quad \text{on} \quad \overline{\Omega}_0 \, . \qquad (2.6.2)$$

If $s \in \mathbb{N}$ there is a constant $C = C(L, s)$ so that for all f, g and $0 \le t \le T$,

$$\sum_{|\alpha| \le s} \|\partial_y^\alpha u(t)\|_{L^2(\Omega_t)} \le C \Big(\sum_{|\alpha| \le s} \|\partial_y^\alpha u(0)\|_{L^2(\Omega_t)} + \int_0^t \sum_{|\alpha| \le s} \|\partial_y^\alpha f(\sigma, x)\|_{L^2(\Omega_\sigma)} \ d\sigma \Big) \, . \qquad (2.6.3)$$

Proof. The first step is to construct a symmetric hyperbolic operator $\widetilde{L}$ defined everywhere on $\mathbb{R}^{1+d}$ and equal to L on $\overline{\Omega}$. To do this first extend the coefficients A_μ to smooth hermitian valued functions on $\mathbb{R}^{1+d}$ with uniformly bounded derivatives. Extend B similarly but without symmetry requirements.

By continuity, the extended coefficient A_0 is strictly positive definite on a neighborhood of $\overline{\Omega}$. This allows one to construct a possibly different extension with bounded derivatives which is strictly postive everywhere. That completes the construction of $\widetilde{L}$.

Extend f and g to $\widetilde{f} \in C_0^\infty(\mathbb{R}^{1+d})$ and $\widetilde{g} \in C_0^\infty(\mathbb{R}^d)$. Solving the tilde initial value problem on $\mathbb{R}^{1+d}$ constructs a solution.

The energy estimate for $\widetilde{L}$ implies that

$$\sum_{|\alpha| \le s} \|\partial_y^\alpha u(t)\|_{L^2(\Omega_t)} \le C(s, \widetilde{L}) \Big(\sum_{|\alpha| \le s} \|\partial_y^\alpha \widetilde{g}\|_{L^2(\mathbb{R}^d)} + \int_0^t \sum_{|\alpha| \le s} \|\partial_y^\alpha \widetilde{f}(\sigma, x)\|_{L^2(\mathbb{R}^d)} \, d\sigma \Big) .$$

The standard extension process for Sobolev functions shows that the infinum of the right hand side over extensions $\widetilde{f}$ and $\widetilde{g}$ is a norm equivalent to the right hand side of (2.6.3).

To prove uniqueness, reason as follows. If u is a solution, choose an extension $\widetilde{u} \in C_0^\infty(\mathbb{R}^{1+d})$. Then $\widetilde{L}\widetilde{u}$ vanishes in $\overline{\Omega}$ and $\widetilde{u}|_{t=0}$ vanishes in $\overline{\Omega}_0$.

The domain of determinacy hypothesis implies that for $y \in \overline{\Omega}$ and $0 \le t \le T$,

$$\mathcal{E}^-(y, \widetilde{L}) \;=\; \mathcal{E}^-(y, L) \;\subset\; \overline{\Omega} .$$

Exercise 2.6.1. *Prove this.*

The sharp finite speed result for $\widetilde{L}$ implies that $\widetilde{u}|_{\overline{\Omega}} = 0$. Since $\widetilde{u}$ is equal u on $\overline{\Omega}$, this completes the proof. ∎

Remark. Solutions with finite regularity can be constructed by an approximation argument using (2.6.3).

Appendix 2.I. Constant coefficient hyperbolic systems

For constant coefficients one can make a rough classification of hyperbolic systems. This appendix describes, largely without proof, such results.

Consider the Cauchy problem for the differential operator

$$L \;=\; A_0 \partial_t \;+\; \sum_{j=1}^d A_j \, \partial_j \;+\; B ,$$

where A_μ and B are constant $N \times N$ matrices. The hyperbolic systems will be those for which the initial value problem

$$L\,u = f, \qquad u|_{t=0} = g,$$

has a unique solution for f, g being arbitrary elements of a suitably large family of functions.

The first observation is that $t = 0$ *must be noncharacteristic, that is* A_0 *must be invertible.* In the opposite case, $\mathrm{Rg}\, A_0$ is a proper subspace of $\mathbb{C}^N$. The differential equation at $t = 0$ then implies that

$$\sum_{j=1}^d A_j \partial_j g + Bg - f(0, x) \;\in\; \mathrm{Rg}\, A_0 .$$

This is a nontrivial linear constraint on the data f, g. So to have solvability for reasonably arbitrary data it is necessary that A_0 is invertible. In that case, multiplying by A_0^{-1} reduces to the case $A_0 = I$ which we assume in the remainder of this Appendix.

The Fourier transform yields the solution of the Cauchy problem with $f = 0$,

$$\hat{u}(t,\xi) \;=\; e^{t\left(-i\sum_{j=1}^{d} A_j\xi_j - B\right)}\;\hat{g}(\xi)\,.$$

Hyperbolic systems are those for which this product makes sense for a large class of g. At the very least one would like to solve with g for which the values of g in a neighborhood of a point of x are independent of the values at $\underline{x} \neq x$. This property is not shared by the real analytic data for which solvability is a consequence of the Cauchy-Kovaleskaya Theorem. The problem of identifying hyperbolic system at least requires finding solvability in a class of functions without analyticity properties.

Considering $B = 0$ one sees that it is bad if $\sum A_j\xi_j$ has an eigenvalue λ with nonvanishing imaginary part. Replacing ξ by $-\xi$ one may suppose that $\operatorname{Im}\lambda > 0$. Then the matrix $e^{-i\sum A_j\xi_j}$ grows exponentially on a conic neighborhood $a\xi$ with $a \to \infty$.. Thus for $e^{-i\sum A_j\xi_j}\,\hat{g}(\xi)$ to be the transform of a nice object, the transform $\hat{g}$ must decay exponentially in such directions. This is a microlocal real analyticity condition on g which shows that such systems must be rejected. The argument works as well when a lower order term $B \neq 0$ is added as the amplification matrix still grows exponentially. The conclusion is that *only systems so that for real ξ, $\sum_j A_j\xi_j$ has only real eigenvalues should be called hyperbolic.*

It is not difficult to show that the condition of real spectrum is equivalent to a bound

$$\exists C, \quad \forall \xi \in \mathbb{R}^d,\ 0 \le t \le 1 \quad \left\| e^{t\left(-i\sum A_j\xi_j - B\right)}\right\| \;\le\; C\,e^{(|\xi|^{(N-1)/N})}\,.$$

Thus, for such operators the Cauchy problem is solvable for data whose Fourier transform decays as $e^{-|\xi|^{\nu}}$ with $1 > \nu > (N-1)/N$. By definition, this is the class of Gevrey data $G^{1/\nu}$. This class is good in the sense that there are Gevrey partitions of unity, and the values at distinct points are entirely independent. A profound result of Bronshtein proves that variable coefficient problems whose coefficients are $G^{1/\nu}$ smooth and so that $\sum_j A_j(t,x)\xi_j$ has only real eigenvalues yield good Cauchy problems for $G^{1/\nu}$ data. The result has the weakness that the value of a solution at t, x with $t > 0$ depends on an infinite number of derivatives of the data and coefficients.

The next class of hyperbolic sytems, introduced and analysed by Gårding with earlier contributions of Petrowsky are defined so that the dependence is reduced to a finite number of derivatives. For this it is necessary and sufficient that one has a bound

$$\exists C, m \ \ \forall \xi \in \mathbb{R}^d,\ 0 \le t \le 1, \quad \left\| e^{t\left(-i\sum A_j\xi_j - B\right)}\right\| \;\le\; C\,\langle\xi\rangle^m\,. \tag{2.I.1}$$

When (2.I.1) is satisfied, the Cauchy problem is solvable with loss of no more than m derivatives in the sense that if $g \in H^s(\mathbb{R}^d)$ then there is a solution $u \in C(\mathbb{R}\ ;\ H^{s-m}(\mathbb{R}^d))$.

The bound (2.I.1) is equivalent to the following eigenvalue condition which depends on the lower order term B,

$$\exists C, \quad \forall \xi \in \mathbb{R}^d, \qquad \text{the eigenvalues of } \ \Big(\sum_j A_j\xi_j + iB\Big) \ \text{ satisfy } \ |\operatorname{Im}\lambda| \le C\,. \tag{2.I.2}$$

This is the *standard definition of hyperbolicity for constant coefficient systems.*

Example. The system

$$\partial_t \ + \ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \partial_x \tag{2.I.3}$$

satisfies (2.I.1) with $m = 1$, but

$$\partial_t \ + \ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \partial_x \ + \ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.I.4}$$

which is obtained by adding a lower order term does not satisfy for any m.

Exercise 2.I.1. *Verify.*

Denote the variables as (u, v). The system (2.I.4) is equivalent to,

$$u_t + v_x \ = \ 0, \qquad v_t + u \ = \ 0\,. \tag{2.I.5}$$

Eliminating v yields $u_{tt} + u_x = 0$. The Cauchy provlem for this equation (and the original system) is solvable for data in $G^{1/\nu}$ for $\nu < 1/2$, and not for data with only a finite number of derivatives. This equation is the sideways heat equation discussed in [R 1991, §3.9]. The system (2.I.3) yields the hyperbolic equation, $u_{tt} = 0$. The solution of the Cauchy problem

$$u_{tt} \ = \ 0, \qquad u\big|_{t=0} \ = \ g_0, \quad u_t\big|_{t=0} \ = \ g_1,$$

is $u = g_0 + t g_1$. If $g \in H^s$ and $g_1 \in H^{s-1}$, the solution is continuous with values in H^{s-1}. This is a loss of one derivative compared to what one would have for the wave equation. This loss reflects the fact that (2.I.1) is satisfied for $m = 1$ and for no smaller value.

The strongest notion of hyperbolicity corresponds to solvability without loss of derivatives. This requires the bound,

$$\sup_{\xi \in \mathbb{R}^d,\ 0 \le t \le 1} \quad \left\| e^{t\left(-i \sum A_j \xi_j - B\right)} \right\| \ < \ \infty\,, \tag{2.I.6}$$

Estimate (2.I.6) is the case $m = 0$ of (2.I.1).

Proposition 2.I.1. *Condition (2.I.6) is equivalent to,*

$$\sup_{\xi \in \mathbb{R}^d} \quad \left\| e^{-i \sum A_j \xi_j} \right\| \ < \ \infty\,. \tag{2.I.7}$$

Proof. We prove that (2.I.6) implies (2.I.7). The opposite implication is similar. Condition (2.I.6) is a uniform estimate

$$\sup_{0 \le t \le 1,\ \xi \in \mathbb{R}^d} \left\| e^{t(A-B)} \right\| \ \le \ M, \quad \text{where} \qquad A = \sum -i A_j \xi_j.$$

We want an estimate $\sup_{\xi \in \mathbb{R}^d} \|u(1)\| < \infty$, where u satisfies $u' = Au$ with unit length initial data. Write the equation for u as

$$u' = (A - B)u \ + \ Bu,$$

so

$$u(t) \ = \ e^{t(A-B)} u(0) \ + \ \int_0^t e^{(t-s)(A-B)}\, B\, u(s)\ ds\,.$$

For $t \le 1$ one has

$$\|u(t)\| \ \le\ M\ +\ \int_0^t M\,\|B\|\,\|u(s)\|\,ds\,.$$

Gronwall's inequality implies that $\|u(1)\| \le M\,e^{M\|B\|}$ proving (2.I.7). ∎

In particular, the conditions (2.I.6) is stable under lower order perturbations.

In order for (2.I.7) to be satisfied it is neccessary that for all real ξ the matrix $\sum_j A_j\xi_j$ is similar to a real diagonal matrix. This follows by considering only the restriction to real mulitples of the given ξ. In particular the system (2.I.3) does not satisfy (2.I.6).

When (2.I.7) holds, $f = 0$, and, $g \in H^s(\mathbb{R}^d)$, then there is a solution continuous in time with values lie in H^s. A Gronwall argument shows that this property is also valid if one adds a variable coefficient lower order term, that is for $L = \partial_t + \sum A_j\partial_j + B(t,x)$ with B satisfying (2.1.6).

Kreiss' Matrix Theorem 2.I.2. *If V is a complex normed vector space, then $A \in \mathrm{Hom}(V)$ satisfies $\sup_{\sigma\in\mathbb{R}} \|e^{i\sigma A}\| < \infty$ if and only if A is diagonalisable with real eigenvalues. Write $A = \sum_j \lambda_j\pi_j$ with distinct real λ_j and π_j the projector along $\mathrm{Rg}\,(A-\lambda_j I)$ onto $\ker\,(A-\lambda_j I)$. Then*

$$\max_j\ \|\pi_j\|\ \le\ \sup_{\sigma\in\mathbb{R}} \|e^{i\sigma A}\|\ \le\ \sum_j\ \|\pi_j\|\,. \tag{2.I.8}$$

Proof. The diagonalisability characterisation is immediate from the Jordan form.

To prove (2.I.8) write

$$e^{i\sigma A}\ =\ \sum_j e^{i\sigma\lambda_j}\,\pi_j\,. \tag{2.I.9}$$

The triangle inequality shows that,

$$\forall\sigma \in \mathbb{R}, \qquad \left\|e^{i\sigma A}\right\|\ \le\ \sum_j\ \|\pi_j\|\,.$$

For the other half of (2.I.8), mulitply (2.1.8) by $e^{-\sigma\lambda_k}$ and integrate $d\sigma$ to show that,

$$\pi_k\ =\ \lim_{T\to\infty}\ \frac{1}{T}\int_0^T\ e^{-i\sigma\lambda_k}\ e^{iA\sigma}\ d\sigma\,.$$

The integral triangle inequality implies that $\left\|\pi_k\right\| \le \sup_{\sigma\in\mathbb{R}} \left\|e^{i\sigma A}\right\|$. ∎

The result (2.I.8) is often rephrased as follows. The map

$$V \ni u \quad\mapsto\quad K(u)\ :=\ (\pi_1 u\,,\,\pi_2 u\,,\,\dots\,,\,\pi_m u)\ \in\ \oplus\ker(A-\lambda_j I)$$

has norm ≤ 1 if the direct sum is normed by the maximum of the norms. Since $u = \sum_j \pi_j u$ and there are at most $N := \dim V$ summands, one has $\max_j \|\pi_j u\| \ge \|u\|/N$ proving that $\|K^{-1}\| \le N$. And KAK^{-1} is diagonal. Thus A is diagonalized by a transformation with $\|K\|\,\|K^{-1}\| \le N\sup_\sigma \|e^{i\sigma A}\|$. The last condition is invariant when K is replaced by $c\,K$.

Therefore, (2.I.7) is satisfied if and only if *there is an invertible matrix valued $K(\xi)$ with K and K^{-1} in $L^\infty(\mathbb{R}^d)$, and so that for all $\xi \in \mathbb{R}^d$, $K(\xi)\,(\sum A_j\xi_j)\,K^{-1}(\xi)$ is diagonal and real.*

Remarks. 1. The condition in italics is satisfied when the A_j are hermitian symmetric in which case K can be chosen unitary and the projectors have norm 1.

2. By homogeneity it suffices to consider ξ with $|\xi| = 1$.

3. The condition is satisfied when for all ξ with $|\xi| = 1$, $\sum_j A_j\xi_j$ is diagonalisable and the multiplicity of its eigenvalues is independent of ξ. In this case $K(\xi)$ and $\pi_j(\xi)$ can be chosen smooth on $\mathbb{R}^d \setminus 0$.

4. A special case of **3** is when $\sum A_j\xi_j$ has N distinct real eigenvalues for all $\xi \neq 0$. Such systems are called **strictly hyperbolic**.

5. We prove in §5.4.4 that when $d > 1$ and for most systems satisfying (2.I.6), the map $u(0) \mapsto u(\underline{t})$ is unbounded on L^p for $p \neq 2$.

Exercise 2.I.2. (i.) *Prove* **3.**

(ii.) *Prove that when* **3** *holds,* K *can be chosen smooth and homogeneous of degree 0 on* $\xi \neq 0$.

Appendix 2.II. Functional analytic proof of existence

This appendix proves Theorem 2.2.1 from the *a priori* estimate (2.1.18) by an abstract argument. The idea of using the Sobolev spaces for negative s to give a particularly elegant version dates at least to [Lax, 1955]. The argument uses Lax's duality in the form $L^1([0,T]\,;H^s(\mathbb{R}^d))' = L^\infty([0,T]\,;\,H^{-s}(\mathbb{R}^d))$. We abuse notation in the usual way by writing the duality of H^s and H^{-s} as an integral.

Example. For $\delta' \in H^{-2}(\mathbb{R})$ and $f \in H^2(\mathbb{R})$, $\int \delta'(x)\, f(x)\, dx = f'(0)$ is not an integral.

Proof of Theorem 2.2.1. Step 1. *If* f, g *are as in Theorem 2.2.1, then there is a solution*

$$u \in L^\infty([0,T]\,;\,H^s(\mathbb{R}^d)) \;\cap\; C([0,T]\,;\,H^{s-1}(\mathbb{R}^d))\,.$$

Let

$$\Psi \;:=\; \Big\{\psi \in \cap_k C^k([0,T]\,;, H^k(\mathbb{R}^d)) \;:\; \psi(T) = 0\Big\},$$

and

$$L^\dagger w \;:=\; -\partial_t - \sum \partial_j(A_j w) + B^\dagger,$$

the transposed operator so

$$\int_{\mathbb{R}^{1+d}} L\phi\;\psi\;dt\,dx \;=\; \int_{\mathbb{R}^{1+d}} \phi\;L^\dagger\psi\;dt\,dx,$$

for all smooth ϕ, ψ whose supports intersect in a compact set. Then $L^\dagger$ is symmetric hyperbolic. Proposition 2.1.1 with initial time T shows that,

$$\forall\,\psi \in \Psi\,, \qquad \sup_{0\le t\le T} \|\psi(t)\|_{H^{-s}(\mathbb{R}^d)} \;\le\; C(s,L) \int_0^T \|L^\dagger\psi(\sigma\|_{H^{-s}(\mathbb{R}^d)}\;d\sigma\,. \tag{2.II.1}$$

In particular, $L^\dagger$ in injective on Ψ. Let $V := L^\dagger\Psi$ a linear subspace of $L^1([0,T]\,;\,H^{-s}(\mathbb{R}^d))$. Estimate (2.II.1) asserts that $(L^\dagger)^{-1} : V \to C([0,T]\,;\,H^{-s}(\mathbb{R}^d))$ is continuous. Since $f \in L^1([0,T]\,;\,H^s(\mathbb{R}^d))$, the linear functional $\ell : V \to \mathbb{C}$ defined at $v = L^\dagger\psi$ as,

$$\ell(v) \;:=\; \int_0^T \psi\,f\;dt\,dx \;-\; \int \psi(0,x)\,g(x)\;dx,$$

is continuous. The Hahn-Banach Theorem [†] implies that there is an extension of ℓ to all of $L^1([0,T]\,;\,H^{-s}(\mathbb{R}^d))$ so there is a $u \in L^\infty([0,T]\,;\,H^s(\mathbb{R}^d))$ so that

$$\ell(v) \;=\; \int_0^T u(t,x)\,v(t,x)\;dt\,dx\,.$$

This proves that for all $\psi \in \Psi$,

$$\int_0^T\!\!\int u(t,x)\;L^\dagger\psi(t,x)\;dt\,dx \;=\; \int_0^T\!\!\int f(t,x)\;\psi(t,x)\;dt\,dx \;-\; \int \psi(0,x)\,g(x)\;dx\,. \qquad (2.II.2)$$

Exercise 2.II.1. *Prove that* $u \in C([0,T]\,;\,H^{s-1}(\mathbb{R}^d))$ *and (2.II.2) implies that* $Lu = f$ *and* $u|_{t=0} = g$. **Warning.** The x integrals in (2.II.2) are pairings of $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$ not integrals.

This completes the first Step.

Step 2. For f, g as in the Theorem, choose

$$f_n \in C_0^\infty(\mathbb{R}^{1+d}), \qquad g_n \in C_0^\infty(\mathbb{R}^d),$$

with

$$f_n \to f \quad \text{in} \quad L^1([0,T]\,;\,H^s(\mathbb{R}^d)), \qquad g_n \to g \quad \text{in} \quad H^s(\mathbb{R}^d).$$

Denote by $u_n \in \cap_s C^s([0,T]\,;\,H^s(\mathbb{R}^d))$ the solution with data f_n, g_n constructed in Step 1. Proposition 2.1.1 applied to $u_n - u_m$ proves that $\{u_n\} \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ is a Cauchy sequence. The limit $u \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ of this sequence is the desired solution, proving existence.

Step 3. Uniqueness. Uniqueness is proved as in the earlier proof. ∎

[†] One can avoid the Hahn-Banach Theorem (and therefore uncountable choice) by using continuity in $L^2[0,T]\,;\,H^s(\mathbb{R}^d))$. In this Hilbert space choose the unique extension which vanishes on $V^\perp$. This yields a $u \in L^2([0,T]\,;\,H^{-s}(\mathbb{R}^d))$ which requires small modifications in the end of the proof.

3.1. Orientation.

In this chapter we return to Fourier analysis techniques as in §1.3, §1.4. The Fourier transform of the solution is written exactly and then analysed. The results show how the geometry of the characteristic variety of $L = L_1(\partial_y)$ is reflected in qualitative properties of the solutions of $Lu = 0$. The main idea is that when the characteristic variety is curved, the corresponding solutions tend to spread out in space. This dispersive behavior is reflected in solutions becoming smaller in $L^\infty(\mathbb{R}^d)$ in contast to $L^2(\mathbb{R}^d)$ conservation.

Three simple examples illustrate the theme. The scalar advection operator

$$L \ := \ \partial_t \ + \ \mathbf{v}.\partial_x \,, \tag{3.1.1}$$

in dimension d and the system

$$\frac{\partial v}{\partial t} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial v}{\partial x} = 0 \tag{3.1.2}$$

in dimension $d = 1$ have only purely translating modes. The characteristic variety of (3.1.1) is the hyperplane $\tau + \mathbf{v}.\xi = 0$ and for (3.1.2) it is the pair of lines $\tau \pm \xi = 0$. The variety is not curved at all.

The system analogue of $\square_{1+2}$,

$$L \ := \ \partial_t \ + \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 \ + \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_2 \tag{3.1.3}$$

behaves differently. Each component satisfies $\square_{1+2} u = 0$. For smooth compactly supported data, they decay (in sup norm) as $t^{-1/2}$. The characteristic variety is $\tau^2 - |\xi|^2 = 0$. Since all charateristic varieties are conic their Gauss curvatures vanish. The present variety intersects $\tau = 1$ in a strictly convex set. So the variety is as curved as a conic set can be. The system is maximally dispersive.

Exercise 3.1.1. *Prove the decay rate for compactly supported solutions of $\square_{1+2} u = 0$ by expressing solutions as convolutions with fundamental solutions.* **Discussion.** An alternative proof uses the stationary phase inequality. That method is systematically exploited in §3.4.

For all three examples the $L^2(\mathbb{R}^d)$ norm is preserved during the time evolution.

For the solutions of the transport equation associated to (3.1.1), the size of the support of solutions does not change in time. For (3.1.3), solutions spread out over a set whose two dimensional area grows with time. The spread together with L^2 conservation, explains the decay.

In optics, the word **dispersion** is used to mean that the speed of light depends on its wavelength. In that sense, none of the above models is dispersive. The dispersion relations of the first and third models are all positive homogeneous of degree one in ξ. The velocity at $\sigma\xi$ is independent of σ so the standard optical definition classifies them as nondispersive. However for (3.1.2), the velocity depends strongly on ξ, though not on $|\xi|$. The fact that the group velocities point in different directions has the effect of spreading the solution, and for large time the solutions decay.

The variation of the group velocity with ξ is given by the matrix of second derivatives $\nabla^2_\xi \tau$. For our homogeneous operators, $\nabla_\xi \tau$ is homogeneous of degree zero, so ξ belongs to the kernel of matrix. The rank can be at most $d - 1$. The D'Alembertian $\square_{1+d}$ achieves this maximal rank so is as dispersive as a homogeneous operator can be.

At the extreme opposite is $\nabla^2_\xi \tau \equiv 0$, in which case the dispersion relation is linear in ξ. The associated graph is a hyperplane that belongs to the characteristic variety. The characteristic variety for (3.1.1) and (3.1.2) consist of hyperplanes while for (3.1.3) the variety is curved. On a hyperplane, $\{\tau = -\mathbf{v}.\xi\}$, contained in the characteristic variety, the group velocity is identically equal to $\mathbf{v}$ so does not depend on ξ. This is the completely nondispersive situation. Solutions translate without spread.

If the variety contains no hyperplanes, the variation of the group velocity spreads wavepackets. We will show that as $t \to \infty$, solutions decay in L^∞. These results, presented in §3.2-§3.3, are taken from [Joly, Métivier and Rauch, Indiana J., 1998].

An even stronger notion of uniform dispersion is when the rank of $\nabla^2_\xi \tau$ is everywhere equal to $d-1$. In this case, the sheets of the characteristic variety are uniformly convex cones and smooth compactly supported solutions decay at the rate $t^{-(d-1)/2}$ as $t \to \infty$. This is investigated in §3.4. In §3.4.1 $L^1 \to L^\infty$ decay estimates are proved. These are applied in §6.7 to prove global solvability of for nonlinear problems with small initial data and high dimension. In §3.4.3 the $L^1 \to L^\infty$ estimates are used to derive Strichartz estimates. In §6.8, these estimates are applied to prove global solvability of the nonlinear Klein-Gordon equation in the natural energy space.

§3.2. Spectral decomposition of solutions.

Since $(\tau, 0)$ is noncharacteristic for L, any hyperplane $\{a\tau + b.\xi = 0\}$ contained in the characteristic variety must have $a \neq 0$. Therefore, it is necessarily a graph $\{\tau = -\mathbf{v}.\xi\}$.

Over each $\xi \in \mathbb{R}^d$ there are at most N points in the characteristic variety. Therefore, the number of distinct hyperplanes in the variety can be no larger than N. Denote by $0 \leq M \leq N$ the number of such hyperplanes, $H_1, \dots, H_M$,

$$H_j = \{\, (\tau, \xi) \,:\, \tau = -\mathbf{v}_j.\xi \,\}\,, \qquad j = 1, \dots, M \leq N\,. \tag{3.2.1}$$

Examples. 1. When $d = 1$ the characteristic variety is a union of lines so consists only of hyperplanes. There are no curved sheets.

2. The characteristic variety of the operator (3.1.3) is the light cone, $\{\tau^2 = |\xi|^2\}$. There are no hyperplanes.

3. The characteristic varieties of Mawell's Equations and the the linearization at $u = 0$ of the compressible Euler equations are the union of a convex light cone and a single horizontal hyperplane $\tau = 0$.

Convention. In this chapter we assume that $L(\partial_t, \partial_x)$ is constant coefficient, homogeneous, symmetric, and, $A_0 = I$.

Definition. *An $\underline{\xi} \in \mathbb{R}^d \setminus \{0\}$ is* **good wave number** *when there is a neighborhood Ω of $\underline{\xi}$ and a finite number of real valued real analytic functions $\lambda_1(\xi) < \lambda_2(\xi) < \cdots < \lambda_m(\xi)$ so that the spectrum of $\sum_{j=1}^d A_j \xi_j$ is $\{\lambda_1(\xi), \dots, \lambda_m(\xi)\}$ for $\xi \in \Omega$. The complementary set consists of* **bad wave numbers**. *The set of bad wave numbers is denoted $\mathcal{B}(L)$.*

Over a good ξ, the characteristic variety of L cotains exactly m nonintersecting sheets $\tau = -\lambda_j(\xi)$. At bad wave numbers, eigenvalues cross and multiplicities change. The examples above have no bad points.

Examples. Consider the characteristic equation $(\tau^2 - |\xi|^2)(\tau - c\xi_1) = 0$ with $c \in \mathbb{R}$. If $|c| < 1$ then the variety is a cone and a hyperplane intersecting only at the origin and all wave numbers are good. If $|c| > 1$ the plane and cone intersect in a cone whose projection on ξ space is the set of bad wave numbers,

$$\mathcal{B} = \left\{ \xi \,:\, (c^2-1)\xi_1^2 = \xi_2^2 + \ldots + \xi_d^2 \right\}.$$

When $|c| = 1$, $\mathcal{B}(L)$ degenerates to a line of tangency. ∎

Proposition 3.2.1. i. *$\mathcal{B}(L)$ is a closed conic set of measure zero in $\mathbb{R}^d \setminus \{0\}$.*

ii. *The complementary set, $\mathbb{R}^d \setminus (\mathcal{B} \cup \{0\})$, is the disjoint union of a finite family of conic connected open sets $\Omega_g \subset \mathbb{R}^d \setminus \{0\}$, $g \in \mathcal{G}$.*

iii. *The mulitplicity of $\tau = -\mathbf{v}_j.\xi$ as a root of $\det L(\tau,\xi) = 0$ is independent of $\xi \in \mathbb{R}^d \setminus (\mathcal{B} \cup \{0\})$.*

iv. *If $\lambda(\xi) \in C^\omega(\Omega_g)$ is an eigenvalue of $\sum A_j \xi_j$ depending real analytically on ξ, then either there is $j \in \{1,\ldots,M\}$ such that $\lambda(\xi) = -\mathbf{v}_j \cdot \xi$ or $\nabla^2\lambda \neq 0$ almost everywhere on Ω_g.*

Proof. i. Use the basic stratification theorem of real algebraic geometry (see [Benedettin and Rissler], [Basu, Pollack, and Roy]). The characteristic variety is a conic real algebraic variety in $\mathbb{R}^{1+d} \setminus \{0\}$.

Over each ξ it contains at least 1 and at most N points. Therefore its projection on $\mathbb{R}^d_\xi$ is the whole space so the variety has dimension at least d. On the other hand it has measure zero by Fubini's Theorem so the dimension is at most d, since $d+1$ dimensional algebraic sets contain open sets.

The singular points are therefore a stratum of dimension at most $d-1$. The bad wave numbers are exactly the projection of this singular locus and so is a real algebraic subvariety of $\mathbb{R}^d_\xi$ of dimension at most $d-1$ and **i** follows.

ii. That there are at most a finite number of components in the complementary set is a classical theorem of Whitney proved in the reference cited in **i.**

iii. Denote by m the mulitplicity on Ω_g and m' the mulitplicity on $\Omega_{g'}$. By definition of mulitplicity,

$$\xi \in \Omega_g \quad \text{and} \quad k < m \quad \Longrightarrow \quad \frac{\partial^k \det L(\tau,\xi)}{\partial \tau^k}\bigg|_{\tau = -\mathbf{v}_j.\xi} = 0\,. \tag{3.2.2}$$

Then $\partial_\tau^k L(-\mathbf{v}_j.\xi,\xi)$ is a polynomial in ξ which vanishes on the nonempty open set Ω_g, so must vanish identically. Thus it vanishes on $\Omega_{g'}$ and it follows that $m' \geq m$. By symmetry one has $m \geq m'$.

iv. If λ is a linear function $\lambda = -\mathbf{v}.\xi$ on Ω_g, then $\det L(-\mathbf{v}.\xi,\xi) = 0$ for $\xi \in \Omega_g$ so by analytic continuation, must vanish for all ξ. It follows that the hyperplane $\tau = -\mathbf{v}.\xi$ lies in the characteristic variety and therefore that $\lambda = -\mathbf{v}_j.\xi$ for some j.

If λ is not a linear function, then the matrix $\nabla^2_\xi \lambda$ is a real analytic function on Ω_g which is not identically zero and therefore vanishes at most on a set of measure zero in Ω_g. ∎

Definitions. *Enumerate the roots of* $\det\, L(\tau,\xi) = 0$ *as follows. Let $\mathcal{A}_f := \{1,\ldots,M\}$ denote the indices of the* **flat parts**, *and for $\alpha \in \mathcal{A}_f$, $\tau_\alpha(\xi) := -\mathbf{v}_\alpha.\xi$. For $g \in \mathcal{G}$ and $\xi \in \Omega_g$, number the roots other than the $\{\tau_\alpha : \alpha \in \mathcal{A}_f\}$ in order $\tau_{g,1}(\xi) < \tau_{g,2}(\xi) < \cdots < \tau_{g,M(g)}$. Multiple roots are not repeated in this list. Let $\mathcal{A}_c$ denote the indices of the* **curved sheets**

$$\mathcal{A}_c := \left\{ (g,j) \,:\, g \in \mathcal{G} \ \text{ and } \ 1 \leq j \leq M(g) \right\}. \tag{3.2.3}$$

Let $\mathcal{A} := \mathcal{A}_f \cup \mathcal{A}_c$. For $\alpha \in \mathcal{A}_f$ and $\xi \in \mathbb{R}^d$ define $E_\alpha(\xi) := \pi(-\mathbf{v}_j.\xi,\xi)$. For $\alpha \in \mathcal{A}_c$ define

$$E_\alpha(\xi) := \begin{cases} \pi\big(\tau_\alpha(\xi),\xi\big) & \text{for} \quad \xi \in \Omega_g \\ \\ 0 & \text{for} \quad \xi \notin \Omega_g\,. \end{cases} \tag{3.2.4}$$

The next proposition decomposes an arbitrary solution of $Lu = 0$ as a finite sum of simpler waves.

Proposition 3.2.2. 1. *For each $\alpha \in \mathcal{A}$, $E_\alpha(\xi) \in C^\omega(\mathbb{R}^d \setminus (\mathcal{B} \cup \{0\}))$ is an orthogonal projection valued function positive homogeneous of degree zero.*
2. *For each $\xi \in \mathbb{R}^d \setminus (\mathcal{B} \cup \{0\})$, $\mathbb{C}^N$ is equal to the orthogonal direct sum*

$$\mathbb{C}^N = \oplus_{\alpha \in \mathcal{A}} \operatorname{Image} E_\alpha(\xi)\,. \tag{3.2.5}$$

3. *The operators $E_\alpha(D_x) := \mathcal{F}^* E(\xi) \mathcal{F}$ are orthogonal projectors on $H^s(\mathbb{R}^d)$, and for each $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ is equal to the orthogonal direct sum,*

$$H^s(\mathbb{R}^d) \;=\; \oplus_{\alpha \in \mathcal{A}} \operatorname{Image} E_\alpha(D_x)\,. \tag{3.2.6}$$

4. *If $f \in \mathcal{S}'(\mathbb{R}^d)$ has Fourier transform equal to a locally integrable function, then the solution of the initial value problem*

$$L(\partial_y)\, u \;=\; 0\,, \qquad u|_{t=0} \;=\; f \tag{3.2.7}$$

is given by the formula

$$\hat u(t,\xi) \;=\; \sum_{\alpha \in \mathcal{A}} \hat u_\alpha(t,\xi) \;:=\; \sum_{\alpha \in \mathcal{A}} e^{it\tau_\alpha(\xi)}\, E_\alpha(\xi)\, \hat f(\xi)\,. \tag{3.2.8}$$

Remarks. 1. The last decomposition is also written

$$u \;:=\; \sum_{\alpha \in \mathcal{A}} u_\alpha \;:=\; \sum_{\alpha \in \mathcal{A}} e^{it\tau_\alpha(D_x)}\, E_\alpha(D_x) f\,.$$

2. Since τ_α is real valued on the support of $E_\alpha(\xi)$ the operator $e^{it\tau_\alpha(D_x)}\, E_\alpha(D_x)$ is a contraction on $H^s(\mathbb{R}^d)$ for all s.
3. If $\alpha \in \mathcal{A}_f$ then $-i\tau_\alpha(D_x) = \mathbf{v}_\alpha.\partial_x$. For $\alpha = (g,j) \in \mathcal{A}_c$, $|\tau_\alpha(\xi)| \le C|\xi|$, so the operator $\tau_\alpha(D_x) f$ is continuous from H^s to H^{s-1}. The mode $u_\alpha = e^{it\tau_\alpha(D_x)}\, E_\alpha(D_x) f$ satisfies $\partial_t u_\alpha = i\tau_\alpha(D_x) u_\alpha$. For $\alpha \in \mathcal{A}_f$ this is $\big(\partial_t + \mathbf{v}_\alpha.\partial_x\big) u_\alpha = 0$, so

$$u_\alpha \;=\; \Big(E_\alpha(D) f\Big)\big(x - \mathbf{v}_\alpha t\big)\,.$$

4. Over $\mathcal{B}(L)$ only the E_α corresponding to the hyperplanes are defined. One does not have a decomposition of $\mathbb{C}^N$. It is important that $\mathcal{B}$ is a negligible set for $\hat f$. The $\hat f \in L^1_{\rm loc}$ assumption in **4** is essential.

§3.3. Large time asymptotics

Definition. *Define $\mathbb{A}$ as the set of tempered distributions whose Fourier transforms belong to $L^1(\mathbb{R}^d)$. Then $\mathbb{A}$ is a Banach space with norm*

$$\|f\|_{\mathbb{A}} := (2\pi)^{-d/2} \int_{\mathbb{R}^d} |\hat f(\xi)|\; d\xi\,. \tag{3.3.1}$$

The Fourier Inversion Formula implies that $\mathbb{A} \subset L^\infty(\mathbb{R}^d)$ and

$$\|f\|_{L^\infty(\mathbb{R}^d)} \ \le\ \|f\|_{\mathbb{A}}\,. \tag{3.3.2}$$

The elements of $\mathbb{A}$ are continuous and tend to zero as $x \to \infty$. Moreover, the Fourier transform of f^2 is a multiple of $\hat f * \hat f$ and therefore in L^1, so $\mathbb{A}$ is an algebra. It is called the *Wiener algebra*. It is a centerpiece of the Tauberian Theorems of N. Wiener.

Theorem 3.3.1 (L^∞ asymptotics for symmetric systems). *Suppose that $f \in \mathbb{A}$ and u is the solution of the initial value problem $L(\partial_x)u = 0$, $u|_{t=0} = f$. Then with the notation introduced in the preceding section,*

$$\lim_{t\to\infty}\ \Big\| u(t) - \sum_{\alpha\in\mathcal{A}_f} \left(E_\alpha(D_x) f\right)(x - \mathbf{v}_\alpha t)\Big\|_{L^\infty(\mathbb{R}^d)} = 0\,. \tag{3.3.3}$$

Remarks. 1. This result shows that a general solution of the Cauchy problem is the sum of M rigidly translating waves, one for each hyperplane in the characteristic variety, plus a term which tends to zero in sup norm. The last part decays because of the dispersion of waves.

2. The Theorem does not extend to f whose Fourier Transform is a bounded measure. For example, $u := \big(e^{i(x_1 - t)}, 0\big)$ satisfies $Lu = 0$ with $\hat f$ equal to a point mass. The characteristic variety contains no hyperplanes so (3.3.3) asserts that solutions with $\hat f \in L^1$ tend to zero in $L^\infty(\mathbb{R}^d)$ while $u(t)$ has sup norm equal to 1 for all t.

Proof of Theorem. Step 1. Approximation-decomposition. Symmetric hyperbolicity implies that for each t, ξ, $\exp\big(it\sum A_j\xi_j\big)$ is unitary on $\mathbb{C}^N$. Therefore $S(t) := \exp\big(-t\sum_j A_j\partial_j\big)$ is isometric on $\mathbb{A}$. Since the family of linear maps

$$f \ \longmapsto\ S(t)f - \sum_{\alpha\in\mathcal{A}_f} \left(E_\alpha(D_x)f\right)(x - \mathbf{v}_\alpha t)$$

is uniformly bounded from $\mathbb{A}$ to $L^\infty(\mathbb{R}^d)$, it suffices to prove (3.3.3) for a set of f dense in $\mathbb{A}$.

For $\alpha \in \mathcal{A}_c$, Propostion 3.2.1.iv shows that the matrix of second derivatives, $\nabla_\xi^2\tau_\alpha$ can vanish at most on a closed set of measure zero. The set of f we choose is those with

$$\hat f \ \in\ C_0^\infty\big(\mathbb{R}^d \setminus \big\{\, \mathcal{B}\ \cup\ \{0\}\ \cup\ \bigcup_{\alpha\in\mathcal{A}_c} \{\xi\in\Omega_g \,:\, \nabla_\xi^2\tau_\alpha(\xi) = 0\,\}\ \big\}\big)\,.$$

Since the removed set is a closed null set, these f are dense.

To prove (3.3.3) for such f decompose

$$f = \sum_{\alpha\in\mathcal{A}} f_\alpha := \sum_{\alpha\in\mathcal{A}} E_\alpha(D_x)\, f\,, \qquad u(t) = S(t)f = \sum u_\alpha(t) := \sum S(t)\, f_\alpha\,. \tag{3.3.4}$$

For $\alpha \in \mathcal{A}_f$, $u_\alpha(t) = \left(E_\alpha(D_x)f\right)(x - \mathbf{v}_\alpha t)$ which recovers the summands in (3.15). To prove (3.15) it suffices to show that for $\alpha \in \mathcal{A}_c$

$$\lim_{t\to\infty} \big\|u_\alpha(t)\big\|_{L^\infty(\mathbb{R}^d)} \ =\ 0\,. \tag{3.3.5}$$

Step 2. Stationary and nonstationary phase. Part 4 of Proposition 3.2. shows that for $\alpha \in \mathcal{A}_c$,

$$u_\alpha(t,x) = \int_{\Omega_g} e^{i(\tau_\alpha(\xi)t + x.\xi)}\, \hat{f}_\alpha(\xi)\, d\xi\,, \qquad \hat{f}_\alpha \in C_0^\infty(\Omega_g)\,. \tag{3.3.6}$$

For each ξ in the support of $\hat{f}_\alpha$, there is a vector $\mathbf{r} \in \mathbb{R}^d$ so that $\langle \nabla_\xi^2 \tau(\xi)\, \mathbf{r}, \mathbf{r}\, \rangle \neq 0$ on a neighborhood of ξ. Using a partition of unity we can write $\hat{f}_\alpha$ as a finite sum of functions $h_\mu \in C_0^\infty(\Omega_g)$ so that for each μ there is a $\mathbf{r}_\mu \in \mathbb{C}^N$ so that on an open ball containing the support of h_μ, $\langle \nabla_\xi^2 \tau(\xi)\, \mathbf{r}_\mu, \mathbf{r}_\mu\, \rangle \neq 0$. It suffices to show that for each μ

$$\lim_{t\to\infty}\ \sup_{x\in\mathbb{R}^d} \int e^{i(\tau_\alpha(\xi)t + x.\xi)}\, h_\mu(\xi)\, d\xi \ = \ 0\,. \tag{3.3.7}$$

For ease of reading we suppress the subscripts. Write $x = tz$. For each $t > 0$, the supremum in x is equal to the supremum in z so it suffices to show that

$$\lim_{t\to\infty}\ \sup_{z\in\mathbb{R}^d} \Big| \int e^{it(\tau(\xi) + z.\xi)}\, h(\xi)\, d\xi \Big| \ = \ 0\,.$$

Choose

$$\sigma \ > \ \sup_{\xi \in \operatorname{supp} h} \big|\nabla_\xi \tau(\xi)\big|\,.$$

There is a $\delta > 0$ so that for all $|z| \geq \sigma$,

$$\big|\nabla_\xi(\tau(\xi) + z.\xi)\big| \ \geq \ \delta\,.$$

The method of nonstationary phase implies that

$$\forall N > 0,\ \exists C_N,\ \forall |z| \geq \sigma,\ t > 1, \qquad \Big| \int e^{it(\tau(\xi) + z.\xi)}\, h(\xi)\, d\xi \Big| \ \leq \ C_N\, t^{-N}\,.$$

It remains to show that

$$\lim_{t\to\infty}\ \sup_{|z|\leq\sigma} \Big| \int e^{it(\tau(\xi) + z.\xi)}\, h(\xi)\, d\xi \Big| \ = \ 0\,. \tag{3.3.8}$$

Make a linear change of variables in ξ so that $\mathbf{r} = (1, 0, \dots, 0)$ and therefore

$$\frac{\partial^2 \tau}{\partial^2 \xi_1} \neq 0\,, \qquad \text{on} \qquad \operatorname{supp}\, h\,.$$

Choose $R > 0$ so that for $\xi \in \operatorname{supp} h$, $|\xi| \leq R$. Set

$$\Gamma \ := \ \{|z_1| \leq \sigma\} \times \{|\xi_2, \dots, \xi_d| \leq R\} \ \subset \ \mathbb{R}^1 \times \mathbb{R}^{d-1}\,.$$

Define

$$\begin{aligned} K(t) \ &:= \sup_{|z|\leq\sigma,\ |\xi_2,\dots,\xi_d|\leq R} \Big| \int e^{it(\tau(\xi) + z_1.\xi_1)}\, h(\xi)\, d\xi_1 \Big| \\ &= \ \sup_{\Gamma} \Big| \int e^{it(\tau(\xi) + z_1.\xi_1)}\, h(\xi)\, d\xi_1 \Big|\,. \end{aligned}$$

Then

$$\sup_{|z|\le\sigma} \Big| \int e^{it(\tau(\xi)+z.\xi)}\, h(\xi)\, d\xi \Big|$$
$$\le \int_{|\xi_2,\dots,\xi_d|\le R} e^{i(z_2\xi_2+\dots+z_d.\xi_d)} \Big(\int e^{it(\tau+z_1\xi_1)}\; h(\xi)\; d\xi_1 \Big) d\xi_2\dots d\xi_d$$
$$\le \Big| \big\{ |\xi_2,\dots,\xi_d| \le R \big\} \Big| \; K(t)\,.$$

It therefore suffices to show that

$$\lim_{t\to\infty} K(t) \;=\; 0\,. \tag{3.3.9}$$

The points of Γ are split according to whether the phase $\tau(\xi)+z_1\xi_1$ has a stationary point with respect to ξ_1 or not. If $\underline{\gamma}\in\Gamma$ is such that

$$\Big| \frac{\partial\tau}{\partial\xi_1} \;+\; z_1 \Big| \;>\; \delta \;>\; 0 \qquad \text{for all} \quad |z_1|\le\sigma,\ |\xi|\le R,$$

the same is true on a neighborhood of $\underline{\gamma}$. The principal of nonstationary phase shows that

$$\int e^{it(\tau_\alpha(\xi)+z.\xi)}\; \hat h_\mu(\xi)\; d\xi_1 \;=\; O(t^{-N})$$

uniformly on such a neighborhood.

On the other hand if for $\underline{\gamma}$ there is a stationary point, then the strict convexity of τ in ξ_1 shows that it is unique and nondegenerate. Therefore for nearby γ there is a nearby unique and nondegenerate stationary point. The inequality of stationary phase (see Appendix) implies that

$$\int e^{it(\tau_\alpha(\xi)+z.\xi)}\; \hat h_\mu(\xi)\; d\xi_1 \;=\; O(t^{-1/2})$$

uniformly on a neighborhood of $\underline{\gamma}$.

Covering the compact set Γ by a finite family of neighborhoods proves (3.3.9) and therefore the Theorem. ∎

Definition. *The operator L is* **purely dispersive** *when its characteristic variety contains no hyperplanes. It is call* **nondispersive** *when its characteristic variety is equal to a union of hyperplanes.*

The nondispersive operators have a discrete set of group velocities. The characteristic variety of purely dispersive operators have only curved sheets. The latter name is justified by the next Corollary.

Corollary 3.3.2. *If $L = L_1(\partial_x)$ is a constant coefficient homogeneous symmetric hyperbolic operator, then the following are equivalent.*

1. *The characteristic variety of L contains no hyperplanes (i.e. L is purely dispersive).*

2. *Every solution of $Lu=0$ with $u\big|_{t=0}\in C_0^\infty(\mathbb{R}^d)$ satisfies,*

$$\lim_{t\to\infty} \|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\to\; 0\,. \tag{3.3.10}$$

3. *Every solution of $Lu = 0$ with $u\big|_{t=0} \in \mathbb{A}$ satisfies (3.3.10).*

4. *If $\tau(\xi)$ is a C^∞ solution of $\det L(\tau, \xi) = 0$ defined on a open set of $\xi \in \mathbb{R}^d$ then for every $\mathbf{v} \in \mathbb{R}^d$, $\{\xi \in \mathbb{R}^d \,:\, \nabla_\xi \tau = -\mathbf{v}\}$ has measure zero.*

Proof. Theorem 3.3 shows that $\mathbf{1} \Leftrightarrow \mathbf{3}$. To complete the proof we show that $\mathbf{3} \Leftrightarrow \mathbf{2}$ and $\mathbf{1} \Leftrightarrow \mathbf{4}$.

The assertions **2** and **3** are equivalent because the family of mappings $u(0) \mapsto u(t)$ is uniformly bounded from $\mathbb{A} \to L^\infty$, and C_0^∞ is dense in $\mathbb{A}$.

That $\sim \mathbf{1} \Longrightarrow \sim \mathbf{4}$ is immediate.

If **4** is violated there is a smooth solution τ so that $\nabla_\xi \tau = -\mathbf{v}$ on a set of positive measure. It follows from the stratification theorem of real algebraic geometry that $\nabla_\xi \tau = -\mathbf{v}$ on a conic open real algebraic set of dimension d in $\mathbb{R}^d \setminus 0$. Then $\tau = -\mathbf{v}.\xi$ on this set and we conclude that the polynomial $\det L(-\mathbf{v}.\xi, \xi)$ vanishes on this set and therefore everywhere. Thus the hyperplane $\{\tau = -\mathbf{v}.\xi\}$ is contained in the characteristic variety and **1** is violated.

Thus **1** and **4** are equivalent. ∎

Remarks. 1. Part four of this Corollary shows that for any velocity $\mathbf{v}$ the group velocity $-\nabla_\xi \tau$ associated to a curved sheet of the characteristic variety takes the value $\mathbf{v}$ for at most a set of frequencies ξ of measure zero.

2. If $\Omega \subset \mathbb{R}^d$ is a set of finite measure, estimate using the Cauchy-Shwartz inequality,

$$\int_\Omega |u(t,x)|^2 \, dx \;\leq\; \|u(t)\|_{L^\infty(\mathbb{R}^d)} \int_\Omega |u| \, dx \;\leq\; \|u(t)\|_{L^\infty(\mathbb{R}^d)}\| \; |\Omega|^{1/2} \; \|u\|_{L^2(\mathbb{R}^d)},$$

shows that for Cauchy data in $\mathbb{A}$ the L^2 norm in any tube of rays tends to zero at $t \to \infty$.

The nondispersive evolutions are described in the next results.

Corollary 3.3.3. *If $L = L_1(\partial_y)$ is a constant coefficient homogeneous symmetric hyperbolic operator with $A_0 = I$, then the following are equivalent.*

1. *The characteristic variety of L is a finite union of hyperplanes.*

2. *(Motzkin and Tausky) The matrices A_j commute.*

3. *If u satisfies $Lu = 0$ with $u(0) \in \mathbb{A}$ and $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $t \to \infty$, then u is identically equal to zero.*

Proof. 2 $\Rightarrow$ **3**. A unitary change of variable $u = Vv$ replaces the equation $Lu = 0$ with the equivalent equation $\tilde{L}v = 0$ with $\tilde{A}_j := V^* A_j V$. When the A_j commute, V can be chosen so that the $\tilde{A}_j$ are all real diagonal matrices. Property **3** is clear for the tilde equation as each component of the solution rigidly translates as time goes on. The only way its supremum can tend to zero at $t \to \infty$ is for it to vanish.

3 $\Rightarrow$ **1**. This is a consequence of Theorem 3.3.1.

1 $\Rightarrow$ **2**. This result of Motzkin and Tausky is proved next completes the proof. ∎

Theorem 3.3.4. (Motzkin and Tausky) *Suppose that A and B are hermitian $N \times N$ matrices. The eigenvalues of $\xi A + \eta B$ are linear functions of ξ, η if and only if A and B commute.*

Proof. We must show that linear eigenvalue implies commutation. The proof is by induction on N. The case $N = 1$ is trivial. We suppose that $N > 1$ and the result is known for dimensions $\leq N - 1$.

Consider the characteristic variety $\det(\tau + \xi A + \eta B) = 0$. Choose a good wave number $(\underline{\xi}, \underline{\eta})$ so that above this point the variety has $k \le N$ real analytic sheets. If $\eta = 0$, leave the spatial coordinates as they are. If $\eta \ne 0$, change orthogonal coordinates in $\mathbb{R}^2$ so that $(\underline{\xi}, \underline{\eta})$ is a mutiple of dy_1. We can without loss of generality assume that above $\eta = 0$ the variety consists of k real analytic sheets.

For s small the eigenvalues of $A + sB$ are real analytic function $\lambda_j(s)$ with $\lambda_j(0) < \lambda_{j+1}(0)$ for $1 \le j < k-1$. Denote by μ_j the multiplicity of $\lambda_j(0)$ and therefore of $\lambda_j(s)$ for s small. By hypothesis the $\lambda_j(s)$ are affine functions of s so $\lambda'' = 0$. We use this only at $s = 0$.

By a unitary change of variable in $\mathbb{C}^N$ one can arrange that A is block diagonal with diagonal entries $\lambda_j(0) I_{\mu_j \times \mu_j}$.

Corresponding to this block structure and the eigenvalue λ_1 one has,

$$\begin{aligned} \pi &= \operatorname{diag}\Big(I_{\mu_1\times\mu_1}, 0_{\mu_2\times\mu_2}, \dots, 0_{\mu_k\times\mu_k}\Big)\,, \\ Q &= \operatorname{diag}\Big(0_{\mu_1\times\mu_1}, \frac{1}{\lambda_2-\lambda_1} I_{\mu_2\times\mu_2}, \dots, \frac{1}{\lambda_k-\lambda_1} I_{\mu_k\times\mu_k}\Big)\,. \end{aligned} \tag{3.3.11}$$

The matrix B has block structure

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} & \dots\dots & B_{1,k} \\ B_{2,1} & B_{2,2} & \dots\dots & B_{2,k} \\ & & & \\ B_{k,1} & B_{k,2} & \dots. & B_{k,k} \end{pmatrix}\,,$$

with B_{ij} a $\mu_i \times \mu_j$ matrix and $B_{ij} = B_{ji}^*$.

The fundamental formula of second order perturbation theory (3.I.3) from Appendix 3.I, yields $\lambda''\pi = 2\pi BQB\pi$. By hypothesis this is equal to zero.

Straightforward calculation shows that

$$\pi B = \begin{pmatrix} B_{1,1} & B_{1,2} & \dots\dots & B_{1,k} \\ 0 & 0 & \dots\dots & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & \dots. & 0 \end{pmatrix}\,, \qquad QB\pi = \begin{pmatrix} 0 & 0 & \dots. & 0 \\ \frac{1}{\lambda_2-\lambda_1} B_{2,1} & 0 & \dots. & 0 \\ & & & \\ \frac{1}{\lambda_k-\lambda_1} B_{k,1} & 0 & \dots. & 0 \end{pmatrix}\,.$$

Therefore, the $\mu_1 \times \mu_1$ upper left hand block block of $\pi QBQ\pi$ is equal to

$$\sum_{j=2}^{k} \frac{1}{\lambda_j - \lambda_1} B_{1,j} B_{1,j}^*\,.$$

Conclude that this sum of positive square matrices vanishes. Thus, for $j \ge 2$, $B_{1,j} = 0$ and $B_{j,1} = 0$.

Thus B and A are reduced by the splitting

$$\mathbb{C}^N = \mathbb{C}^{\mu_1} \times \mathbb{C}^{N-\mu_1}\,.$$

The commutation then follows by the inductive hypothesis applied to the diagonal blocks. This proves the case N and completes the induction. ∎

Example. The implication **1** $\Rightarrow$ **2** is not true without the symmetry hypothesis. For example, the hypberbolic system

$$\partial_t \ + \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 \ + \ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \partial_2$$

has flat characteristic variety with equation

$$(\tau + \xi_1 + \xi_2)(\tau - \xi_1 + \xi_2) \ = \ 0\,,$$

and the coefficient matrices do not commute. The conclusion is correct assuming that the hyperbolic system generates a semigroup in $L^2(\mathbb{R}^d)$ (see [Gues and Rauch]).

Theorem 3.3.5. (P. Brenner). *If $L = L(\partial_y)$ is a constant coefficient homogeneous symmetric hyperbolic operator with $A_0 = I$, then the conditions of Corollary 3.3.3 are equivalent to each of the following.*

i. *For all $t \in \mathbb{R}$ and $p \in [1, \infty]$ the Fourier multiplication operator*

$$S(t) := \mathcal{F}^{-1}\, e^{-it\sum A_j\xi_j}\, \mathcal{F}$$

is a bounded from $L^p(\mathbb{R}^d)$ to itself.

ii. *For some $\underline{t} \in \mathbb{R} \setminus 0$ and $2 \neq \underline{p} \in [1, \infty]$ the operator $S(\underline{t})$ is bounded from $L^{\underline{p}}(\mathbb{R}^d)$ to itself.*

Remark. The Fourier multiplication operators are unitary on L^2. Property **ii** means that the restriction to $\mathcal{S}(\mathbb{R})$ extends to bounded operators on L^p, equivalently

$$\sup_{f \in \mathcal{S}(\mathbb{R}^d)\setminus 0} \frac{\|S(t)f\|_{L^p(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}} \ < \ \infty\,.$$

Proof. The conditions of Corollary 3.3.3 imply that after an orthogonal change of basis, the A_j are all real diagonal matrices. It is then elementary to verify that **i** is satisfied.

Clearly **i** implies **ii**. It remains to show that **ii** implies the conditions of Corollary 3.3.3. Equivalently, if the conditions of the the Corollary are violated, then **ii** is violated. First remark that **ii** is stronger than it appears. Since $S(\underline{t})$ is unitary on L^2, if **ii** is satisfied then $S(t)$ is bounded on L^p for all p between 2 and $\underline{p}$. Thus we may assume that $\underline{p}$ in not equal to 1 or ∞.

For $\sigma \in \mathbb{R} \setminus 0$, $Lu = 0$ if and only if $u^\sigma(t, x) := u(\sigma t, \sigma x)$ satisfies $Lu^\sigma = 0$. It follows that if **ii** is satisfied then

$$\|S(t)\|_{\mathrm{Hom}(L^{\underline{p}})} \ = \ \|S(\underline{t})\|_{\mathrm{Hom}(L^{\underline{p}})} \ < \ \infty\,, \qquad \forall\, t \neq 0\,. \tag{3.3.12}$$

If $\underline{q}$ is the conjugate index to $\underline{p}$, that is $\frac{1}{\underline{p}} + \frac{1}{\underline{q}} = 1$, then

$$\begin{aligned}
\|S(t)\|_{\mathrm{Hom}(L^{\underline{q}}(\mathbb{R}^d))} \ &= \ \sup_{f,g \in \mathcal{S}\setminus 0} \frac{\big(S(t)f\,,\, g\big)}{\|f\|_{L^{\underline{q}}(\mathbb{R}^d)}\ \|g\|_{L^{\underline{p}}(\mathbb{R}^d)}} \\
&= \ \sup_{f,g \in \mathcal{S}\setminus 0} \frac{\big(f\,,\, S(-t)g\big)}{\|f\|_{L^{\underline{q}}(\mathbb{R}^d)}\ \|g\|_{L^{\underline{p}}(\mathbb{R}^d)}} \ = \ \|S(-t)\|_{\mathrm{Hom}(L^{\underline{p}}(\mathbb{R}^d)}\,.
\end{aligned}$$

Thus when **ii** is satisfied for $\underline{p}$ it is satisfied for $\underline{q}$ so we may suppose that $\infty > \underline{p} > 2$.

When the conditions of Corollary 3.3.3 are violated, there is a conic set of good wave numbers Ω_g and a sheet $\tau = \tau(\xi)$ over Ω_g with $\nabla^2_{\xi\xi}\tau \neq 0$ for almost all $\xi \in \Omega_g$. Denote by $\pi(\xi)$ the associated spectral projection. Choose an $f \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{f}$ compactly supported in Ω_g. Replacing $\hat{f}$ by $\pi(\xi)\hat{f}$ we may assume that $\pi(D)f = f$. Theorem 3.3.1 implies that

$$\lim_{t\to\infty} \|S(t)\, f\|_{L^\infty(\mathbb{R}^d)} \;=\; 0\,.$$

Then

$$\|S(t)f\|^{p}_{L^{\underline{p}}(\mathbb{R}^d)} \;\le\; \|S(t)f\|^{p-2}_{L^\infty(\mathbb{R}^d)}\;\|S(t)f\|^2_{L^2(\mathbb{R}^d)} \;=\; \|S(t)f\|^{p-2}_{L^\infty(\mathbb{R}^d)}\;\|f\|^2_{L^2(\mathbb{R}^d)} \;\to\; 0\,,$$

as $t \to \infty$.
Therefore,

$$\big\|S(-t)\big\|_{\mathrm{Hom}(L^{\underline{p}})} \;\ge\; \frac{\big\|S(-t)\big(S(t)f\big)\big\|_{L^{\underline{p}}}}{\big\|S(t)f\big\|_{L^{\underline{p}}}} \;=\; \frac{\big\|f\big\|_{L^{\underline{p}}}}{\big\|S(t)f\big\|_{L^{\underline{p}}}} \;\to\; \infty\,.$$

Thus (3.3.12) is violated and the proof is complete. ∎

Example. It may seem that (3.3.12) together with $\lim_{t\to 0} S(t)f = f$ might imply that $S(t)$ has norm equal to 1. That this is not true can be seen from the one dimensional example

$$\partial_t \;+\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \partial_x\,,$$

and L^p norm defined by

$$\|(u_1,u_2)\|_p \;:=\; \int \|(u_1,u_2)\|^p\; dx\big)^{1/p}\,, \qquad \|(u_1,u_2)\| := \big(|u_1|^2 + |u_2|^2\big)^{1/2}\,,$$

so that for $p = 2$ one has unitarity. Choosing $u_1(0) = u_2(0) = f \in C_0^\infty(\{|x| \le \rho\})$ one has

$$\|u(0)\|^p_p \;=\; (\sqrt{2})^p \|f\|^p_p\,,$$

and for $|t| > \rho$,

$$\|u(t)\|^p_p = 2\,\|f\|^p_p\,.$$

It follows that for all $t \neq 0$ and $p < 2$, $\|S(t)\|^p_{\mathrm{Hom}(L^p)} \ge 2^{1-\,p/2} > 1$. Reversing time, treats $p > 2$.

§3.4. Maximally dispersive systems

§3.4.1. The $L^1 \to L^\infty$ decay estimate

If $\tau = \tau(\xi)$ parametrizes a real analytic patch of the characteristic variety of a hyperbolic operator then τ is homogeneous of degree 1 in ξ. The group velocity $\mathbf{v}(\xi) = -\nabla_\xi \tau(\xi)$ is homogeneous of degree 0. Therefore $\xi.\nabla_\xi \mathbf{v} = 0$ so ξ belongs to the kernel of the symmetric matrix $\nabla_\xi \mathbf{v}(\xi) = \nabla^2_\xi \tau(\xi)$. Thus the rank of $\nabla^2_\xi \tau$ is at most $d-1$. When the rank is equal to $d-1$ the group velocity depends as strongly on ξ as possible. The dispersion is as strong as possible.

Definition. *The homogeneous constant coefficient symmetric hyperbolic operator is* **maximally dispersive** *when*

$$\mathrm{Char}\, L \;=\; \cup_{j=1}^m \Big\{(\tau,\xi) \;:\; \tau = \tau_j(\xi)\Big\}$$

where for $\xi \in \mathbb{R}^d \setminus 0$

$$\tau_1(\xi) < \tau_2(\xi) \ldots < \tau_m(\xi)\,,$$

the τ_j *are real analytic, positive homogeneous of degree one in* ξ, *and*

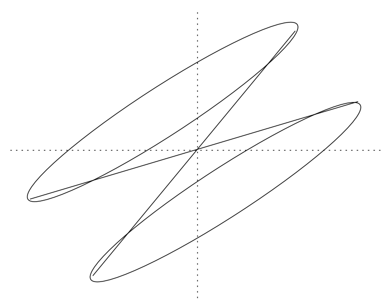
$$\forall j, \quad \forall\, \xi \in \mathbb{R}^d \setminus 0\,, \qquad \operatorname{rank} \nabla^2_\xi \tau(\xi) \;=\; d-1\,. \tag{3.4.1}$$

Examples. i. The simplest example is

$$(\tau^2 - |\xi|^2)\,(\tau^2 - c^2|\xi|^2) \;=\; 0\,, \qquad 0 < c \neq 1\,.$$

The variety in this case consists of two sheets $\tau = |\xi|$ and $\tau = c|\xi|$ which have $d-1$ strictly positive principal curvatures. The other sheets bound $\tau \le -|\xi|$ and $\tau \le -c\,|\xi|$ and have $d-1$ strictly negative curvatures.

ii. The next figure gives an example with two sheets bounding strictly convex regions for which the functions τ_j change sign. In particular the generator $G = -\sum A_j \partial_j$ is not elliptic since the points where the cone crosses $\tau = 0$ are characteristic for G.



The next result is closely related to Hadamard's ovaloid theorem which is proved in Appendix 3.III.

Propostion 3.4.1. *If* $\tau(\xi)$ *is smooth in* $\xi \neq 0$, *homogeneous of degree one and the hessian has rank equal to* $d-1$ *at all points, then the nonzero eigenvalues of* $\nabla^2_\xi \tau$ *have the same sign. When they are positive (resp. negative)* τ *is convex (resp. concave).*

Proof. When $d = 2$, $\nabla^2_\xi \tau$ has only one nonzero eigenvalue and the result is immediate.

For $d \ge 3$, consider the mapping

$$\Gamma(\xi) \;:=\; \mathbf{v}(\xi) \;=\; -\nabla_\xi \tau(\xi)\,.$$

The differential of the mapping Γ is equal to $-\nabla^2_\xi \tau$ so ξ is in its kernel and it is invertible when restricted to the orthogonal to ξ.

Since Γ is homogeneous of degree 0, it is natural to consider Γ as a map from $S^{d-1} = \{|\xi| = 1\}$. As such it is an immersion onto a compact $d-1$ dimensional manifold, $\mathcal{M}$. The image is oriented by the image of the orientation of S^{d-1}.

Since ξ is orthogonal to the image of $-\nabla^2_\xi \tau(\xi)$ it follows that the ξ is the unit normal to $\mathcal{M}$ at $\Gamma(\xi)$. Thus, at least locally, Γ is the inverse of the Gauss map of $\mathcal{M}$. Since the differential is invertible it follows that the Gauss curvature of $\mathcal{M}$ is nowhere vanishing.

Since $\xi \in \ker(\nabla^2_\xi \tau(\xi)$, the unit normal to $\mathcal{M}$ at $\mathbf{v}(\xi)$ is equal to ξ. Since the map from $\xi \in S^{d-1}$ to $\mathbf{v}(\xi)$ has invertible jacobian, the Gauss curvature of $\mathcal{M}$ is nowhere vanishing.

Since $d \geq 3$, it follows from Hadamard's ovaloid theorem, that $\mathcal{M}$ is the boundary of a strictly convex set and $\Gamma : S^{d-1} \to \mathcal{M}$ is a diffeomorphism.

Thus each value $-\nabla_\xi \tau(\xi) \in \mathcal{M}$ is attained at a unique $\xi \in S^{d-1}$.

The normals to $\tau = \tau(\xi)$ are the nonzero multiples of $(1, v(\xi))$. Thus, the hyperplane $\{\tau + v(\underline{\xi}).\xi = 0\}$ is tangent at $\tau = \tau(\underline{\xi})$ and at no other point $\tau = \tau(\underline{\xi}')$ with $\underline{\xi}' \in S^{d-1}$.

It follows that the cone $\tau = \tau(\xi)$ is strictly convex in the sense that its intersection with its tangent plane conists exactly of the line $(\mathbb{R} \setminus 0)(\tau(\xi), \xi)$.

This implies that the $d-1$ nonzero eigenvalues must have one sign. ∎

Examples. The characteristic variety of a maximally dispersive system consists of m disjoint sheets, each the boundary of a strictly convex cone.

Lemma 3.4.2 (Pointwise decay). *If $d \geq 2$, τ is as above and $k \in C_0^\infty(\mathbb{R}^d \setminus 0)$ then there is a constant C so that*

$$u(t,x) \ := \int e^{it\tau(\xi)} \ e^{ix.\xi} \ k(\xi) \ d\xi \,,$$

satisfies

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \ \leq \ C\,(1+|t|)^{-(d-1)/2} \,. \tag{3.4.2}$$

Remark. This is the decay rate for solutions of $\Box_{1+d} u = 0$ which corresponds to the choice $\tau(\xi) = \pm|\xi|$.

Proof. The easy estimate

$$\|u(t,x)\|_{L^\infty(\mathbb{R}^d)} \ \leq \int |k(\xi)| \ d\xi \,,$$

shows that only the decay for $|t| \geq 1$ needs to be proved.

Let

$$y \ := \ \frac{x}{t} \,, \qquad x \ = \ ty \,.$$

Then

$$\sup_x \big|u(t,x)\big| \ = \ \sup_y \big|u(t,ty)\big| \ = \ \sup_y \Big| \int e^{it(\tau(\xi)+y.\xi)} \ k(\xi) \ d\xi \Big| \,.$$

The phase $\tau(\xi) + y.\xi$ is stationary when

$$-\nabla_\xi \tau(\xi) \ = \ y \,.$$

The left hand side is the group velocity.

As in Lemma 3.4.1, denote by $\mathcal{M}$ the set of attained group velocities which is an embedded strictly convex compact $d-1$ manifold.

For any open neighborhood $\mathcal{O}$ of $\mathcal{M}$, the method of nonstationary phase shows that for any N,

$$\sup_{y \in \mathbb{R}^d \setminus \mathcal{O}} \ \Big| \int e^{it(\tau(\xi)+y.\xi)} \ k(\xi) \ d\xi \Big| \ \leq \ C_N \, |t|^{-N} \,,$$

as $t \to \infty$.

Choose $0 < r_1 < r_2$ so that

$$\operatorname{supp} k \;\subset\; \{r_1 \;\le\; |\xi| \;\le\; r_2\}\,.$$

Write

$$\int e^{it(\tau(\xi)+y.\xi)}\; k(\xi)\; d\xi \;=\; \int_{r_1}^{r_2} \Big(\int_{|\xi|=1} e^{it(\tau(\xi)+y.\xi)}\; k(r\,\xi)\; d\sigma(\xi) \Big)\; r^{d-1}\; dr\,.$$

It suffices to show that for any $\underline{y} \in \mathcal{M}$ and $\underline{r} \in [r_1, r_2]$ one has

$$\int_{|\xi|=1} e^{it(\tau(\xi)+y.\xi)}\; k(r\,\xi)\; d\sigma(\xi) \;\le\; C\,|t|^{-(d-1)/2}\,,$$

uniformly for r, y in a neighborhood of $\underline{r}, \underline{y}$.

For $\underline{r}, \underline{y}$ fixed, there is a unique $\underline{\xi}$ with $|\underline{\xi}| = \underline{r}$ for which the phase is stationary and the stationary point is nondegenerate because of the rank equal to $d-1$ hypothesis. It follows that for r, y in a neighborhood, there is a unique uniformly nondegenerate stationary point. The desired estimate follows from the inequality of stationary phase Theorem 3.II.1. ∎

Proposition 3.4.3. *Suppose that $0 < R_1 < R_2 < \infty$ and $\omega := \{\xi \in \mathbb{R} \,:\, R_1 < |\xi| < R_2\}$. There is a constant C so that for all $f \in L^1(\mathbb{R}^d_x)$ with $\operatorname{supp} \hat f \;\subset\; \overline{\omega}$,*

$$u(t,x) \;:=\; (2\pi)^{-d/2} \int e^{i(t\tau_j(\xi)+x.\xi)}\; \hat f(\xi)\; d\xi \;:=\; e^{it\tau_j(D_x)}\, f$$

satisfies

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\le\; C\,(1+|t|)^{-(d-1)/2}\, \|f\|_{L^1(\mathbb{R}^d)}\,. \tag{3.4.3}$$

The proof is based on a simple idea. The solution u is equal to the convolution of the fundamental solution with f. The Fourier transform of the fundamental solution at $t=0$ is equal to a constant. To have an analogous but more regular representation, it is sufficient that one convolve with a solution whose initial data has Fourier Transform equal to this constant on the spectrum of f.

Proof. Choose a $k \in C_0^\infty(\mathbb{R}^d \setminus 0)$ with k equal to $(2\pi)^{-d/2}$ on a neighborhood of $\overline{\omega}$. Define G so that $\hat G := k$. Then since $(2\pi)^{d/2}\, k\, \hat f = \hat f$ one has $G * f = f$. Since $e^{it\tau(D_x)}$ is a Fourier multiplier, one has

$$u(t) := e^{it\tau(D_x)}\, f = e^{it\tau(D_x)} \big(f * G\big) = f * \big(e^{it\tau(D_x)}\, G\big)\,.$$

Then

$$\|u(t)\|_{L^\infty} \;\le\; \|f\|_{L^1}\, \|e^{i\tau(D_x)}\, G\|_{L^\infty}\,.$$

The preceding Lemma shows that

$$\|e^{i\tau(D_x)}\, G\|_{L^\infty} \;\le\; C\,(1+|t|)^{-(d-1)/2}\,. \qquad ∎$$

The next subsections consist of two different paths for exploiting the estimates just proved. The first is more elementary and will be used in Chapter 6 to derive, in the spirit of John-Klainerman, that in high dimensions there is global solvability for maximally dispersive nonlinear problems with small data. The second is devoted to Strichartz estimates which are important in trying to treat existence problems with low regularity data. That in turn is important in trying to pass from local solvability to global solvability for nonlinear problems for which the natural *a priori* estimates control few derivatives.

§3.4.2. Fixed time dispersive Sobolev estimates*

First find decay estimates for $\|u(t)\|_{L^1}$ for sources with Fourier transform supported in $\lambda\overline{\omega}$ for $0<\lambda$. The starting point is

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\le\; C\,|t|^{-(d-1)/2}\,\|f\|_{L^1(\mathbb{R}^d)}\,, \qquad \operatorname{supp}\hat f\subset\omega\,. \tag{3.4.4}$$

Proposition 3.4.4. *There is a constant C so that for all $\lambda>0$ and $f\in L^1$ with $\operatorname{supp}\hat f\subset\lambda\omega$, the solution of*

$$L\,u \;=\; 0\,,\qquad u\big|_{t=0} \;=\; f\,,$$

satisfies

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\le\; C\,|t|^{-(d-1)/2}\,\|\,|D|^{(d+1)/2}f\|_{L^1(\mathbb{R}^d)}\,. \tag{3.4.5}$$

Next perform a dimensional analysis of the homogeneous estimate (3.4.5). With t,x having the dimensions of a length ℓ, the factor $|t|^{(d-1)/2}$ has dimension $\ell^{(d-1)/2}$. On the other hand, in

$$\|\,|D|^\gamma f\|_{L^1(\mathbb{R}^d)} \;=\; \int \big||D|^\gamma f\big|\;dx$$

the integrand has dimension $\ell^{-\gamma}$ and dx has dimension ℓ^d. In total the right hand side of (3.4.5) has dimension $\ell^{d-\gamma-(d-1)/2}$. It is dimensionless as is the left hand side exactly when

$$\gamma \;:=\; \frac{d+1}{2}\,.$$

Proof. Choose $\psi\in C_0^\infty(\mathbb{R}^d_\xi)$ so that $\psi_\pm=|\xi|^{\pm\gamma}$ on $\overline{\omega}$. Then

$$|D|^\gamma f \;=\; C\,\hat\psi_+ * f\,,\qquad \text{and}\qquad f \;=\; C\,\hat\psi_- * (|D|^\gamma f)\,.$$

Young's inequality implies that $\|\,|D|^\gamma f\|_{L^1}$ is a norm equivalent to that on the right in (3.4.4) so

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;\le\; C\,|t|^{-(d-1)/2}\,\|\,|D|^\gamma f\|_{L^1(\mathbb{R}^d)}\,,\qquad \operatorname{supp}\hat f\subset\omega\,.$$

If $u_\lambda(t,x):=u(\lambda t,\lambda x)$ then, $Lu_\lambda=0$ if and only if $Lu=0$ and $\hat u_\lambda(\lambda t,\xi)=\lambda^{-d}\hat u(t,\xi/\lambda)$. The spectrum of u is contained in ω if and only if the spectrum of u_λ is contained in $\lambda\omega$.

Exercise 3.4.1. *Show that if $f_\lambda(x):=f(\lambda x)$,*

$$|D|^\gamma f_\lambda(x) \;=\; \lambda^{-\gamma}\big(|D|^\gamma f\big)(\lambda x)\,.$$

---

* The material in this subsection is not needed for the Strichartz estimates in the next subsection

The change of variable $z = \lambda x$ yields

$$\| \, |D|^\gamma f_\lambda \|_{L^1(\mathbb{R}^d)} \;=\; \int \lambda^{-\gamma} \big(|D|^\gamma f \big)(\lambda x)\ dx \;=\; \lambda^{-\gamma - d} \| \, |D|^\gamma f \|_{L^1(\mathbb{R}^d)} \, .$$

Then (3.4.3) yields

$$\begin{aligned} \big\| u_\lambda(t) \big\|_{L^\infty} \;&=\; \big\| u(\lambda t) \big\|_{L^\infty} \;\le\; C \, \big| \lambda t \big|^{-(d-1)/2} \, \| \, |D|^\gamma f \|_{L^1(\mathbb{R}^d)} \\ &=\; C \, \big| \lambda t \big|^{-(d-1)/2} \, \| \, |D|^\gamma f \|_{L^1(\mathbb{R}^d)} \\ &=\; C \, \lambda^{-(d-1)/2 + \gamma + d} \big| t \big|^{-(d-1)/2} \| \, |D|^\gamma f_\lambda \|_{L^1(\mathbb{R}^d)} \, . \end{aligned}$$

The choice $\gamma = (d+1)/2$ is made so that the λ factors cancel. ∎

Since $\hat{u}$ and $\hat{f}$ are locally integrable functions, the point $\xi = 0$ is negligible so we have the Littlewood-Paley decompositions

$$u \;=\; \sum_{j=-\infty}^{\infty} \chi(2^{-j} D_x) \, u \;:=\; \sum_{j=-\infty}^{\infty} u_j \, , \qquad f \;=\; \sum_{j=-\infty}^{\infty} \chi(2^{-j} D) \, f \;:=\; \sum_{j=-\infty}^{\infty} f_j \, ,$$

where the dyadic decomposition from Lemma 3.II.2. This expresses a solution of $Lu = 0$ as a sum of spectrally localized solutions. The estimates of the next exercise show that $|D|^\sigma$ acts like multiplication by $2^{\sigma j}$ on f_j.

Exercise 3.4.2. *Show that there is an integer k and a constant C depending on σ and χ so that for $p \in [1, \infty]$*

$$\big\| \, |D|^\sigma f_j \big\|_{L^p} \;\le\; C \; 2^{\sigma j} \sum_{|n-j| \le k} \| f_n \|_{L^p} \, , \tag{3.4.6}$$

$$\big\| \, f_j \big\|_{L^p} \;\le\; C \, 2^{-\sigma j} \sum_{|n-j| \le k} \| \, |D|^\sigma f_n \|_{L^p} \, . \tag{3.4.7}$$

Theorem 3.4.5. i. *If $Lu = 0$ and $u\big|_{t=0} = f$ then,*

$$\| u \|_{L^\infty} \;\le\; C \, |t|^{-(d-1)/2} \sum_{j=-\infty}^{\infty} \| \, |D|^\gamma f_j \|_{L^1} \, , \qquad \gamma \;=\; \frac{d+1}{2} \, . \tag{3.4.8}$$

ii. *If $0 < \delta < \gamma$ there is a constant $C(\gamma, \delta)$ so that*

$$\sum_{j=-\infty}^{\infty} \| \, |D|^\gamma f_j \|_{L^1} \;\le\; C \, \Big(\big\| \, |D|^{\gamma - \delta} f \big\|_{L^1(\mathbb{R}^d)} \;+\; \big\| \, |D|^{\gamma + \delta} f \big\|_{L^1(\mathbb{R}^d)} \Big) \, . \tag{3.4.9}$$

Remarks. 1. The sum on the right of (3.4.8) is the definition of the norm in the homogeneous Besov space $\dot{B}^\gamma_{1,1}$. Estimate (3.4.9) yields a bound which is not as sharp but avoids these spaces.

2. A slightly weaker estimate than (3.4.8-3.4.9) was proved by [Lucente-Ziliotti].

3. It is impossible to have a decay estimate of the form

$$\|u(t)\|_{L^\infty} \ \le\ g(t)\|f\|_{H^s}\,, \qquad \lim_{t\to\infty} g(t) = 0\,,$$

with a conserved norm on the right hand side. If there were such an estimate one can apply it to $v(t) = v(t-T)$ at $t = T \to \infty$ to find

$$\|u(0)\|_{L^\infty} \le g(T)\,\|f\|_{H^s} \ \to\ 0\,.$$

The appearance of norms which are not propagated by the equation is necessary.

4. An L^1 condition encodes more rapid decay as $|x| \to \infty$ than an L^2 condition. This is natural since the energy in a ring $R < |x| < R+1$ can focus at time $t \sim R$ into a ball of radius $O(1)$. If the amplitude in the initial ring in $\sim a$ the L^2 norm is $\sim a^2\,R^{d-1}$. If the focused amplitude is $\sim A$ one obtains $A^2 \sim a^2\,R^{d-1}$. If this focussing is to take place at $t \sim R$ and also $A^2 \le t^{-(d-1)}$ one must have $a \le R^{-(d-1)}$. This is on the L^1 borderline. Thus, one cannot have $t^{-(d-1)/2}$ decay estimates as in the Theorem with L^p norms on the right with $p > 1$.

Proof of Theorem. i. Estimate (3.4.5) implies

$$\|u_j(t)\|_{L^\infty} \ \le\ C\,|t|^{-(d-1)}\,\|\,|D|^\gamma f_j\|_{L^1}\,.$$

Summing yields

$$\|u\|_{L^\infty} \ \le\ \sum \|u_j\|_{L^\infty} \ \le\ C\,|t|^{-(d-1)} \sum \|\,|D|^\gamma f_j\|_{L^1}\,.$$

ii. For $j \ge 0$, estimate (3.4.6) implies

$$\big\|\,|D|^\gamma f_j\big\|_{L^1} \ \le\ C\;2^{\gamma j} \sum_{|n-j|\le k} \big\|f_n\big\|_{L^1}\,.$$

Estimate (3.4.7) implies

$$\big\|f_n\big\|_{L^1} \ \le\ C\;2^{-\sigma n} \sum_{|m-n|\le k} \big\|\,|D|^\sigma f_m\big\|_{L^1}\,.$$

Finally,

$$\big\|\,|D|^\sigma f_m\big\|_{L^1} \ \le\ C\,\big\|\,|D|^\sigma f\big\|_{L^1}\,.$$

Combining yields

$$\sum_{j\ge0} \|\,|D|^\gamma f_j\|_{L^1} \ \le\ C\,\big\|\,|D|^\sigma f\big\|_{L^1} \sum_{j\ge0}\ \sum_{|n-j|\le k} 2^{\gamma j - \sigma n}\,.$$

With $\sigma = \gamma + \delta$, the sum is finite, so

$$\sum_{j\ge0} \|\,|D|^\gamma f_j\|_{L^1} \ \le\ C\,\big\|\,|D|^{\gamma+\delta} f\big\|_{L^1}\,.$$

Exercise 3.4.3. *Prove the complementary low frequency estimate*

$$\sum_{j<0} \|\,|D|^\gamma f_j\|_{L^1} \ \le\ C\,\big\|\,|D|^{\gamma-\delta} f\big\|_{L^1}\,.$$

This completes the proof. ∎

Corollary 3.4.6. *For any $d/2 > \delta > 0$ there is a constant C so that if $Lu = 0$, then*

$$\begin{aligned}\|u(t)\|_{L^\infty(\mathbb{R}^d)} \ \leq\ C\,\langle t\rangle^{-(d-1)/2}\Big(&\|f\|_{H^{d/2+\delta}(\mathbb{R}^d)} \\ &+ \|\,|D|^{(d+1)/2+\delta} f\|_{L^1(\mathbb{R}^d)} \ +\ \|\,|D|^{(d+1)/2-\delta} f\|_{L^1(\mathbb{R}^d)}\Big)\,.\end{aligned} \tag{3.4.10}$$

Remark. The smaller is $\delta > 0$ the stronger is the conclusion.

Proof. Sobolev's inequality yields

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \ \leq\ C\,\|u(t)\|_{H^{\delta+d/2}(\mathbb{R}^d)} \ =\ C\,\|f\|_{H^{\delta+d/2}(\mathbb{R}^d)}\,.$$

This yields (3.4.10) for $|t| \leq 1$.
For $|t| \geq 1$ use the two estimates of the Theorem. ∎

§3.4.3. Strichartz estimates

The estimates involve norms

$$\|u\|_{L^q_t L^r_x} \ :=\ \Big(\int_0^\infty \|u(t)\|^q_{L^r(\mathbb{R}^d_x)}\ dt\Big)^{1/q}$$

which integrate over space and time. If such a norm is finite, then the integrand must be small for large times. This requires $r > 2$. The estimates express time decay because of dispersion.

The group velocities lie on the strictly convex manifold $\mathcal{M}$. For a typical Fourier transform, an open set of these velocites is sampled. The method of nonstationary phase shows that for large time the solution is concentrated on the rays with these speeds, starting from the support of the initial data. Thus, a solution is expected to be concentrated on and spread over a region of measure which grows like t^{d-1}. An example is concentration in an annulus $\rho_1 < |x| - t < \rho_2$. Or even finer, concentration on that part of the annulus subtending a fixed solid angle.

Conservation of $L^2(\mathbb{R}^d)$ and also Lemma 3.4.2 show that the expected amplitude is $O(t^{-(d-1)/2})$. Then

$$\|u(t)\|^r_{L^r} \ \sim\ t^{-r(d-1)/2}\ t^{d-1}\,,$$

so

$$\|u\|^q_{L^q_t L^r_x} \ \sim\ \int_1^\infty \Big(t^{-r(d-1)/2}\ t^{d-1}\Big)^{q/r}\ dt\,.$$

The limiting indices are those for which the power of t is equal to -1, that is with

$$\sigma \ :=\ d-1\,,$$

$$\Big(\frac{-r\sigma}{2} + \sigma\Big)\frac{q}{r} \ =\ -1\,, \qquad \text{equivalently,} \qquad \frac{-\sigma}{2} \ +\ \frac{\sigma}{r} \ =\ \frac{-1}{q}\,.$$

The admissible indices are those for which the power is less than or equal to -1,

$$\frac{-\sigma}{2} \ +\ \frac{\sigma}{r} \ \leq\ \frac{-1}{q}\,.$$

Definitions. *The pair $2 < q, r < \infty$ is* **σ-admissible** *if*

$$\frac{1}{q} + \frac{\sigma}{r} \le \frac{\sigma}{2}.$$

It is **sharp σ-admissible** *when equality holds.*

The estimates involve the homogeneous Sobolev norms

$$\| \, |D|^\gamma f \|_{L^2} := \Big(\int \big| \, |\xi|^\gamma \hat f(\xi) \big|^2 \, d\xi \Big)^{1/2}.$$

Theorem 3.4.7 (Strichartz inequality). *Suppose that $L(\partial)$ is maximally dispersive, $\sigma = d-1$, q, r is σ-admissible, and γ is the solution of*

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma.$$

There is a constant C so that for $f \in L^2$ with $\| \, |D|^\gamma f \|_{L^2} < \infty$, the solution of $Lu = 0$, $u|_{t=0} = f$ satisfies

$$\big\| u \big\|_{L^q_t L^r_x} \le C \, \big\| \, |D|^\gamma f \big\|_{L^2(\mathbb{R}^d)}. \tag{3.4.11}$$

There are two complicated relations in this assertion. The first is the definition of admissibility. It is the crucial one which encodes the rate of decay of solutions. The second is the definition of γ. Once admissible q, r are chosen, γ is forced so that the two sides of (3.4.11) scale the same under dilatation, $(t, x) \mapsto (at, ax)$. From this perspective the dispersion is key as it constrains the q, r.

There is a diametrically opposite perspective which starts from the scaling relation which is independent of the dispersion. For example if you are obliged to work with a specific γ (e.g. when we treat the energy space in §6.8) then the scaling restricts $1/q, 1/r$ to lie on a line. The admissability chooses an interval on that line. Changing the dispersion, for example considering a problem with the same scaling but weaker dispersion leaves the line fixed but constrains the $1/q, 1/r$ to lie on a smaller subinterval.

We follow the proof of [Keel-Tao]. ([Ginibre-Velo] is a second standard reference). The limit point case (not discussed here) is treated in the first reference. The key step is an estimate for spectrally localized data.

Lemma 3.4.8. *Suppose that $\sigma := d - 1$, q, r is σ-admissible, and ω is as in the Proposition 3.4.3. There is a constant C so that for all $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} \hat f \subset \overline{\omega}$,*

$$u(t) := e^{it\tau_j(D_x)} f := U(t) f, \qquad U(t)^* = U(-t),$$

satisfies

$$\big\| u \big\|_{L^q_t L^r_x} \le C \, \big\| f \big\|_{L^2}. \tag{3.4.12}$$

Futhermore, for all $F \in L^{q'}_t L^{r'}_x$ with $\operatorname{supp} \hat F(t, \cdot\,) \subset \overline{\omega}$,

$$\Big\| \int_0^\infty U(s)^* \, F(s) \, ds \Big\|_{L^2(\mathbb{R}^d)} \le C \, \big\| F \big\|_{L^{q'}_t L^{r'}_x} \tag{3.4.13}$$

Remark. The estimate is true in the sharp admissible case even though for the heuristics given before the definition, the integral diverged. It is not possible to achieve the concentration suggested in the heuristics with data which has spectrum with support in an annulus. For example, if one considers the wave operator $\Box$ on $\mathbb{R}^{1+3}$ with data supported in $|x| \le 1$ the solutions are supported in $|x| - t \le 1$ and decay along with their derivatives exactly as in the heuristic. Thus one gets divergent integrals. However, compact support and compactly supported Fourier transform are not compatible, and the compact spectrum is enough to overcome the logarithmic divergence of $\int^\infty 1/t\, dt$ which appears in the heuristics.

Proof. Denote by (,) the $L^2(\mathbb{R}^d)$ scalar product. Since.

$$\int_0^\infty (U(t)\, f\,,\, F(t))\ dt \ =\ \int_0^\infty (f\,,\, U(t)^*\, F(t))\ dt \ =\ \Big(f\,,\, \int_0^\infty U(t)^*\, F(t)\ dt\Big)\,,$$

estimates (3.4.12) and (3.4.13) are equivalent thanks to the duality representations of the norms,

$$\Big\| \int_0^\infty U(t)^*\, F(t)\ dt \Big\|_{L^2(\mathbb{R}^d)} \ =\ \sup\Big\{ \Big(f\,,\, \int_0^\infty U(t)^*\, F(t)\ dt\Big)\ :\ \hat f \in C_0^\infty(\omega),\ \ \|f\|_{L^2} = 1\Big\}\,,$$

$$\Big\| U(t) f\Big\|_{L^q L^r} \ =\ \sup\Big\{ \int_0^\infty (U(t)\, f\,,\, F(t))\ dt\ :\ \hat F \in C_0^\infty\big(]0,\infty[\times\omega\big),\ \ \|F\|_{L^{q'} L^{r'}} = 1\Big\}\,.$$

Estimate (3.4.13) holds if and only if

$$\Big(\int_0^\infty (U(t)^* F(t))\ dt\ ,\ \int_0^\infty (U(s)^* G(s))\ ds\Big)$$

is a continuous bilinear form on $L^{q'} L^{r'}$, that is

$$\Big| \int_0^\infty \int_0^\infty \big(U(s)^*\, F(s)\,,\, U(t)^*\, G(t)\big)\ ds\, dt\Big| \ \le\ C\, \|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{q'} L_x^{r'}}\,. \tag{3.4.14}$$

Unitarity implies that

$$\forall s, t\,, \quad B \ :=\ U(t) U^*(s)\,, \qquad \text{satisfies} \qquad \|Bf\|_{L^2} \ \le\ \|f\|_{L^2}\,.$$

The dispersive estimate (3.4.3) is

$$\forall s, t\,, \quad \|Bf\|_{L^\infty} \ \le\ C \langle t - s\rangle^{-\sigma} \|f\|_{L^1}\,.$$

With $r' \in]1, 2[$ the dual index to r, choose $\theta \in]0, 1[$ so that

$$\frac{1}{r'} \ =\ \theta\, \frac{1}{1} \ +\ (1 - \theta)\, \frac{1}{2}\,, \qquad \text{then,} \qquad \theta = \frac{2 - r'}{r'} = \frac{r - 2}{r}\,. \tag{3.4.15}$$

The Riesz-Thorin Theorem implies that

$$\|Bf\|_{L^r} \ \le\ C^\theta\, \langle t - s\rangle^{-\sigma\theta}\, \|f\|_{L^{r'}}\,.$$

With Hölder's inequality, this yields the interpolated bilinear estimate,

$$\Big|\big(U(s)^*\,F(s)\,,\,U(t)^*\,G(t)\big)\Big|\ \le\ C^\theta\,\langle t-s\rangle^{-\sigma\theta}\,\|F(s)\|_{L^{r'}}\|G(t)\|_{L^{r'}}\,.$$

Admissibility implies that

$$\frac{1}{q}\ \le\ \sigma\Big(\frac12-\frac1r\Big)\ =\ \sigma\Big(\frac{r-2}{2r}\Big)\ =\ \frac{\sigma\,\theta}{2}\,.$$

When strict inequality holds in the definition of admissibility, $\langle t-s\rangle^{-\sigma\theta}\in L^{q/2}(\mathbb{R}_t)$. The hypothesis $q>2$ is used here. For the limiting case, it is nearly so. The Hardy-Littlewood inequality shows that convolution with $|t|^{-2/q}$ has the L^p mapping properties that convolution with an element of $L^{q/2}(\mathbb{R})$ would have.

The Hausdorff-Young inequality shows that

$$L^{p_1}\ *\ L^{p_2}\ \subset\ L^{p_3}\,,\qquad \text{provided}\qquad \frac{1}{p_1}\ +\ \frac{1}{p_2}\ =\ 1\ +\ \frac{1}{p_3}\,. \tag{3.4.16}$$

The Hardy Littlewood inequality asserts that when $1<p_1,p_2,p_3<\infty$

$$\frac{1}{\langle t\rangle^{1/p_1}}\ *L^{p_2}(\mathbb{R})\ \subset L^{p_3}(\mathbb{R})\,,\qquad \text{provided}\qquad \frac{1}{p_1}\ +\ \frac{1}{p_2}\ =\ 1\ +\ \frac{1}{p_3}\,. \tag{3.4.17}$$

Set

$$p_1\ =\ \frac{q}{2}\,,\qquad p_2\ =\ q',\qquad \text{and,}\qquad p_3\ =\ q\,. \tag{3.4.18}$$

The index conditions in (3.4.16)-(3.4.17) become

$$\frac{2}{q}\ +\ \frac{1}{q'}\ =\ 1+\frac{1}{q}\,,$$

which holds by definition of q'. Then (3.4.16) in the admissible case and (3.4.17) in the sharp admissible case imply that

$$\Big\|\int_{-\infty}^{\infty}\langle t-s\rangle^{-\sigma\theta}\ \|F(s)\|_{L^{r'}}\ ds\Big\|_{L^q(\mathbb{R}_t)}\ \le\ C\,\big\|F\big\|_{L^{q'}_tL^{r'}_x}\,. \tag{3.4.19}$$

Hölder's inequality yields

$$\int_0^\infty\Big(\int_0^\infty\langle t-s\rangle^{-\sigma\theta}\ \big\|F(s)\big\|_{L^{r'}}\ ds\Big)\ \big\|G(t)\big\|_{L^{r'}}\ dt\ \le\ C\,\big\|F\big\|_{L^{q'}_tL^{r'}_x}\ \big\|G\big\|_{L^{q'}_tL^{r'}_x}\,.$$

This proves the desired estimate (3.4.14). ∎

A scaling yields estimates for sources with Fourier transform suppoerted in $\lambda\overline{\omega}$ for $0<\lambda$.

Lemma 3.4.9. *With q,r,ω,σ as in the previous lemma and γ as in the Theorem, there is a C so that for all $0<\lambda$ and $f\in L^2$ with $\operatorname{supp}\hat f\subset\lambda\overline{\omega}$,*

$$u(t)\ :=\ e^{it\tau_j(D_x)}f\ :=\ U(t)f\,,$$

satisfies

$$\big\|u\big\|_{L^q_t L^r_x} \;\le\; C \big\|\,|D|^\gamma f\big\|_{L^2}\,. \tag{3.4.20}$$

Proof of Lemma. If $u_\lambda(t,x) := u(\lambda t, \lambda x)$ then, $Lu_\lambda = 0$ and the spectrum of u_λ is contained in $\overline{\omega}$.

The two sides of (3.4.12) scale differently. Compute

$$\big\|u_\lambda(t)\big\|_{L^r} \;=\; \Big(\int |u_\lambda(t,x)|^r\;dx\Big)^{1/r} \;=\; \Big(\int |u(\lambda t\,,\,\lambda x)|^r\;dx\Big)^{1/r}.$$

The substitution $y = \lambda x$, $dx = \lambda^{-d} dx$ yields

$$=\; \lambda^{-d/r}\Big(\int |u(\lambda t\,,\,y)|^r\;dy\Big)^{1/r} \;=\; \lambda^{-d/r}\big\|u(\lambda t)\big\|_{L^r}\,.$$

A similar change of variable for the time integral shows that

$$\big\|u_\lambda\big\|_{L^q_t L^r_x} \;=\; \lambda^{-1/q\,-\,d/r}\big\|u\big\|_{L^q_t L^r_x}\,.$$

For any γ, $\|\,|D|^\gamma f\|_{L^2}$ is a norm equivalent to the norm on the right hand side for sources with spectrum in $\overline{\omega}$. Compute

$$\begin{aligned}\|\,|D|^\gamma f_\lambda\|_{L^2} \;&=\; \Big(\int |\xi|^{2\gamma}\big|\hat f_\lambda(\xi)\big|^2 d\xi\Big)^{1/2} = \Big(\int |\xi|^{2\gamma}\big|\lambda^{-d}\hat f(\xi/\lambda)\big|^2 d\xi\Big)^{1/2}\\ &=\; \lambda^{\gamma\,-\,d/2}\Big(\int |\xi|^{2\gamma}\big|\hat f(\xi)\big|^2 d\xi\Big)^{1/2} \;=\; \lambda^{\gamma\,-\,d/2}\|\,|D|^\gamma f\|_{L^2}\,.\end{aligned}$$

Given q, r, the γ of the Theorem is the unique value so that the two norms scale the same. Therefore the estimate of the present Lemma follows from the preceding Lemma. ∎

Proof of Theorem. With χ from the dyadic partition of unity for $\mathbb{R}^d_\xi \setminus 0$ from Lemma 3.II.2. Introduce the Littlewood-Paley decomposition of tempered distributions

$$g \;=\; \sum_{J\in\mathbb{Z}} g_j\,, \qquad g_j \;:=\; \chi(D/2^j)\,g \;:=\; (2\pi)^{-d/2}\int e^{ix\xi}\;\chi(\xi/2^j)\;\hat g(\xi)\;d\xi\,.$$

Then for $1 < r < \infty$ the square function estimate (see [Stein, 1970]) asserts that there is a $C = C(p) > 1$ so that

$$C^{-1}\|g\|_{L^r} \;\le\; \Big\|\Big(\sum_{j\in\mathbb{Z}} |g_j|^2\Big)^{1/2}\Big\|_{L^r} \;\le\; C\,\|g\|_{L^r}\,.$$

Lemma 3.4.10. *If $2 \le q, r < \infty$, there is a constant C so that*

$$\big\|F\big\|^2_{L^q_t L^r_x} \;\le\; C\sum_{j\in\mathbb{Z}} \big\|F_j\big\|^2_{L^q_t L^r_x}\,, \tag{3.4.21}$$

where $F(t) = \sum_j F_j(t)$ *is the Littlewood-Paley decomposition in* x.

Proof of Lemma. The square function estimate yields

$$\big\|F(t)\big\|^2_{L^r_x} \;\le\; C \int \Big(\sum_j |F_j(t)|^2\Big)^{r/2}\, dx \;=\; C\,\Big\|\sum_j |F_j(t)|^2\Big\|_{L^{r/2}}\,.$$

Minkowski's inequality in $L^{r/2}$ shows that this is

$$\le\; C\sum_j \big\|F_j(t)^2\big\|_{L^{r/2}} \;=\; C\sum_j \big\|F_j(t)\big\|^2_{L^r}\,.$$

Using this yields

$$\big\|F\big\|^2_{L^q_t L^r_x} \;\le\; C\Big(\int_0^\infty \Big(\sum_j \big\|F_j(t)\big\|^2_{L^r}\Big)^{q/2}\, dt\Big)^{2/q} \;=\; C\,\Big\|\sum_j \|F_j(t)\|^2_{L^r(\mathbb{R}^d_x)}\Big\|_{L^{q/2}(\mathbb{R}_t)}\,.$$

Minkowski's inequality in $L^{q/2}(\mathbb{R}_t)$ shows this is

$$\le\; C\sum_j \Big\|\|F_j(t)\|^2_{L^r(\mathbb{R}^d_x)}\Big\|_{L^{q/2}(\mathbb{R}_t)} \;=\; C\sum_j \|F_j(t)\|^2_{L^q_t L^r_x}\,. \qquad \blacksquare$$

Return now to the proof of the Theorem. Associate to the sheet $\tau = \tau_k(\xi)$ the projector $\pi_k(\xi) := \pi(\tau_k(\xi), \xi)$ from §3.2. The π_k are real analytic on $\xi \ne 0$ and homogeneous of degree 0 in ξ. In addition $\sum_k \pi_k = I$. The solution u satisfies

$$u \;=\; \sum_k e^{it\tau_k(D)}\, \pi_k(D)\, f \;:=\; \sum_k u_k\,.$$

Apply (3.4.21) to u_k to find using (3.4.20)

$$\big\|u_k\big\|^2_{L^q_t L^r_x} \;\le\; C\sum_j \|u_{k,j}\|^2_{L^q_t L^r_x} \;\le\; C'\sum_j \|\,|D|^\gamma \pi_k(D) f_j\|^2_{L^2} \;\le\; C'\,\|\,|D|^\gamma f\|^2_{L^2}\,.$$

The finite sum on k completes the proof of the Theorem. $\blacksquare$

Corollary 3.4.11. *Denote by* $S(t)$ *the* L^2 *unitary mapping* $u(0) \mapsto u(t)$ *for solutions of* $Lu = 0$. *With the indices of the Theorem one has*

$$\Big\|\int_0^\infty S(s)^*\, F(s)\, ds\Big\|_{L^2(\mathbb{R}^d)} \;\le\; C\,\big\||D_x|^\gamma\, F\big\|_{L^{q'}_t L^{r'}_x}\,. \tag{3.4.22}$$

Proof. Estimate (3.4.22) is equivalent to the Strichartz estimate (3.4.11) by a duality like that used to establish the equivalence of (3.4.12) and (3.4.13). $\blacksquare$

Exercise 3.4.4. *Prove the following complement to (3.4.21) which comes from the other side of the square function inequality. If* $1 < p \le 2$ *and* $1 \le r \le 2$ *then there is a* C *so that*

$$\sum_{j=-\infty}^{\infty} \big\|F_j\big\|^2_{L^r_t L^p_x} \;\le\; C\,\big\|F\big\|^2_{L^r_t L^p_x}\,. \tag{3.4.23}$$

Appendix 3.I. Perturbation theory for semisimple eigenvalues

The computation of the form of the operator $\pi\, L\, \pi$ requires formulas from the perturbation theory of eigenvalues. These results for multiple eigenvalues which are semisimple is not that well known. The key idea is that one should NOT make a choice of basis of eigenfunctions, but work systematically with the spectral projections.

Definition. *An eigenvalue λ of a matrix A is* **semisimple** *when the kernel and range of $A - \lambda I$ are complementary subspaces. In this case denote by π the* **spectral projection** *onto the kernel of $A - \lambda I$ along its range and by Q the* **partial inverse** *defined by*

$$Q\,\pi \;=\; 0\,, \qquad Q\left(A - \lambda\, I\right) = I - \pi\,. \tag{3.I.1}$$

Examples. i. Every eigenvalue of a hermitian symmetric or normal matrix is semisimple.

ii. More generally, a matrix is similar to a diagonal matrix if and only if each of its eigenvalues is semisimple.

Theorem 3.I.1. *Suppose that $\Omega \subset \mathbb{R}^m$ is open and $M(Y) \in C^\infty(\Omega, \mathrm{Hom}\,(\mathbb{C}^N))$ is a matrix valued function. Suppose that there is a disk $\mathbb{D} \in \mathbb{C}$ so that for every Y there is exactly one eigenvalue, $\lambda(Y)$ of M in $\overline{\mathbb{D}}$ and that eigenvalue is semisimple. Denote by $\pi(Y)$ the projection along the range of $M - \lambda I$ onto the kernel of $M - \lambda I$ and by $Q(Y)$ the partial inverse defined by*

$$Q(Y)\,\pi(Y) \;=\; 0, \qquad Q(Y)\,(M(Y) - \lambda I) \;=\; I - \pi(Y),$$

Then λ, π and Q are smooth functions of Y.

Proof. It suffices to prove smoothness at an arbitrary $\underline{Y}$. Suppose that $\mathbb{D} = \{|z - \underline{z}| < r\}$. Choose $\epsilon > 0$ so that for $|Y - \underline{Y}| < \epsilon$ the disk $|z - \underline{z}| \le r + \epsilon$ contains only one eigenvalue. The regularity of $\pi(Y)$ for those Y follows from the contour integral representation,†

$$\pi(Y) \;=\; \frac{1}{2\pi i} \oint_{|z - \underline{z}| = r + \epsilon} \Big(z - (M(Y) - \lambda(Y) I)\Big)^{-1}\, dz\,.$$

The regularity of Q and λ follow from the identities,

$$Q(Y) = (I - \pi(Y))\left(\pi(Y) + M(Y)\right)^{-1}, \qquad \lambda(Y) \;=\; \frac{\mathrm{trace}(M(Y)\,\pi(Y)}{\mathrm{trace}\,\pi(Y)}\,. \qquad \blacksquare$$

Theorem 3.I.2. *Suppose that $]a, b[\ni s \to A(s)$ is a smooth family of complex matrices so that the disk $\overline{\mathbb{D}} \subset \mathbb{C}$ contains a single semisimple eigenvalue $\lambda(s)$. Denoting d/ds by $'$, one has the following perturbation formulas,*

$$\lambda'(s)\ \pi(s) \;=\; \pi(s)\ A'(s)\ \pi(s)\,, \tag{3.I.2}$$

$$\lambda''\ \pi \;=\; \pi\, A''\, \pi \;-\; 2\, \pi\, A'\, Q\, A'\, \pi\,, \tag{3.I.3}$$

† A short proof is to evaluate the right hand sides in a basis whose first k elements form a basis for $\ker\,(M - \lambda I)$ and whose last elements are a basis for the range.

$$\pi' \;=\; -\pi\, A'\, Q - QA'\pi\,. \tag{3.I.4}$$

Example. If A is hermitian symmetric semisimplicity is automatic and π is hermitian. If $\mathbf{v}$ is a unit eigenvector, then multiplying (3.I.2) on the right by $\mathbf{v}$ and then taking the scalar product with $\mathbf{v}$ yields the standard formula, $\lambda' = \langle \mathbf{v}\,,\, A'\,\mathbf{v}\rangle$.

Proof. The smoothness of $\pi(s), Q(s), \lambda(s)$ follows from Theorem 3.I.1.
The formulas (3.I.2-3.I.4) are proved by differentiating the identity $(A-\lambda)\pi = \pi(A-\lambda) = 0$ with respect to s. The equation for each d^j/ds^j is analysed by considering its projections π and $I-\pi$. Equivalently, each equation is multiplied first by π, then by Q.
Differentiating $(A-\lambda)\pi$ yields,

$$(A-\lambda)'\;\pi \;+\; (A-\lambda)\;\pi' \;=\; 0\,. \tag{3.I.5}$$

Mulitplying on the left by π eliminates the second term to yield,

$$\pi\;(A-\lambda)'\;\pi \;\;=\; 0\,, \tag{3.I.6}$$

which is (3.I.2).
Multiply equation (3.I.5) on the left by Q to find,

$$(I-\pi)\pi' \;=\; -Q\,\big(A-\lambda\big)'\,\pi\,.$$

Since $Q\,\pi = 0$ this simplifies to,

$$(I-\pi)\pi' \;=\; -QA'\pi\,. \tag{3.I.7}$$

Equation (3.I.5) is exhausted and we take a second derivative,

$$(A-\lambda)''\pi + 2(A-\lambda)'\pi' + (A-\lambda)\pi'' = 0\,.$$

Mutiply on the left by π to eliminate the last term,

$$\pi(A-\lambda)''\pi + 2\pi(A-\lambda)'\pi' \;=\; 0\,.$$

Subtract $2(\pi(A-\lambda)'\pi)\pi' = 0$ to find,

$$\pi(A-\lambda)''\pi + 2\pi(A-\lambda)'(I-\pi)\pi' \;=\; 0\,.$$

Then (3.I.7) yields

$$\pi(A-\lambda)''\pi + 2\pi(A-\lambda)'(-QA'\pi) \;=\; 0\,. \tag{3.I.8}$$

Since $\pi\,Q = 0$ one has

$$2\,\pi\lambda'(-QA'\pi) \;=\; 0\,. \tag{3.I.9}$$

Adding (3.I.8) and (3.I.9) yields (3.I.3).
To prove (3.I.4) knowing (3.I.7), what is needed is $\pi\,\pi'$. Differentiate $\pi^2 = \pi$ to find,

$$\pi\,\pi' + \pi'\,\pi \;=\; \pi'\,, \qquad \text{whence,} \qquad \pi\,\pi' \;=\; \pi'(I-\pi)\,. \tag{3.I.10}$$

Differentiate $\pi\,(A-\lambda) = 0$ to find,

$$\pi'(A-\lambda) + \pi\,(A-\lambda)' \;=\; 0\,.$$

Mulitply on the right by Q to find,

$$\pi'(I-\pi) \;=\; -\pi\,(A'-\lambda')\,Q\,.$$

Use (3.I.10) and simplify using $\pi\,Q = 0$ to find,

$$\pi\,\pi' \;=\; \pi'(I-\pi) \;=\; -\pi\,A'\,Q\,.$$

Adding this to (3.I.7) completes the proof. ∎

Chapter 3. Appendix II. The stationary phase inequality

Definition. *A point $\underline{x}$ in an open subset $\Omega \subset \mathbb{R}^d$ is a* **stationary point** *of $\phi \in C^\infty(\Omega\,;\,\mathbb{R})$ when $\nabla_x\phi(\underline{x}) = 0$. It is a* **nondegenerate stationaray point** *when the matrix of second derivatives at $\underline{x}$ is nonsingular.*

When $\underline{x}$ is a nondegenerate stationary point the map $x \mapsto \nabla_x\phi(x)$ has nonsingular Jacobian at $\underline{x}$. It follows that the map is a local diffeomorphism and in particular the stationary point is isolated. Taylor's Theorem shows that

$$\nabla_x\phi(x) \;=\; \frac{1}{2}\,\nabla_x^2\phi(\underline{x})\,(x-\underline{x}) \;+\; O(|x-\underline{x}|^2)\,.$$

Therefore if $\omega \subset\subset \Omega$ contains $\underline{x}$ and no other stationary point, nondegeneracy implies that there is a constant $C>0$ so,

$$\forall \underline{x} \in \omega\,, \qquad \big|\nabla_x\phi(x)\big| \;\geq\; C\,|x-\underline{x}|\,. \tag{3.II.1}$$

We estimate the size of oscillatory integrals whose phase has a single nondegenerate stationary point. These integrals have a complete asymptotic expansion. I learned the a dyadic argument using the method of nonstationary phase from G. Métivier. See [Stein, Real Variable Methods] for an alternate proof.

Theorem 3.II.1. *Suppose that $\phi \in C^\infty(\Omega\,;\,\mathbb{R})$ has a unique stationary point $\underline{x} \in \Omega$. Suppose that $\underline{x}$ is nondegenerate and let m denote the smallest integer strictly larger than $d/2$. Then for any $\omega \subset\subset \Omega$ there is a constant C so that for all $f \in C_0^m(\omega)$, and $0<\epsilon<1$,*

$$\Big|\int e^{i\phi/\epsilon} f(x)\;dx\Big| \;\leq\; C\,\epsilon^{d/2} \sup_{|\alpha|\leq m} \|\partial^\alpha f(x)\|_{L^\infty(\omega)}\,. \tag{3.II2}$$

Lemma 3.II.2. *There is a nonnegative $\chi \in C_0^\infty(\mathbb{R}^d\setminus 0)$ so that for all $x\neq 0$, $\sum_{k=-\infty}^{\infty}\chi(2^k\,x) = 1$.*

Proof of Lemma. Choose nonnegative $g \in C_0^\infty(\mathbb{R}^d\setminus 0)$ so that $g\geq 1$ on $\{1\leq |x|\leq 2\}$. Define the locally finite sum

$$G(x) \;:=\; \sum_{k=-\infty}^{\infty} g(2^k x)\,, \qquad G(2^k x) = G(x)\,.$$

Then $G \in C^\infty(\mathbb{R}^d\setminus 0)$, and $G\geq 1$. The function $\chi := g/G$ has the desired properties. ∎

Proof of Theorem. Translating coordinates we may suppose that $\underline{x} = 0$. Choose χ as in the lemma and write

$$\int e^{i\phi/\epsilon} f(x)\;dx \;=\; \sum_{k=-\infty}^{\infty} \int \chi(2^k\,x)\;e^{i\phi/\epsilon}\;f(x)\;dx \;:=\; \sum_{k=-\infty}^{\infty} I(k)\,.$$

The half sum $\sum_{k<0}\chi(2^k x)$ is a smooth function on $\mathbb{R}^d$ which vanishes on a neighbhorhood of the origin and is identically equal to 1 outside a large ball. The nonstationary phase Lemma 1.2.2 implies that

$$\Big|\int\; e^{i\phi/\epsilon}\Big(\sum_{k<0}\chi(2^k\,x)\Big)\;f(x)\;dx\Big| \;\leq\; C\,\epsilon^m \sup_{|\alpha|\leq m}\|\partial^\alpha f(x)\|_{L^1(\omega)}\,.$$

The sum $\sum_{2^k\epsilon^{1/2}\geq 1}\chi(2^k\,x)$ is a bounded function supported in a ball $|x|\leq C\epsilon^{1/2}$ so

$$\Big|\int\ e^{i\phi/\epsilon}\Big(\sum_{2^k\epsilon^{1/2}\geq 1}\chi(2^k\,x)\Big)\ f(x)\ dx\Big|\ \leq\ C\,\epsilon^{d/2}\,\|f(x)\|_{L^\infty(\omega)}\,.$$

There remains the sum over $1\leq 2^k<\epsilon^{-1/2}$. The change of variable $y=2^k\,x$ yields

$$I(k)\ =\ 2^{-kd}\int\ \chi(y)\ e^{i\phi_k(y)/(2^{2k}\epsilon)}\ f(2^{-k}y)\ dy\,,\qquad \phi_k(y)\ :=\ 2^{2k}\,\phi(2^{-k}\,y)\,.$$

It follows from (3.II.1) that there is a constant $c>0$ so that on the support of χ,

$$c^{-1}\leq\big|\nabla\phi_k\big|\leq c\,.$$

In addition there is are constants $C(\alpha)$ independent of $k\geq 0$ so that $|\partial^\alpha\phi_k|\ \leq\ C_\alpha$. The method of nonstationary phase shows that there is a constant independent of $k\geq 0$ so that

$$\Big|\int\ \chi(y)\ e^{i\phi_k(y)/(2^{2k}\epsilon)}\ f(2^{-k}y)\ dy\Big|\ \leq\ C\,\big(2^{2k}\,\epsilon\big)^m\ \sup_{|\alpha|\leq m}\ \|\partial^\alpha f(x)\|_{L^1(\omega)}\,.$$

Therefore

$$\sum_{1\leq 2^k<\epsilon^{-1/2}}|I(k)|\ \leq\ C\,\epsilon^m\sum_{1\leq 2^k<\epsilon^{-1/2}}2^{-kd}\ 2^{2km}\ \sup_{|\alpha|\leq m}\ \|\partial^\alpha f(x)\|_{L^1(\omega)}\,.$$

The finite geometric sum has ratio $r=2^{2m-d}>1$. If K is the largest index,

$$r^K\ \leq\ 1+r+r^2\ldots+r^K\ =\ \frac{r^{K+1}-1}{r-1}\ <\ \frac{r}{r-1}\,r^K\ :=\ C(r)\,r^K\,.$$

The sum is comparable to the last term. Therefore, with $C=C(m,d)=r/(r-1)$,

$$\epsilon^m\sum_{1\leq 2^k<\epsilon^{-1/2}}2^{-kd}\ 2^{2km}\ \leq\ C\,\epsilon^m\ \big(2^K\big)^{2m-d}\ \leq\ C\ \epsilon^m\ \big(\epsilon^{-1/2}\big)^{2m-d}\ =\ C\,\epsilon^{d/2}\,.$$

This completes the proof. ∎

Corollary 3.II.3. *Suppose that $\phi(x,\zeta)$ is a family of phases depending smoothly on ζ on a neighborhood of $0\in\mathbb{R}^q$ and that $\phi(x,0)$ satisfies the hypotheses of the preceding Theorem. Then there is a neighborhood $0\in\mathcal{O}$ so that the hypotheses are satisfied for $\zeta\in\mathcal{O}$ and the estimate (3.II.1) holds with a constant independent of $\zeta\in\mathcal{O}$.*

Proof. The first assertion follows from the implicit function theorem applied to the system of equations $\nabla_x\phi(x,\zeta)\ =\ 0$. The estimates of the proof are all locally uniform which proves the second assertion. ∎

Chapter 3. Appendix III. Hadamard's ovaloid theorem.

Theorem 3.III.1. (Hadamard). *If $d \geq 3$ and $\mathcal{M}$ is an oriented connected compact immersed hypersurface of $\mathbb{R}^d$ whose Gaussian curvature is nonzero at all points, then $\mathcal{M}$ is the boundary of a strictly convex set.*

Proof. Consider the Gauss map $\mathcal{N}$ from $\mathcal{M}$ to S^{d-1} which takes a point to its unit normal consistent with the orientation.

The nonvanishing curvature is equivalent to the differential of $\mathcal{N}$ being invertible at all points. The inverse function theorem shows that this is equivalent to $\mathcal{N}$ being a local diffeomorphism.

For any $\xi \in S^{d-1}$ the point(s) $\underline{x} \in \mathcal{M}$ where $x.\xi$ is maximal have normal equal to ξ so $\mathcal{N}$ is surjective.

The number of preimages of points is finite and locally constant, hence constant. Therefore $\mathcal{N}$ is a covering map.

Since S^{d-1} is simply connected, it follows that $\mathcal{N}$ is a homeomorphism and therefore a diffeomorphism. We recall the proof.

It suffices to show that $\mathcal{N}$ is injective. If $\mathcal{N}(m_1) = \mathcal{N}(m_2) = p \in S^{d-1}$ choose a curve $\gamma_0 : [a,b] \to \mathcal{M}$ connecting m_1 to m_2. The image $\mathcal{N} \circ \gamma$ is a closed curve μ_0 in S^{d-1} beginning and ending at p.

Simple connectivity implies that there is a homotopy of closed curves μ_t for $0 \leq t \leq 1$ beginning and ending at p with μ_1 reducing to the constant path p.

Since $\mathcal{N}$ is a covering, the homotopy lifting lemma shows that there is a homotopy γ_t, $0 \leq t \leq 1$ so that $\mathcal{N} \circ \gamma_t = \mu_t$.

The point $\gamma_t(a)$ is a point of $\mathcal{M}$ depending continuously on t with $\mathcal{N}(\gamma_t(a)) = p$. It follows that $\gamma_t(a)$ is constant and therefore equal to m_1. Similarly $\gamma_t(b) = m_2$. In particular this holds for $t = 1$.

But γ_1 is a lifting of the constant map μ_1 and is therefore constant. Therefore

$$m_1 \ = \ \gamma_1(a) \ = \ \gamma_1(b) \ = \ m_2$$

proving injectivity.

Thus each vector in S^{d-1} is normal to $\mathcal{M}$ at exactly one point. This shows that $\mathcal{M}$ is strictly convex in the sense that it intersects each tangent plane in exactly one point.

That it is strictly convex in the stronger sense of osculating ellipsoids, then follows from the nonvanishing Gaussian curvature. ∎

Example. A curve in $\mathbb{R}^2$ with positive curvature and looping twice about the origin shows that the result is not true when $d = 2$.

Chapter 4. Linear Elliptic Geometric Optics

The study of oscillatory solutions of elliptic equations is easier than the corresponding hyperbolic theory. The reason is simple. Oscillations propagate in the hyperbolic case, and have only local effects in the elliptic case. Nevertheless, the elliptic case is a good starting point for several reasons. First, it is easier to introduce some of the basic notions in this case. Second, the elliptic results are needed eventually in the proofs of nonlinear hyperbolic results.

For the analysis of this section there is no need for symmetry or any other hypothesis of hyperbolicity. Similarly the independent variable y is not split into space and time. The partial differential operator

$$L(y, \partial_y) = \sum A_\mu(y) \frac{\partial}{\partial y_\mu} + B(y)$$

is assumed to have smooth matrix valued coefficients on an open set $\mathcal{O} \subset \mathbb{R}^n$.

The fundamental dichotomy is between oscillations with phases ϕ such that $(y, d\phi(y))$ is characteristic or not. For elliptic operators only the second possibility occurs while for hyperbolic problems both are possible.

When the phase is noncharacteristic one can have oscillatory solutions only when there are oscillatory sources.

§4.1. Euler's method and elliptic geometric optics with constant coefficients

Our starting point is the elementary theory of constant coefficient ordinary differential operators

$$L(d/dt) \ := \ p_m \frac{d^m}{dt^m} + p_{m-1} \frac{d^{m-1}}{dt^{m-1}} + \cdots + p_1 \frac{d^1}{dt^1} + p_0 \,, \qquad a_m \neq 0\,.$$

Euler taught that since $L(d/dt)e^{i\tau t} = L(i\tau)e^{i\tau t}$, a solution of,

$$Lu = b\, e^{i\tau t}$$

is given by

$$u = L(i\tau)^{-1}\, b\, e^{i\tau t}, \qquad \text{provided that,} \qquad L(i\tau) \neq 0\,. \tag{4.1.1}$$

Since

$$|L(\tau)| \geq |p_m\, \tau^m| - \sum_0^{m-1} |p_j\, \tau|^j \,, \qquad p_m \neq 0\,,$$

$L(i\tau) \neq 0$ for large τ so this method suffices for rapidly oscillating sources.

Consider a localized strongly oscillatory source,

$$L(d/dt)u \ = \ b(t)\, e^{i\phi(t)/\epsilon}\,, \qquad b \in C_0^\infty(\mathbb{R})\,, \qquad 0 < \epsilon << 1\,.$$

To guarantee oscillation suppose that $\phi' \neq 0$ on $\operatorname{supp} b$. For t outside of $\operatorname{supp} b$, u satisfies $Lu = 0$. The general solution of this homogeneous equation is a linear combination of m solutions of the form $q(t)\, e^{rt}$ with polynomial q and roots r of $L(r) = 0$. In particular it does not oscillate on scale ϵ for small ϵ. Oscillations do not propagate beyond the support of b. We will see that elliptic partial differential equations behave like this while in the hyperbolic case, oscillations propagate.

For a constant coefficient partial differential operator Euler's identity is, $L(\partial_y)e^{iy.\eta} = L(i\eta)e^{iy.\eta}$, so a particular solution of a constant coefficient system of partial differential equations,

$$L(\partial_y)\, u = b\, e^{iy.\eta}\,, \qquad b \in \mathbb{C}^N\,, \tag{4.1.2}$$

is given by

$$u = L(i\eta)^{-1}\, b\, e^{iy.\eta}, \qquad \text{provided that}, \qquad \det L(i\eta) \neq 0\,.$$

To study the case of a first order system in the limit of small wavelength, use η/ϵ in place of η and consider $\epsilon \to 0$. Since

$$L(i\eta/\epsilon) \;=\; L_1(i\eta/\epsilon) + L_0 \;=\; \frac{1}{\epsilon}\,\big(L_1(i\eta) + \epsilon L_0\big)\,,$$

it follows that if η is not characteristic then $L(i\eta/\epsilon)$ is invertible for ϵ small. For such η, ϵ, an explicit solution of (4.2) is given by

$$u = e^{iy.\eta/\epsilon}\,\big(L_1(i\eta/\epsilon) + L_0\big)^{-1}\, b = \epsilon\, e^{iy.\eta/\epsilon}\,\big(L_1(i\eta) + \epsilon\, L_0\big)^{-1}\, b\,.$$

Write

$$L_1(i\eta) + \epsilon\, L_0 \;=\; L_1(i\eta)\big(I \;+\; \epsilon\, L_1(i\eta)^{-1}\, L_0\big),$$

to show that for ϵ small, the inverse is given by a convergent Neumann series,

$$u(y) = \epsilon\, e^{iy.\eta/\epsilon} \sum_{n=0}^{\infty} \big(-\epsilon\, L_1(i\eta)^{-1}\, L_0\big)^n\, L_1(i\eta)^{-1}\, b$$
$$= \epsilon\, e^{iy.\eta/\epsilon}\,\big(L_1(i\eta)^{-1}\, b + \text{higher order terms}\big)\,.$$

The form of the solution is a series

$$e^{iy.\eta/\epsilon}\,\Big(\epsilon\, a_1 + \epsilon^2 a_2 + \cdots\Big)$$

where the vector valued summands are each obtained by multiplying b by a finite product of matrices.

Multiplying the source and the solution by $e^{ic/\epsilon}$ with real c shows that the computation above works for the affine phase $\phi = c + y.\eta$, in which case $\eta = d\phi$.

A key feature of this solution is that the leading term depends only on the principal symbol L_1. The reason is simple. For the highly oscillatory solutions, the derivatives are of order $1/\epsilon$ larger than u. Thus so long as the combination of derivatives represented by $L_1(i\eta)\, u$ is nonzero it will be dominant. When η is noncharacteristic, $L_1(i\eta)$ is invertible and this condition is satisfied. The general principle here is that for noncharacteristic short wavelength oscillations, the principal symbol dominates. In contrast when $L_1(i\eta)$ is not invertible, the lower order terms play an important role (see §1.4 and Chapter 5).

§4.2. Iterative improvement for variable coefficients and nonlinear phases

The next step is a key insight. Suppose that one considers source terms which are rapidly oscillating with possibly nonlinear phase, and a source with amplitude which depends on y and a differential operator with possibly variable coefficients

$$L(y, \partial_y)\, u = b(y)\, e^{i\phi(y)/\epsilon}\,. \tag{4.2.1}$$

The phase ϕ is a smooth real valued function whose gradient is assumed to be nonvanishing on the support of $b(y)$. The coefficients of L are assumed to be smooth on a neighborhood of this support.

Imagine an observer who looks at u near a point $\underline{y}$. Suppose that the observation region is large compared to ϵ but small compared to the scale on which b, the coefficients of L, and $d\phi$ vary. To such an observer these quantities appear constant and the differential equation looks like

$$L(\underline{y}, \partial_y)u \;=\; b(\underline{y})\, e^{i(\phi(\underline{y})+d\phi(\underline{y}).(y-\underline{y}))/\epsilon}\,. \tag{4.2.2}$$

If $(\underline{y}, d\phi(\underline{y}))$ is not in the characteristic variety of L, the previous analysis shows that for ϵ small an approximate solution on this region is given by

$$u_{\text{approx}} \sim \epsilon\, e^{i(\phi(\underline{y})+d\phi(\underline{y}).(y-\underline{y}))/\epsilon}\, L_1(\underline{y}, id\phi(\underline{y}))^{-1}\, b(\underline{y}) \approx \epsilon\, e^{i\phi(y)/\epsilon}\, L_1(\underline{y}, id\phi(\underline{y}))^{-1}\, b(\underline{y})\,.$$

These computations suggest that

$$u(y) \;=\; \epsilon\, e^{i\phi(y)/\epsilon}\, a_1(y)\,, \qquad a_1(y) := L_1(y, id\phi(y))^{-1}\, b(y)\,, \tag{4.2.3}$$

defines a reasonable approximate solution.

The idea leading to this guess was that in the limit of very small wavelength the problem can be replaced by an approximate problem with constant coefficients, a source with constant amplitude, and, an affine phase. To assess the accuracy of the approximation, take u as defined in (4.2.3) and apply $L(y, \partial)$. The largest $O(1/\epsilon)$ arise when a derivative falls on the exponential factor where $L_1(y, \partial)e^{i\phi/\epsilon} = L(y, id\phi/\epsilon)\, e^{i\phi/\epsilon}$ yielding,

$$L(y, \partial_y)\big(\epsilon\, e^{i\phi/\epsilon} a_1(y)\big) = e^{i\phi/\epsilon}\big(L_1(y, id\phi)\, a_1 + L(y, \partial)a_1\big) = e^{i\phi/\epsilon}\big(b(y) + \epsilon b_1(y)\big), \tag{4.2.4}$$

where

$$b_1(y) := L(y, \partial_y)\, a_1(y)\,. \tag{4.2.5}$$

The error on the right hand side is of the same order of magnitude as the approximate solution which is not an auspicious start. The good news, is that the previous computation tells us a corrector. It suffices to subtract from the approximate solution an approximate solution, v, to $L\, v = \epsilon b_1\, e^{i\phi/\epsilon)}$. Thus, take

$$u \;:=\; e^{i\phi(y)/\epsilon}\left(\epsilon\, a_1(y) + \epsilon^2\, a_2(y)\right), \qquad a_2(y) \;:=\; -L_1(y, id\phi(y))^{-1}\, b_1(y)\,, \tag{4.2.6}$$

to find

$$L(y, \partial_y)\, u = e^{i\phi(y)/\epsilon}\left(b(y) + \epsilon^2 b_2(y)\right), \tag{4.2.7}$$

where

$$b_2(y) := L(y, \partial_y)\, a_2(y)\,. \tag{4.2.8}$$

This process, by induction on m, then proves the following theorem.

Theorem 4.2.1. *Suppose that $m \geq 1$ is an integer, $\Omega \subset \mathbb{R}^n$ is a bounded open set, $b(y)$ is a smooth amplitude on Ω and ϕ is a smooth real valued phase such that for all $y \in \Omega$, $d\phi(y) \neq 0$ and $(y, d\phi(y)) \notin \mathrm{Char}\, L$. Then, there are uniquely determined smooth amplitudes a_j on Ω so that*

$$u^\epsilon := e^{i\phi(y)/\epsilon}\left(\epsilon\, a_1(y) + \epsilon^2\, a_2(y) + \cdots + \epsilon^m\, a_m(y)\right) \tag{4.2.9}$$

satisfies

$$Lu^\epsilon = e^{i\phi(y)/\epsilon}\, b(y) + \epsilon^m\, e^{i\phi(y)/\epsilon}\, r^\epsilon(y)\,, \tag{4.2.10}$$

with

$$\forall\alpha \qquad \sup_{(\epsilon,y)\in]0,1]\times\Omega} \left|\partial_y^\alpha r^\epsilon(y)\right| \;<\; \infty\,.$$

The principal amplitude is given by $a_1 = L(y, id\phi(y))^{-1}\, b(y)$.

We have followed a path leading from the method of Euler for constant coefficient ordinary differential equations to these expansions (4.2.9) of WKB type.

§4.3. Formal asymptotics approach

Once the form of the expansion (4.2.9) is known, the exact coefficients can be computed without going through the above recursion. We treat a more general situation where a sequence of smooth amplitudes b_j are given and we seek amplitudes a_j so that

$$L(y,\partial_y)\Big(e^{i\phi(y)/\epsilon}\left(\epsilon\, a_1(y)+\epsilon^2 a_2(y)+\cdots\;\right)\Big) \sim e^{i\phi(y)/\epsilon}\left(\,b_0(y)+\epsilon^1 b_1(y)+\cdots\;\right). \tag{4.3.1}$$

The previous analysis was the case where the $b_j = 0$ for $j \ge 1$. The source on the right hand side is $O(1)$ while the response in $O(\epsilon)$ as it was in the preceding sections. Computing the left side as in (4.2.4) yields,

$$e^{i\phi(y)/\epsilon}\left(L_1(y,id\phi(y))a_1(y)+\sum_{j=1}^{\infty}\epsilon^j\left(L_1(y,id\phi(y))a_{j+1}(y)+L(y,\partial_y)a_j(y)\right)\right). \tag{4.3.2}$$

The equations determining the a_j are then read off to be

$$L_1(y,id\phi(y))\,a_1(y) = b_0(y)\,, \tag{4.3.3}$$

and for $j > 1$,

$$L_1(y,id\phi(y))\,a_j(y) = -L(y,\partial_y)\,a_{j-1}(y)+b_{j-1}\,. \tag{4.3.4}$$

The relation (4.3.1) was left purposely vague to show that the formal computations are straightforward. To put meat on the bones of the formal series one has to give meaning to the sums in (4.3.1). These sums do not usually converge but represent asymptotic expansions as $\epsilon \to 0$. The interpretation is like that of Taylor expansions of smooth but not analytic functions.

Definitions. 1. *If* $\mathcal{O}$ *is an open set in* $\mathbb{R}^n$, $b_j(y)$ *is a sequence of smooth functions on* $\mathcal{O}$, *and* $b \in C^\infty(]0,1[\times\mathcal{O}\,;\,\mathbb{C}^N)$, *the asymptotic relation*

$$b(\epsilon,y) \sim b_0(y)+\epsilon\, b_1(y)+\epsilon^2\, b_2(y)+\cdots \quad \text{in} \quad C^\infty(\mathcal{O}) \tag{4.3.5}$$

means that for every integer $m \ge 0$, *every multiindex* $\alpha \in \mathbb{N}^n$, *and every compact subset* $K \subset \mathcal{O}$,

$$\sup_K\;\left|\,\partial_y^\alpha\Big(b-\sum_{j=0}^{m}\epsilon^j\, b_j(y)\Big)\,\right| = O(\epsilon^{m+1}) \tag{4.3.6}$$

as $\epsilon \to 0$. *If* $c \in C^\infty(]0,1[\times\mathcal{O})$, *the asymptotic relation*

$$b(\epsilon,y) \sim c(\epsilon,y)$$

means that $b - c \sim \sum \epsilon^j\, 0$.

2. *If $\mathcal{O}$ is a bounded open set then $b \sim \sum \epsilon^j b_j$ in $C^\infty(\overline{\mathcal{O}})$ is defined similarly by replacing the supremum over compact subsets K by the supremum over $\overline{\mathcal{O}}$.*

Remarks. 1. Instead of $\sim \sum \epsilon^j\, 0$ we write ~ 0.

2. If b is smooth on a neighborhood of $\epsilon = 0$, then Taylor's theorem shows that (4.3.5) is equivalent to

$$b_j(y) = \frac{1}{j!}\,\frac{\partial^j b(0,y)}{\partial \epsilon^j}\,.$$

The definition still leaves the interpretation of (4.3.1) up in the air since the construction went from b_j to a_j with no mention of functions of $a(\epsilon, y)$ and $b(\epsilon, y)$. The key link is Borel's Theorem.

Borel's Theorem 4.3.1. *Given a sequence b_j of smooth functions on the open set $\mathcal{O} \subset \mathbb{R}^n$ there is a smooth function $b(\epsilon, y)$ defined on $\mathbb{R} \times \mathcal{O}$ so that*

$$b(\epsilon, y) \ \sim\ b_0(y) + \epsilon b_1(y) + \epsilon^2 b_2(y) + \cdots .$$

Remarks. 1. If $\tilde b(\epsilon, y)$ is a second such function then $\tilde b \sim b$.

2. Returning to the discussion of the Definition, Borel's Theorem implies that one can choose $c(\epsilon, y) \in C^\infty(\mathbb{R} \times \mathcal{O})$ with $c \sim \sum \epsilon^j\, b_j(y)$. Then $b \sim c$ and $j!\, b_j = \partial^j c(0, y)/\partial\epsilon^j$. This shows that the smooth in ϵ case of Remark 3 is the general case.

The proof of Borel's theorem is a direct generalization of the proof of the following seemingly much more special result.

Borel's Theorem 4.3.2. *Given a sequence b_j, $0 \le j$ of complex numbers there is a smooth function $b(\epsilon)$ on $\mathbb{R}$ whose Taylor series at the origin is $\sum \epsilon^j\, b_j$.*

Proof. The idea is to set $b(\epsilon) = \sum \epsilon^j\, b_j$. However, this series has no reason to converge since the b_j may grow arbitrarily rapidly. The clever idea is to cut off the summands so that they live only where $|\epsilon|$ is so that the ϵ^j compensate the b_j.

Choose a function $\chi \in C_0^\infty(]-1, 1[)$ such that $\chi(\epsilon) = 1$ for $|\epsilon| \le 1/2$.

The summand $\epsilon^j b_j$ is replaced by $\epsilon^j\, \chi(M_j\epsilon) b_j$ where the sequence of positive numbers M_j is chosen as follows.

Set $M_0 = 1$. For $j \ge 1$ choose $M_j \ge 1$ so that for $m = 0, 1, 2, \dots, j-1$ and all $\epsilon \in \mathbb{R}$,

$$\left|\frac{d^m}{d\epsilon^m}\big(\chi(M_j\epsilon)\,\epsilon^j\, b_j\big)\right| \ \le\ \frac{1}{2^j}\,. \tag{4.3.7}$$

This is possible since when the derivatives are expanded there are a finite number of terms. Each term is a bounded function of ϵ times

$$b_j\, \epsilon^{j-l}\, M_j^k\, \frac{d^k\chi}{d\epsilon^k}(M_j\epsilon)\,, \qquad k + l = m \le j - 1\,. \tag{4.3.8}$$

In the support of the $\chi^{(k)}$ term, $\epsilon < 1/M_j$. Thus the term (4.3.8) is bounded by

$$\frac{c(\chi, j)|b_j|}{M_j^{j-k-l}} \ \le\ \frac{c(\chi, j)|b_j|}{M_j}\,,$$

so (4.3.7) can be achieved by choosing M_j sufficiently large.

Then

$$\sum \epsilon^j\,\chi(M_j\epsilon)\,b_j$$

converges uniformly with all of its derivatives to a function $b(\epsilon)$. That it satisfies the conditions of the theorem is immediately verified by differenting term by term and setting $\epsilon = 0$. ∎

Exercise 4.3.1. *Prove Theorem 4.3.1.*

Given the language of asymptotic expansions in ϵ the computations prove the following result. In the construction ellipticity was used only to ensure that $L_1(y, d\phi)$ was invertible. The next result is stated for possibly nonelliptic operators for which this is true.

Theorem 4.3.3. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set,*

$$b(\epsilon, y) \;\sim\; \sum_{j=0}^{\infty} \epsilon^j\, b_j(y)$$

is a smooth family of amplitudes on Ω, and ϕ is a smooth real valued phase such that for all $y \in \Omega$ $d\phi(y) \neq 0$ and $(y, d\phi(y)) \notin \operatorname{Char} L$. Then, there is a smooth

$$a(\epsilon, y) \;\sim\; \sum_{j=1}^{\infty} \epsilon^j\, a_j(y)$$

such that

$$L(y, \partial_y)\left(e^{i\phi(y)/\epsilon}\, a(\epsilon, y)\right) \;-\; b(\epsilon, y)e^{i\phi(y)/\epsilon} \;\sim\; 0\,.$$

The amplitude a is unique in the sense that if $\tilde a(\epsilon, y)$ is a second solution then $a(\epsilon, y) - \tilde a(\epsilon, y) \sim 0$. In particular the $a_j(y)$ are uniquely determined and the principal amplitude is given by (4.2.3).

Remark. One can take both $a(\epsilon, y)$ and $b(\epsilon, y)$ as smooth functions on $[0, \infty[\,\times\Omega$. However, neither the source $e^{i\phi(y)/\epsilon}\, b(\epsilon, y)$ nor the response $e^{i\phi(y)/\epsilon}\, a(\epsilon, y)$ is smooth up to $\epsilon = 0$. This follows from

$$\frac{d^k}{d\epsilon^k}\, e^{i\phi(y)/\epsilon}\, b(\epsilon, y) = \frac{(-1)^{k-1}(k-1)!}{\epsilon^k}\, \phi(y)\, e^{i\phi(y)/\epsilon}\, b_0(y) + O(\epsilon^{1-k})\,,$$

and a similar expression differing by a power of ϵ for the response. These derivatives diverge to infinity as $\epsilon \to 0$.

Exercise 4.3.2. *Compute two terms of an asymptotic solution of*

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial u}{\partial x_1} = e^{ix.\xi/\epsilon}\;.$$

It is interesting to ask whether the asymptotic solution $u(\epsilon, y) := a(\epsilon, y)e^{i\phi(y)/\epsilon}$ can be corrected by a term $c(\epsilon, y) \sim 0$ so that

$$L(u + c) \;=\; e^{i\phi/\epsilon}\, a\,.$$

Define the residual by

$$r(\epsilon, y) \;:=\; L(y, \partial_y)\, u(\epsilon, y) - e^{i\phi/\epsilon}\, b \;\sim\; 0\,.$$

One needs to solve

$$L(y,\partial_y)\, c(\epsilon,y) \;=\; -r(\epsilon,y)\,, \qquad \text{with} \qquad c \;\sim\; 0\,.$$

Under a variety of conditions this is possible. For example, if L is symmetric hyperbolic it suffices to supplement the equation for c with the initial condition $c|_{t=0} = 0$. A similar argument works for parabolic equations determining c in $t \geq 0$.

If L is elliptic, then inhomogeneous equations like that for c are solvable on sufficiently small neighborhoods of arbitrary points. Thus the asymptotic expansion can be locally corrected.

If L has constant coefficients, one can choose a fundamental solution E and a plateau cutoff χ and set $c := E * (\chi r)$. This works on compact subsets of space time.

However, [Levy] showed that linear partial differential equations with variable coefficients, even with polynomial coefficients, need not be locally solvable. In such cases the equation for c need not have solutions, and the construction of an asymptotic solution is the best that one can do.

§4.4. Perturbation approach

The fundamental equations, (4.3.3), (4.3.4) have now been derived two different ways, one inductive and one by plugging in the right *ansatz*. Here is a third derivation which has more the feel of perturbation theory. It is useful to have a variety of approaches for at least three reasons. First one sees that they are all versions of the same thing. In much of the mathematical and scientific literature these different computations are confused as fundamentally different things. Second, in extending these ideas sometimes one or the other point of view is more easily adaptable. Finally, different arguments appeal to different people and you can choose your favorite.

Suppose that as $\epsilon \to 0$,

$$b(\epsilon,y) \sim\; b_0(y) + \epsilon b_1(y) + \epsilon^2\, b_2(y) + \cdots\,. \tag{4.4.1}$$

Seek $u(\epsilon,y)$ solving

$$L(y,\partial_y)\, u \sim e^{i\phi(y)/\epsilon}\, b\,. \tag{4.4.2}$$

Motivated by the case of constant coefficient ordinary differential equations try

$$u = e^{i\phi(y)/\epsilon}\, a(\epsilon,y)\,. \tag{4.4.3}$$

Compute

$$L(y,\partial_y)\,\big(e^{i\phi(y)/\epsilon}\, a\big) = e^{i\phi(y)/\epsilon}\,\Big(\frac{1}{\epsilon}\, L_1(y, id\phi(y)) + L(y,\partial_y)\Big)\, a\,. \tag{4.4.4}$$

Thus (4.4.2) holds if and only if

$$\Big(\, L_1(y, id\phi(y)) + \epsilon L(y,\partial_y)\Big)\, a \sim \epsilon b\,. \tag{4.4.5}$$

If there is a solution a which has derivatives which are $O(1)$ as $\epsilon \to 0$, then there are two terms on the left, one of order 1 and the other of order ϵ. Neglecting the latter yields a first approximation which is identical to (4.2.3). There are at least two natural ways to proceed from here. One is to seek a as an asymptotic (a.k.a. Taylor series) in ϵ

$$a(\epsilon,y) \sim \epsilon\, a_1(y) + \epsilon^2\, a_2(y) + \cdots\,. \tag{4.4.5}$$

Plugging this into (4.4.5) is the method of §4.3.

An alternative is to do fixed point iteration on the equation (4.4.5) generating a sequence of approximations by solving

$$a^\nu(y) = L(y, id\phi(y))^{-1}\Big(\, b - \epsilon L(y,\partial_y)\, a^{\nu-1}\,\Big)\,. \tag{4.4.6}$$

The first approximation is found by dropping the $\epsilon\, L$ term from the right hand side to find

$$a^1 = \epsilon\, L(y, id\phi(y))^{-1}\, b\,. \tag{4.4.7}$$

The same choice is generated by choosing $a^0{=}0$. The iteration implies that

$$a^{\nu+1} - a^\nu = -\epsilon\, L(y, id\phi(y))^{-1}\, L(y,\partial_y)\,\big(a^\nu - a^{\nu-1}\,\big)\,. \tag{4.4.8}$$

This implies that

$$a^\nu = \epsilon\, a_1 + \epsilon^2\, a_2 + \cdots + \epsilon^\nu\, a_\nu + O(\epsilon^{\nu+1})\,, \tag{4.4.9}$$

with the a_j from (4.3.3), (4.3.4).

Having given three distinct approaches to solving (4.4.5) which all lead to the same answer you may have the misimpression that the solution of this equation is trivial. In fact, that is not the case. The differential operator on the left of (4.4.5) is a first order operator with the property that the differentiation terms have a coefficients of size ϵ. The derivative terms which are normally the main terms have a small coefficient and so end up playing the role of corrections. One consequence is that the successive correction terms are generated by applying operators of high order in $\epsilon\partial_y$. This is all to say that the approximation just produced is subtle, and also that convergence of the series in ϵ is unlikely except when the operators and sources satisfy real analyticity hypotheses. Such hypotheses are physically unnatural since they imply that knowledge of sources in one neighborhood determines them everywhere.

§4.5. Elliptic regularity.

A striking application of Theorem 4.3.3 is a proof of the interior ellliptic regularity theorem. The proof is modified in §4.6 to give the microlocal version which is one of the two central results in linear microlocal analysis. The other is proved in Chapter 5.

The most familiar elliptic regularity theorems assert that harmonic functions and solutions of the Cauchy Riemann equations are real analytic. More generally if Δu is real analytic then so is u. Such results extend to elliptic equations with real analytic coefficients.

We will treat sources which are smooth or only finitely differentiable. Elliptic regularity assert that if L is an $m^{\rm th}$ order elliptic operator and Lu has k deriviatives in an appropriate sense then u has $m+k$ derivatives. In dimension greater than one, it is false that if $Lu\in C^k$, then $u\in C^{k+m}$. That solutions of $\Delta u = \rho$ with ρ in the Hölder space $C^{k+\alpha}$, $\alpha\in]0,1[$ satisfy $u\in C^{2+k+\alpha}$ is a classical regularity theorem for Newtonian potentials. This Hölder version extends to general elliptic equations. The version we give is for Sobolev spaces.

Whenever there is elliptic regularity, there is a corresponding estimate. For example, if L is first order on $\overline{\Omega}$ and elliptic on $\omega\subset\subset\Omega$, Theorem 4.5.1 implies that if u and Lu belong to $H^s(\Omega)$ then $u\in H^{s+1}(\omega)$. Thus there is an inclusion

$$\Big\{u\in H^s(\Omega)\ :\ Lu\in H^s(\Omega)\Big\}\quad\to\quad H^{s+1}(\omega)\,.$$

Both sides are Hilbert spaces. The norm for the left hand side is equal to

$$\Big(\|u\|^2_{H^s(\Omega)}\ +\ \|L\,u\|^2_{H^s(\Omega)}\Big)^{1/2}\,.$$

Exercise 4.5.1. *Prove that the inclusion has closed graph so is continuous.*

Therefore, there is a constant $C = C(L, \omega, \Omega, s)$ so that

$$\|u\|_{H^{s+1}(\omega)} \ \leq \ C\left(\|Lu\|_{H^s(\Omega)} + \|u\|_{H^{s-1}(\Omega)}\right). \tag{4.5.1}$$

Such closed graph arguments showing that qualitative results implied quantitative estimates were invented in [Banach]. They show that in practice to prove regularity you must prove the estimate. In some cases, like the proof below, this is done but is not emphasized.

If $L = L_1(\partial)$ is homogeneous with constant coefficients, (4.5.1) cannot hold for nonelliptic operators. In the nonelliptic case, there would be a point $\eta \in \operatorname{Char} L$ and associated plane wave solutions $Lv^\epsilon = 0$,

$$v^\epsilon \ := \ e^{iy.\eta/\epsilon}\, a\,, \qquad a \in \ker L_1(\eta)\,.$$

The functions

$$u^\epsilon \ := \ \psi\ v^\epsilon$$

with $\psi \in C_0^\infty(\Omega)$ violate (4.5.1) in the limit $\epsilon \to 0$. This construction can be lifted to the variable coefficient case.

Exercise 4.5.2. *Show that in the variable coefficient case, if there is a point $(\underline{y}, \underline{\eta}) \in \operatorname{Char} L$ then (4.5.1) cannot be satisfied for any neighborhood ω of $\underline{y}$.* **Hint.** Use plane waves for the operator $L_1(\underline{y}, \partial)$ and a cutoff of the form $\psi\big((y - \underline{y})/\epsilon^\mu\big)$ for suitable $\mu > 0$ and $\psi \in C_0^\infty(\{|y| < 1\})$.

Definition. If $\Omega \subset \mathbb{R}^n$ is open, $\underline{y} \in \Omega$ and u is a distribution on Ω, we say that u **is in** H^s **at** $\underline{y}$ and write $u \in H^s(\underline{y})$, if and only if there is a $\psi \in C_0^\infty(\Omega)$ with $\psi(\underline{y}) \neq 0$ and $\psi u \in H^s(\Omega)$.

Elliptic Regularity Theorem 4.5.1. *Suppose that $\underline{y} \in \Omega$, $L(y, \partial_y)$ is an elliptic operator of order 1 on Ω, u is a distribution on Ω, and, $Lu \in H^s(\underline{y})$. Then $u \in H^{s+1}(\underline{y})$.*

Proof. Let

$$f \ := \ Lu$$

which is defined and H^s on a neighborhood of $\underline{y}$. Choose a smooth $\tilde\psi$, compactly supported in this neighborhood and identically equal to 1 on a smaller neighborhood of $\underline{y}$ so that L is elliptic and $f \in H^s$ on a neighborhood of the support of $\tilde\psi$. Choose a second such function, ψ supported in the set where $\tilde\psi = 1$

Denote by ω the points of the unit sphere, $S^{n-1} \subset \mathbb{R}^n$.

The strategy is to prove that $\psi u \in H^{s+1}$ by studying the Fourier transform $\widehat{\psi u}(\eta)$ for $k = |\eta| \to \infty$. Let $k := 1/\epsilon$ and $\omega := \eta/|\eta|$. Compute

$$\widehat{\psi u}(\eta) = \widehat{\psi u}\,(k\omega) = \big\langle \psi u, e^{-ik\omega.y} \big\rangle = \big\langle u\,,\ e^{-i\omega.y/\epsilon}\psi(y) \big\rangle\,.$$

The differential equation $Lu = f$ assserts that for $v \in C_0^\infty$,

$$\langle u, L^\dagger v\rangle \ = \ \langle f, v\rangle\,. \tag{4.5.2}$$

If $v(\epsilon, \omega, y)$ is a good approximate solution of $L^\dagger(y, \partial_y)\, v = \psi\, e^{-iy.\omega/\epsilon}$ then

$$\widehat{\psi\, u}(\omega/\epsilon) \ = \ \big\langle u, \psi\, e^{-ix.\omega/\epsilon} \big\rangle \ \approx \ \big\langle u, L^\dagger v \big\rangle \ = \ \big\langle f, v(\epsilon, \omega, \cdot) \big\rangle\,.$$

Since L is elliptic, every ω is noncharacteristic. The same is therefore true for the transposed operator $L(y,\partial_y)^\dagger$ since the principal symbol is $-L_1(y,\eta)^\dagger$. Thus we have constructed, for each ω, asymptotic solutions $v(\epsilon,\omega,y)$ to

$$L(y,\partial_y)^\dagger\, v(\epsilon,\omega,y) \;-\; \psi(y)e^{-iy.\omega/\epsilon} \;\sim\; 0\,. \tag{4.5.3}$$

The construction is uniform in the parameters in the sense that it yields $a_j(\omega,y)\in C^\infty(S^{n-1}\times\mathbb{R}^n)$ which vanish for y outside the support of ψ. Borel's theorem yields

$$C^\infty([0,1]\times S^{n-1}\times\mathbb{R}^n)\ni a(\epsilon,\omega,y) \;\sim\; \sum_{j=0}^{\infty}\epsilon^j\, a_j(\omega,y) \quad\text{in}\quad C^\infty(S^{n-1}\times\mathbb{R}^n)\,, \tag{4.5.4}$$

with a vanishing for y outside the support of ψ.
Setting

$$v(\epsilon,\omega,y) \;:=\; \epsilon\, a(\epsilon,\omega,y)\, e^{iy.\omega/\epsilon}\,, \tag{4.5.5}$$

(4.5.3) holds in $C^\infty(S^{n-1}\times\mathbb{R}^n)$. Then

$$\widehat{\psi u}(\eta) \;=\; \big\langle u\,,\, L^\dagger v\big\rangle + \langle u\,,\, \psi e^{ix.\omega/\epsilon} - L^\dagger v\rangle, \tag{4.5.6}$$

and

$$\forall M,\quad \langle u, \psi e^{ix.\omega/\epsilon} - L^\dagger v\rangle = O(|\eta|^{-M})\,.$$

The proof is completed by showing that $\langle\eta\rangle^{s+1}\widehat{\psi u}(\eta)\in L^2$ by showing that $\langle\eta\rangle^{s+1}$ times each of the summands on the right of (4.5.6) belongs to $L^2(\mathbb{R}^n_\eta)$.
For the second summand it suffices to choose M with $2M-2(s+1)>n$ so that,

$$\int_{|\eta|>1}\frac{<\eta>^{2(s+1)}}{|\eta|^{2M}}\,d\eta \;<\;\infty\,.$$

The approximate solution v vanishes for y oustide the support of ψ. Therefore, $\widetilde{\psi}\,v=v$. This with (4.5.2) shows that $\big\langle u\,,\,L^\dagger v\big\rangle \;=\; \big\langle\tilde\psi f\,,\,v\big\rangle$. Formula (4.5.5) shows that the right hand side is equal to

$$\int\tilde\psi\;f\;a\;e^{-iy.\omega/\epsilon}\;dy \;=\; \epsilon\,\mathcal{F}\Big(a(\epsilon,\omega,\cdot)\,\tilde\psi(\cdot)\,f(\cdot)\Big)(\eta)\,,\qquad \eta=\omega/\epsilon\,. \tag{4.5.7}$$

Expressing the Fourier transform of a product as a convolution yields

$$\big\langle u\,,\,L^\dagger v\big\rangle \;=\; \epsilon\, c\int\mathcal{F}(\tilde\psi f)(\zeta)\;\hat a(\epsilon,\omega,\zeta-\eta)\;d\zeta\,. \tag{4.5.8}$$

The smoothness and compact support of a implies that,

$$\forall N,\;\exists\, C_N,\;\forall\,(\epsilon,\omega)\in[0,1]\times S^{n-1},\qquad \big|\hat a(\epsilon,\omega,\zeta)\big| \;\le\; C_N\,\langle\zeta\rangle^{-N}. \tag{4.5.9}$$

Since $\tilde\psi f\in H^s$,

$$\big|\mathcal{F}\big(\tilde\psi f\big)(\zeta)\big| \;=\; \langle\zeta\rangle^{-s}\;g\,,\qquad\text{with}\quad g(\zeta)\in L^2(\mathbb{R}^n)\,. \tag{4.5.10}$$

Estimates (4.5.8-4.5.10) imply that for $|\eta| > 1$

$$\Big|\langle u\,,\,L^\dagger v\rangle\Big| \;\le\; \int \frac{C\,g(\zeta)}{\langle\zeta\rangle^s\;|\eta|\;\langle\zeta-\eta\rangle^N}\;d\zeta\,,$$

where the key is the factor $\epsilon = |\eta|^{-1}$. Multiplying by $\langle\eta\rangle^{s+1}$ taking advantage of the fact that $|\eta| \ge 1$ yields the bound

$$\langle\eta\rangle^{s+1}\;\Big|\langle u\,,\,L^\dagger v\rangle\Big| \;\le\; C\int \frac{\langle\eta\rangle^s}{\langle\zeta\rangle^s\;\langle\zeta-\eta\rangle^N}\;g(\zeta)\;d\zeta\,. \tag{4.5.11}$$

Suppose that $s \ge 0$. Since $(\zeta-\eta)+\zeta = \eta$ either $|\zeta-\eta| > |\eta|/2$ or $|\zeta| > |\eta/2|$ so the integrand is bounded by

$$\frac{\langle\eta\rangle^s}{\langle\zeta\rangle^s\;\langle\zeta-\eta\rangle^s}\;\frac{g(\zeta)}{\langle\zeta-\eta\rangle^{N-s}} \;\le\; C\;\frac{g(\zeta)}{\langle\zeta-\eta\rangle^{N-s}}\,.$$

Choose $N > n+s$ so $\langle\zeta\rangle^{-N+s} \in L^1(\mathbb{R}^n_\zeta)$. Young's inequality implies that

$$\Big\|\langle\eta\rangle^{s+1}\;\langle u\,,\,L^\dagger v\rangle\Big\|_{L^2(\mathbb{R}^n_\eta)} \;\le\; C\,\big\|\langle\zeta\rangle^{N-s} * g\big\|_{L^2(\mathbb{R}^n_\zeta)} \;\le\; C\,\|\langle\zeta\rangle^{-N+s}\|_{L^1(\mathbb{R}^n_\zeta)}\;\|g\|_{L^2(\mathbb{R}^n)} \;<\; \infty\,.$$

This completes the proof when $s \ge 0$.

When $s < 0$ the integrand is bounded by

$$\frac{g(\zeta)\,\langle\zeta\rangle^{|s|}}{\langle\eta\rangle^{|s|}\,\langle\zeta-\eta\rangle^N} \;\le\; \frac{\langle\zeta\rangle^{|s|}}{\langle\eta\rangle^{|s|}\;\langle\zeta-\eta\rangle^{|s|}}\;\frac{g(\zeta)}{\langle\zeta-\eta\rangle^{N-|s|}} \;\le\; C\;\frac{g(\zeta)}{\langle\zeta-\eta\rangle^{N-|s|}}\,,$$

since either $|\eta| \ge |\zeta|/2$ or $|\zeta-\eta| \ge |\zeta|/2$. Young's inequality completes the proof. ∎

Remarks. 1. In the heart of the proof the gain of one derivative came from the factor ϵ in the approximate solution. In a fundamental sense elliptic regularity is a reflection of the fact that a right hand side oscillating with wavelength ϵ yields a response oscillating in the same way whose amplitude is smaller by a factor ϵ.

2. A standard proof of the regularity theorem is to construct a pseudodifferential operator $P(y,\partial)$ of order -1 so that $LP = I + \text{Smoothing}$. Then $P\left(b(y)e^{i\omega/\epsilon}\right)$ is an infinitely accurate approximate solution of $Lu = b(y)\,e^{iy.\omega/\epsilon}$. The computation of the full symbol of P is the same analytic problem as computing the full asymptotic expansion $a(\epsilon, y)\,e^{iy.\omega/\epsilon}$. From this perspective the computation in §4.2 resembles the Levi construction of elliptic parametrices, while the computations in §4.3 and §4.4 resemble more closely the calculations using the symbol calculus for pseudodifferential operators.

3. The heart of the proof is an explicit formula for $\widehat{\psi u}(\eta)$ with error bound $O(|\eta|^{-\infty})$. This is an impressive achievment for a variable coefficient problem.

Corollary 4.5.2. *If $L(y,\partial_y)$ is elliptic on the open set Ω and $u \in \mathcal{D}'(\Omega)$ satisfies $Lu \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.*

Proof. If $y \in \Omega$ then u is $H^s(y)$ for some possibly very negative s. The Theorem implies that $u \in H^{s+1}(y)$. A second application implies that $u \in H^{s+2}(y)$. An induction shows that u belongs to $H^{s+m}(y)$ for all integers m. Sobolev's embedding theorem implies that for every k, $u \in C^k$ on a neighborhood of y. ∎

§4.6. The Microlocal Elliptic Regularity Theorem

This is one of the two basic theorems in microlocal analysis. What needs to be added to the ideas of the last section are two fundamental definitions.

Definition. *If u is a distribution defined on an open set Ω, $\underline{y} \in \Omega$, $s \in \mathbb{R}$ and $\underline{\eta} \in \mathbb{R}^n \setminus 0$, we say that u* **is in H^s microlocally at** $(\underline{y}, \underline{\eta})$ *and write $u \in H^s(\underline{y}, \underline{\eta})$ if and only if there is a $\psi \in C_0^\infty(\Omega)$ with $\psi(\underline{y}) \neq 0$ and a $\chi \in C^\infty(\mathbb{R}^n \setminus 0)$ homogeneous of degree 0 with $\chi(\underline{\eta}) \neq 0$ so that*

$$\int_\Gamma \chi(\eta) \, \big|\widehat{\psi u}(\eta)\big|^2 \, \langle \eta \rangle^{2s} \, d\eta \; < \; \infty \,. \tag{4.6.1}$$

The set of (y, η) so that $u \notin H^s(y, \eta)$ is called the H^s **wave front set** *and is denoted $WF_s(u)$.*

Examples. i. If $u = \delta(x)$ then for $s \geq -d/2$ then $WF_s(\delta) = \{(0, \eta) \; : \; \eta \neq 0\}$. For $s < -d/2$ $u \in H^s$ and WF_s is empty.

ii. Consider $u(y_1, y_2) = h(y_2)$ with h denoting Heaviside's function. The singular support of u is equal to $\{y_2 = 0\}$. Thus if $y_2 \neq 0$, $u \in H^s(y, \eta)$ for all s, η.

For points with $y_2 = 0$, take a cutoff function $\psi = \phi_1(y_1)\phi_2(y_2)$ so

$$\widehat{\psi u}(\eta_1, \eta_2) \;=\; \widehat{\phi}_1(\eta_1) \; \mathcal{F}(\phi_2(y_2) \, h(y_2)) \,.$$

Since $\widehat{\phi}_1(\eta_1)$ is rapidly decreasing, this proves that for any η with $\eta_1 \neq 0$, $u \in H^\infty(y, \eta)$, that is in $H^s(y, \eta)$ for all s.

There remain the points with $y_2 = 0$ and $\eta_1 = 0$. Use $|\mathcal{F}(\phi_2 \, h)(\eta_2)| \sim c/|\eta_2|$. On a conic neighborhood one has

$$\widehat{\psi u}(\eta_1, \eta_2) \;\leq\; \frac{|\hat{\phi}_1(\eta_1)|}{\langle \eta_2 \rangle}.$$

Using Fubini's Theorem shows that u is microlocally $H^s(y, \eta)$ if and only if $s < 1/2$.

Exercise 4.6.1. *Prove that $WF_{s-1}(\partial_j u) \subset WF_s(u)$.*

Exercise 4.6.2. *Prove that if $u \in H^s(\underline{y}, \underline{\eta})$ and $\psi \in C^\infty(\Omega)$ then $\psi u \in H^s(\underline{y}, \underline{\eta})$.* **Hints.** After a cutoff $\psi \in C_0^\infty(\Omega)$ and $u = u_1 + u_2$. $\widehat{u_1} = O(\langle \xi \rangle^M)$ and vanishes on a conic neighborhood of $\underline{\eta}$ while $u_2 \in H^s(\mathbb{R}^d)$. Write the transform of the product as a convolution. Consider only η in a small conic neighborhood of $\underline{\eta}$. For $\widehat{\psi} * \widehat{u}_1$ show that the argument of $\widehat{\psi}$ is $\geq c|\eta|$.

Microlocal Elliptic Regularity Theorem 4.6.1. *Suppose that on a neighborhood of $\underline{y}$, $L(y, \partial_y)$ is a system of differential operators of order 1 with smooth coefficients and that L is noncharacteristic at $(\underline{y}, \underline{\eta})$. If u is a distribution with $Lu \in H^s(\underline{y}, \underline{\eta})$, Then $u \in H^{s+1}(\underline{y}, \underline{\eta})$.*

Proof. The proof follows the proof of Theorem 4.5.1. The change comes in the construction of the approximate solution (4.5.3). In the elliptic case this was done for all ω using the fact that $L_1(y, \omega)$ is invertible for all (y, ω).

In the present context we choose the cutoffs ψ and χ with sufficiently small support so that that (y, η) is noncharacteristic on $\operatorname{supp} \psi \times \operatorname{supp} \chi$.

Then $v(\epsilon, y, \eta)$ is defined for $(y, \eta) \in \operatorname{supp} \psi \times \operatorname{supp} \chi$. Estimates are uniform on the corresponding compact set of (y, ω).

The estimate for the integral (4.5.8) must be changed because it is no longer assumed that $\widetilde{\psi} f \in H^s$. What is true is that one can choose $\widetilde{\psi}$ so that $\mathcal{F}(\widetilde{\psi} f)$ is the sum of two terms. One is as in the right hand side of (4.5.10) and is treated as before. The second term, $h(\zeta)$, vanishes on a conic neighborhood of $\underline{\eta}$ and for some σ, possibly very large,

$$\frac{h(\zeta)}{\langle\zeta\rangle^\sigma} \in L^2(\mathbb{R}^n_\zeta)\,.$$

For this h part of (4.5.8) we must show that

$$\int \frac{h(\zeta)\ \langle\eta\rangle^s}{\langle\zeta\rangle^s\ \langle\zeta-\eta\rangle^N}\ d\zeta$$

is square integrable on a small closed conic neighborhood of $\underline{\eta}$. Choosing that neighborhood small enough it will be disjoint from a closed cone containing the support of h. Then, in the support of the integrand, $|\zeta-\eta| \ge C\max\{|\zeta|,|\eta|\}$. For $s \ge 0$ and ζ in the support of the integrand,

$$\frac{h(\zeta)\ \langle\eta\rangle^s}{\langle\zeta\rangle^s\ \langle\zeta-\eta\rangle^N} \le \frac{h(\zeta)}{\langle\zeta\rangle^\sigma}\ \frac{C}{\langle\zeta-\eta\rangle^{N-\sigma-s}}\,.$$

Choosing $N > \sigma + s + n$, Young's inequality finishes the proof.

For $s < 0$, the integrand is dominated by

$$\frac{h(\zeta)\ \langle\zeta\rangle^{|s|}}{\langle\eta\rangle^{|s|}\ \langle\zeta-\eta\rangle^N} \le \frac{h(\zeta)}{\langle\zeta\rangle^\sigma}\ \frac{C}{\langle\zeta-\eta\rangle^{N-\sigma-|s|}}\,,$$

and choosing $N > \sigma + |s| + n$, Young's inequality completes the proof. ∎

$WF_s(u)$ is a closed conic subset of $\Omega\times\mathbb{R}^n\setminus 0$. Theorem 4.6.1 implies that if $(y,\eta)\notin \mathrm{Char}L$ then

$$(y,\eta)\in WF_{s+1}u \qquad\Longleftrightarrow\qquad (y,\eta)\in WF_s(Lu)\,.$$

The theorem yields the $\Rightarrow$ part. The $\Leftarrow$ implication is not hard to prove using Exercise 4.6.1.

A candidate for a microlocal smoothness set is the set of points (y,η) such that $u\in H^s(y,\eta)$ for all s. This is the complement of $\cup_s WF_s(u)$. There is a stronger notion which leads to a possibly smaller set.

Definition. *If u is a distribution on Ω the* **wavefront set** *of u, denoted $WF(u)$ is a set of points $(y,\eta)\in\Omega\times\mathbb{R}^n\setminus 0$ so that $(y,\eta)\notin WF(u)$ if and only if there is a $\psi\in C_0^\infty(\Omega)$ with $\psi(\underline{y})\ne 0$ and a $\chi\in C^\infty(\mathbb{R}^n\setminus 0)$ homogeneous of degree 0 with $\chi(\underline{\eta})\ne 0$ so that for all $N\in\mathbb{Z}_+$,*

$$\sup_{\eta\in\mathbb{R}^n}\ |\eta|^N\,\mathcal{F}(\chi\,u)(\eta)\ <\ \infty\,.$$

Exercise 4.6.3. *Prove that if $\psi\in C^\infty(\Omega)$ then $WF(\psi u)\subset WF(u)$.* **Hint.** Read the hint in exercise 4.6.2.

Exercise 4.6.4. *Prove that $WF(\partial^\alpha u)\subset WF(u)$.* **Discussion.** Combining with exercise 4.6.3 proves that $WF(p(x,\partial)u)\subset WF(u)$ for partial differential operators p with smooth coefficients. Operators which do not increase wavefront sets are called **microlocal**.

Example. Define $u = \mathcal{F}^{-1}k$ where $k(\eta) \in C^\infty(\mathbb{R}^n)$ is homogeneous of degree σ on $|\eta| \geq 1$. That is, there is a homogeneous function K so that $k = K$ on $|\eta| \geq 1$. Combining the results of the next three exercises shows that $WFu \;=\; \{0\} \times \operatorname{supp} K$.

Exercise 4.6.5 *Prove that the singular support of u is equal to the origin by showing that for any k there is an N so that $|y|^{2N}\, u \in C^k$.*

Exercise 4.6.6. *Show that for any η disjoint from the support of K, $(0, \eta) \notin WF(u)$.* **Hint.** Estimate products as in the proof of Theorem 4.6.1.

Exercise 4.6.7. *If $\psi \in C_0^\infty(\mathbb{R}^2)$ with $\psi(0) = 1$, then*

$$\mathcal{F}(\psi u) - K \;=\; O(|\eta|^{\sigma-1})$$

as $\eta \to \infty$. **Hint.** Write the transform of the product as a convolution. Then take advantage of the fact that $\widehat{\psi}$ is rapidly decreasing.

Microlocal Elliptic Regularity Theorem 4.6.2. *If L is as in Theorem 4.6.1, $(y, \eta) \notin \operatorname{Char} L$, and, $(y, \eta) \notin WF(Lu)$, then $(y, \eta) \notin WF(u)$.*

Exercise 4.6.8. *Prove this by suitably modifying the proof of Theorem 4.6.1.*

Next analyse the wavefront set of a piecewise smooth function by using two methods.

Definition. *If M is a embedded hypersurface, $m \in M$ and u is a distribution defined on a neighborhood of M, then u is* **piecewise smooth** *at m if the restriction of u to each side of M has a smooth extension to a neighborhood of m.*

Exercise 4.6.9. *If u is piecewise smooth at m prove using the definition of WF that on a neighborhood of m, $WF(u)$ is contained in the conormal bundle of M.* **Hint.** By subtracting a function smooth near m reduce to the case of u vanishing on one side. Estimate $\mathcal{F}(\psi u)$ in directions not conormal by the method of nonstationary phase. Compute the boundary terms that appear in the integration by parts. Show that they are small by the method of nonstationary phase.

Remark. For the same u one can prove the slightly weaker result that for all s, $WF_s(u)$ is contained in the conormal variety using only the microlocal! elliptic regularity theorem. If η is not conormal, at m one can find a smooth vector field $V(y, \partial)$ on a neighborhood of m so that V is tangent to M and $V(m, \eta) \neq 0$. Then for any k, $V^k u \in L^2_{\rm loc}$. Microlocal elliptic regularity applied for the operator V implies by induction that $V^{k-j}u \in H^j(m, \eta)$ for $0 \leq j \leq k$.

Chapter 5. Linear Hyperbolic Geometric Optics

§5.1. Introduction

The mathematical subject of geometric optics is devoted to the construction and analysis of asymptotic solutions of partial differential equations which are accurate when wavelengths are small compared to other natural lengths in the problem. Since the wavelength of visible light is of the order of magnitude 5×10^{-5}cm a great deal of what one sees falls into this category. The description of optical phenomena was and is a great impetus to study short wavelength problems.

The key feature of geometric optics, propagation along rays, is not present in the elliptic case of the last section. Rays occur for hyperbolic problems, and in the same spirit, for singular elliptic problems which arise when looking for high frequency time periodic solutions of a hyperbolic equation. An example of the latter is that solutions of the form

$$u(t,x) = e^{i\tau t}\, w(x)\,, \qquad \tau >> 1 \tag{5.1.1}$$

to D'Alembert's equation

$$\Box u = 0 \tag{5.1.2}$$

must satisfy

$$\tau^2\, w + \Delta\, w = 0\,. \tag{5.1.3}$$

The singularity in this elliptic equation is that one is interested in solutions in the limit $\tau\to\infty$ and therefore a coefficient is tending to infinity.

The key dichotomy is that in the (nonsingular) elliptic case, rapid oscillations have only local effects. The values of the coefficients a_j in the neighborhood of a point are determined by the values of the b_j in the same neighborhood. For hyperbolic equations (and their related singular elliptic problems like (5.1.3)) the oscillations may and usually do propagate to distant parts of space time. This is why they are the main carriers of information in both the communications industry and in the universe.

The starting point for the construction of the asymptotic solutions of geometric optics is the observation that if $L = L_1(\partial_y)$ has constant coefficients and no lower order terms then there are plane waves

$$u(y) \ :=\ a(y.\eta)\,, \qquad y.\eta \ :=\ \sum_{\mu=0}^{d} y_\mu \eta_\mu\,.$$

In §2.4 we showed that u satisfies $L\,u=0$ when $\eta\in \mathrm{Char}\, L$ and

$$\forall \sigma\,, \quad a'(\sigma)\in \ker L_1(\eta)\,. \tag{5.1.4}$$

For $\eta\in \mathrm{Char}L$, the plane wave solutions (modulo constants) are parameterized by functions $a:\mathbb{R}\to \ker L(\eta)$. With the notation

$$\eta \ =\ (\eta_0,\eta_1\cdots,\eta_d) \ :=\ (\tau,\xi_1,\cdots,\xi_d)\,, \tag{5.1.5}$$

plane wave solutions have the form

$$u = a(\tau t + x.\xi). \tag{5.1.6}$$

They translates at velocity $\mathbf{v}$ in the sense that $u(t,x) = u(0,x-\mathbf{v}t)$ if and only if $\mathbf{v}$ satisfies $\mathbf{v}.\xi = -\tau$.

The early sections of this chapter construct a generalization of the plane wave solutions which can be localized in space and apply to systems with variable coefficients and lower order terms.

Among the plane wave solutions described above are those with short wavelength sinusoidal oscillatory behavior,

$$u \;=\; e^{iy.\eta/\epsilon}\, a(y.\eta), \tag{5.1.7}$$

where a is smooth and ϵ is small compared to the typical distances on which a varies. Equation (5.1.7) represents wave packets with wavelength $\epsilon << 1$. They have derivatives of order $1/\epsilon >> 1$. A zero order term applied to (5.1.7) is bounded so much smaller. For variable coefficient problems, the coefficients do not vary much on the scale ϵ of a wavelength. This suggests that at least locally, replacing an operator by its constant coefficient part without lower order terms is a reasonable approximation for high frequency solutions. This turns out to be not quite true since in the solutions the large contributions of the highest order terms nearly cancel and the lower order terms can be important as in the striking example of §1.4. In addition, the derivatives of the coefficients affect the local propagation. Only a detailed computation reveals the exact laws.

The model problem of §1.2 is a first such calculation based on the Fourier Transform. In the next section we treat the case of scalar second order equations with constant coefficients. They are interesting in their own right. The scalar results are algebraically a little more straight forward than the systems treated afterward. Thus they are a natural starting point. For many applications, the systems are essential. An impatient or experienced reader can skip directly to §5.3.

§5.2. Second order scalar constant coefficient principal part

Begin with the example of a second order scalar constant coefficient operator which may not be hyperbolic,

$$L \;:=\; \sum_{\mu,\nu=0}^{d} a_{\mu\nu}\partial_\mu\partial_\nu \;+\; \sum_{\mu=0}^{d} b_\mu\partial_\mu \;+\; c\,, \qquad a_{\mu\nu} \;=\; a_{\nu\mu}\,. \tag{5.2.1}$$

Suppose that the principal coefficients $a_{\mu\nu}$ are real and do not depend on y. Otherwise linear phases would be unrealistic, since for variable coefficient operators surfaces of constant phase are unlikely to be planar.

In §1.2 we found short wavelength solutions of the form,

$$e^{iy.\eta/\epsilon}\, a(\epsilon, y)\,, \qquad a(\epsilon,y) \;\sim\; \sum_j \; a_j(y)\,\epsilon^j\,.$$

Similar expansions were successful in Chapter 4. Seeking solutions of that form, compute $L\left(e^{iy.\eta/\epsilon}\, a\right)$. The most singular terms occur when both derivatives fall on the exponent yielding

$$\frac{1}{\epsilon^2}\; \sum a_{\mu\nu}\; (i\eta_\mu)(i\eta_\nu)\; a(\epsilon,y)\,.$$

In order to find solutions one must have

$$\sum a_{\mu\nu}\, \eta_\mu\eta_\nu = 0\,. \tag{5.2.2}$$

This asserts that $\eta \in \operatorname{Char} L$. Recall that for an m^{th} order differential operator, the characteristic variety is defined by the equation $\det L_m(y,\eta) = 0$. The equation (5.2.2) has a solution $0 \neq \eta \in \mathbb{R}^{1+d}$ if and only if L is not elliptic.

Given (5.2.2), there are terms of order ϵ^{-1} when one derivative hits the exponent and terms of order ϵ^0 when all the derivatives fall on a,

$$L(\partial)\left(e^{iy.\eta/\epsilon}\, a(\epsilon,y)\right) = e^{iy.\eta/\epsilon}\left(i\Big(2\sum_{\mu,\nu} a_{\mu,\nu}\,\eta_\nu\,\partial_\mu + \sum_\mu b_\mu\eta_\mu\Big)/\epsilon + L(\partial)\right)a\,.$$

Injecting the expansion of $a(\epsilon,y)$ and setting the coefficient of ϵ^{j-1} equal to zero yields the recurrence,

$$i\Big(2\sum_{\mu,\nu} a_{\mu\nu}\eta_\nu\partial_\mu + b.\eta\Big)a_j + L(\partial)\,a_{j-1} = 0\,, \tag{5.2.3}$$

with the convention that $a_{-1}=0$. Define the vector field V and scalar c by

$$V \;:=\; \sum a_{\mu\nu}\,\eta_\nu\,\partial_\mu, \qquad \gamma \;:=\; \sum_\mu b_\mu\eta_\mu\,. \tag{5.2.4}$$

The leading amplitude $a_0(y)$ must satisfy the transport equation

$$\big(V+\gamma\big)\,a_0 = 0\,. \tag{5.2.5}$$

The integral curves of V are straight lines called **rays**. Equation (5.2.5) shows that the restriction of a_0 to a ray satisfies a first order linear ordinary differential equation whose solutions are exponentials. If $\gamma=0$, then a_0 is constant on rays. The eikonal equation (5.2.2) shows that the phase $\phi(y)=y.\eta$ satisfies $V\phi=0$ so ϕ is constant on rays.

Example of phases and rays. For the operator $\partial_t^2 - c^2\Delta$, and $\eta=(\tau,\xi)$ with $\tau=\pm|c\xi|$ the phase is given by,

$$\phi(y)=\phi(t,x) \;=\; y.\eta \;=\; (t,x).(\tau,\xi) \;=\; \tau t + x.\xi\,.$$

The vector field V is given by,

$$V = \phi_t\partial_t - c^2\nabla_x\phi.\nabla_x = \tau\,\partial_t - c^2\,\xi.\nabla_x = \pm|c\xi|\,\Big(\partial_t \mp c\,\frac{\xi}{|\xi|}\,.\,\nabla_x\Big).$$

The integral curves of V move with speed c in the direction $\mp\xi/|\xi|$. This is equal to the group velocity associated to (τ,ξ) in §2.4. In the present computation there is no hint of the stationary phase argument usually used to introduce the group velocity as in §1.3.

Initial conditions for a_0 can be prescribed, for example, on a hyperplane P transverse to V. The other a_j can be similarly determined from their values on P. For $j>0$ the transport equation for a_j has a source term depending on a_{j-1}.

Proposition 5.2.1. *Suppose that η is characteristic, P is a hyperplane transverse to V in (5.2.4), and $f_j\in C^\infty(P)$ for $j\ge 0$. Then there are uniquely determined a_j with $a_j\big|_P = f_j$ so that (5.2.5) is satisfied. In this case $L\big(e^{iy.\eta/\epsilon}\,a\big)\sim 0$.*

In sharp contrast to the elliptic case, there are oscillatory asymptotic solutions without oscillatory source terms in the equation. To leading approximation, the oscillations propagate with velocity given by the vector field V.

The vector field V has integral curves which move with velocity $\mathbf{v} := (a_1, \ldots, a_d)/a_0$. The eikonal equation implies that this velocity satisfies $\mathbf{v}.\xi = -\tau$, the condition defining the phase velocities. Usually (as in exercise 5.2.2 below), $\mathbf{v}$ is not parallel to ξ .

Exercise 5.2.1. *If the principal symbol* $\sum a_{\mu\nu}\eta_\mu\eta_\nu$ *is a nondegenerate quadratic form on* $\mathbb{R}^{d+1}_\eta$*, it defines an isomorphism from* $\mathbb{R}^{d+1}_\eta$ *to* $\mathbb{R}^{d+1}_y$ *by* $\eta \mapsto y$ *defined by*

$$\forall \zeta, \quad a(\eta, \zeta) \;=\; y.\zeta\,.$$

Prove that the vector V *is the image by this isomorphism of the covector* $\eta \in \operatorname{Char} L$.

Exercise 5.2.2. *Suppose that* $L_2 = \partial_t^2 - c^2\partial_1^2 - \partial_2^2$. *Compute the characteristic variety and the velocity of transport for every* $\eta \in \operatorname{Char} L$. **Discussion.** For $c \neq 1$ the velocity of transport is not parallel to ξ except when $\xi_1\xi_2 = 0$.

Definitions. *The operator (5.2.1) is* **strictly hyperbolic with time variable** t *if and only if* $L_2(1, 0, \ldots, 0) \neq 0$ *and for every* $\xi \in \mathbb{R}^d \setminus 0$, *the equation* $L_2(\tau, \xi) = 0$ *has two distinct real roots. The* **time like cone** *is the connected component in* $\{\eta : L_2(\eta) \neq 0\}$ *containing* $(1, 0, \ldots, 0)$. *A hyperplane is* **space like** *if it has a conormal belonging to the time like cone.*

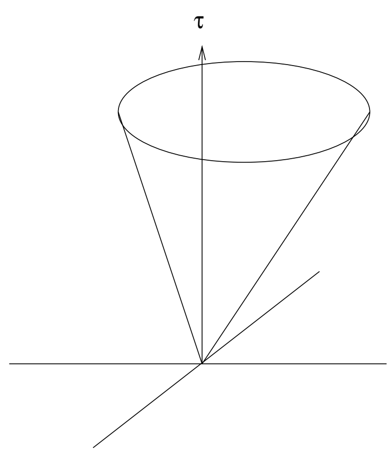


Figure 5.2.1. The time like cone.

Exercise 5.2.3. For what real values γ is the operator $\partial_t^2 + \gamma\partial_t\partial_x - \partial_x^2$ strictly hyperbolic with time t?

Exercise. 5.2.4. *Show that* $L = \partial_t^2 - \sum_{ij} a_{ij}\partial_i\partial_j$ *is strictly hyperbolic with time* t *if and only if* $\sum a_{ij}\xi_i\xi_j$ *is a strictly positive definite symmetric bilinear form.*

Exercise 5.2.5. *Show that the set of operators* L *of form (5.2.1) with real symmetric* $a_{\mu\nu}$ *that are strictly hyperbolic with time* t*, is open.*

Exercise 5.2.6. *Carry out the construction of asymptotic solutions of the Klein-Gordon operator* $L = \partial_t^2 - \sum_{ij} a_{ij}\partial_i\partial_j + m^2$, *with* a_{ij} *real and strictly positive. In particular show that the equations for the leading amplitude do not involve* m. **Discussion.** The propagation of short wavelength oscillations is to leading order unaffected by the m^2u term. This is consistent with the idea that for highly oscillatory solutions, lower order terms are less important. Note however that the first order terms $b.\partial_y$ do affect the leading amplitude a_0. These results are closely related to the fact (see §1.2) that the jump discontinuities in the fundamental solution of $u_{tt} - u_{xx} + m^2u = 0$ are not affected by m.

Exercise 5.2.7. *Prove that if L is strictly hyperbolic with time t, H is a space like hyperplane, and η is characteristic, then the transport vector field is transverse to H.*

The structure of the leading approximation $e^{iy\eta/\epsilon} a_0(y)$ is visualized as follows. The surfaces of constant phase are the hyperplanes $y.\eta = \text{const}$. The eikonal equation (5.2.2) implies that the transport vector field V is tangent to these surfaces. In the absence of lower order terms, the leading amplitude a_0 is constant on the rays which are integral curves of the constant vector field V. In the general case, the restriction of a to a ray is an exponential function. If $a_0\big|_{t=0}$ is supported in a set $E \subset \mathbb{R}^d_x$ then a_0 is supported in the tube of rays

$$\mathcal{T} \;=\; \Big\{(0,x) + tV \;:\; x \in E,\; t \in \mathbb{R}\Big\}\,.$$

The velocity associated to V is called the **group velocity**. From the definition of V one has

$$\text{group velocity} \;:=\; -\frac{\sum_{j=1}^{d} \sum_\mu \, a_{j\mu}\, \eta_\mu \, \partial_j}{\sum_\mu a_{0\mu}\eta_\mu}$$

Exercise 5.2.8. *Show that in the strictly hyperbolic case, the characteristic variety is parameterized as a graph $\tau = \tau(\xi)$, and the group velocity is equal to $-\nabla_\xi \tau$.* **Discussion.** This is the same formula for group velocity found in §2.4.

Examples of raylike solutions. For $\partial_t^2 - c^2\Delta_x$, and $\eta = (\tau, \xi)$ with $\tau = \pm|c\xi|$, ff $a_0(0,x) = f(x)$ then $a_0(t,x) = f(x \mp ct\xi/|\xi|)$. A particularly interesting case is when f has support in a small ball centered at a point $\underline{x}$. In this case the principal term in the approximate solution is supported in a cylinder in space time about the ray starting at $\underline{x}$. If one takes initial data $a_j(0,x) = 0$ for $j \ge 1$, then Theorem 5.2.1 yields approximate solutions u^ϵ which are supported in such a narrow cylinder and whose residual is infinitely small in ϵ.

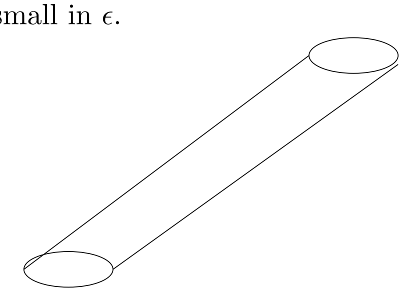


Figure 5.2.2 The cylindrical support of a ray like solution.

This recovers in a different way, the approximate solutions derived by Fourier transform in §1.4.

The reader is encouraged to think of these as flashlight beams or raylike approximate solutions to the wave equation. It is worth noting that the only exact solution of the multidimensional wave equation which is supported in such a cylinder is the solution $u = 0$.

Exercise 5.2.9. *Prove that if $d > 1$ and u is a smooth solution of $\Box u = 0$ supported in a tube*

$$\Big\{(t,x) \;:\; \big|(x_1 - ct, x_2, \ldots, x_d)\big| \le R\Big\}\,,$$

then $u = 0$. **Hint.** Prove that $\|\nabla_{t,x} u(t)\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $t \to \infty$ (actually $= O\big(t^{(1-d)/2)}\big)$). This together with the support condition shows that the energy is $o(1)$ as $t \to \infty$. Conservation of energy implies $u = 0$. **Alternate hint.** The boundary of the tube is characteristic where $x_2 = x_3 = \ldots = x_d = 0$. It is noncharacteristic at the other points. Apply the Global Hölmgren Uniqueness Theorem to prove that $u = 0$ vanishes in the cylinder.

§5.3. Symmetric hyperbolic systems

Convention. *From here on, the underlying operator $L(y, \partial_y)$ is assumed to be a symmetric hyperbolic first order system and $y = (t, x)$.*

With the background of the elliptic case from Chapter 4, and the scalar hyperbolic examples in §5.2, it is natural to seek solutions,

$$L(y, \partial_y) \big(a(\epsilon, y) \, e^{i\phi(y)/\epsilon} \big) \ \sim \ 0 \,. \tag{5.3.1}$$

with vector valued

$$a(\epsilon, y) \ \sim \ \sum_{j=0}^{\infty} \epsilon^j \, a_j(y) \qquad \text{as} \ \ \epsilon \to 0 \,. \tag{5.3.2}$$

Here ϕ is assumed to be a smooth real valued function with $d\phi \neq 0$ on the domain of interest. This guarantees that the solutions are rapidly oscillating as $\epsilon \to 0$. The surfaces of constant phase in space time have conormal vectors equal to $d\phi$.

Computing as in the derivation of (4.3.2) yields

$$\begin{aligned} L(y, \partial_y) \Big(e^{i\phi(y)/\epsilon} \, a \Big) \ &\sim \ e^{i\phi(y)/\epsilon} \left(\frac{1}{\epsilon} \, L_1(y, id\phi(y)) \, a \ + \ L(y, \partial_y) a \right) \\ &\sim e^{i\phi(y)/\epsilon} \left(\frac{1}{\epsilon} \, L_1(y, id\phi(y)) \, a_0 + \sum_{j=0}^{\infty} \epsilon^j \, \big[L_1(y, id\phi(y)) \, a_{j+1}(y) + L(y, \partial_y) \, a_j(y) \big] \right). \end{aligned}$$

The convention $a_{-1} := 0$ yields,

$$L(y, \partial_y) \Big(e^{i\phi(y)/\epsilon} \, a \Big) \ = \ e^{i\phi(y)/\epsilon} \sum_{j=-1}^{\infty} \epsilon^j \, \big[L_1(y, id\phi(y)) \, a_{j+1}(y) + L(y, \partial_y) \, a_j(y) \big] \,. \tag{5.3.3}$$

Therefore, equation (5.3.1) holds if and only if

$$L_1 \big(y, id\phi(y) \big) \, a_j(y) + L(y, \partial_y) \, a_{j-1}(y) \ = \ 0 \,, \qquad \text{for} \quad j = 0, 1, 2, \cdots \,. \tag{5.3.4}$$

The special case $j = 0$ is,

$$L_1(y, id\phi(y)) \, a_0 \ = \ 0 \,. \tag{5.3.5}$$

Proposition 5.3.1 *If*

$$a(\epsilon, y) \ \sim \ \sum_{j=0}^{\infty} \epsilon^j \, a_j(y) \quad \text{in} \quad C^\infty(\Omega)$$

then

$$L(y, \partial_y) \Big(a(\epsilon, y) \, e^{i\phi(y)/\epsilon} \Big) \ \sim \ 0 \quad \text{in} \quad C^\infty(\Omega)$$

if an only if the coefficients $a_j(y)$ and phase ϕ satisfy (5.3.4).

In contrast to the scalar case of the previous section, it is not at all obvious how or whether the equations (5.3.4) determine the amplitudes a_j and phase ϕ. The rest of this section is devoted to that question.

In order for (5.3.5) to admit nonzero solutions, the matrix valued function $L_1(y, d\phi(y))$ must be singular,

$$\det L_1(y, d\phi(y)) \;=\; 0. \tag{5.3.6}$$

This is a nonlinear first order partial differential equation for the real valued phase ϕ. It is called the **eikonal equation**. It asserts that the graph of $d\phi$ belongs to the characteristic variety of L. Such equations are the subject of Hamilton-Jacobi Theory which is recalled in Appendix 5.I.

Example. If L is the Maxwell system the equation (5.3.6) becomes,

$$(\partial_t \phi)^2 \left((\partial_t\phi)^2 - c^2 |\nabla_x \phi|^2 \right)^2 \;=\; 0\,.$$

The phases leading to solutions which satisfy the divergence constraint are those which satisfy

$$(\partial_t \phi)^2 \;=\; c^2 |\nabla_x \phi|^2\,. \tag{5.3.7}$$

For the macroscopic Maxwell's equations in matter with scalar dielectric and magnetic susceptibilties $\varepsilon(y)$ and $\mu(y)$ one obtains the same result with the speed of light $c = c(y)$ depending on position.

Examples of solutions of (5.3.7) with $c = \text{const.}$. 1. We have already encountered the linear phases $\phi = t\tau + x.\xi$, with $\tau^2 = c^2|\xi|^2$ which occur in plane waves.

2. Spherically symmetric solutions satisfy $\phi_t^2 = c^2\phi_r^2$. So long as $d\phi \neq 0$, $\phi_t \neq 0$ and they satisfy one of the two equations $\phi_t = \pm c\phi_r$. The general solution is $\phi = f(c\,t \pm |x|)$, with smooth f with $f' \neq 0$. For the plus sign, the surfaces of constant phase are incoming spheres. The sphere which starts at radius R degenerates at time $t = R$, showing that solutions of the eikonal equation will normally exist only locally in time.

3. An interesting class of solutions to (5.3.7) is those of the form

$$\phi(t,x) := c\,t \pm \psi(x)\,, \qquad \text{with} \qquad |\nabla\psi| = 1\,.$$

Discussion. If ψ is a smooth solution of $|\nabla\psi| = 1$ on a neighborhood of $\underline{x}$, then on a neighborhood of $\underline{x}$, $\psi - \psi(\underline{x})$ is equal to the signed distance from x to $M := \{\psi(x) = \psi(\underline{x})\}$.

Proof. Replace ψ by $\psi - \psi(\underline{x})$ to reduce to the case $\psi(\underline{x}) = 0$. Denote by ϕ the signed distance which is positive on the same side of M as ψ. Then, $|\nabla\phi| = |\nabla\psi| = 1$ and $\phi|_M = \psi|_M$. It follows from the sign condition that $d\phi = d\psi$ on M. The uniqueness from the Hamilton-Jacobi Theory applied to the equation $|\nabla\psi| = 1$ completes the proof. ∎

If the level set $\{\psi = \psi(\underline{x})\}$ in the last example is curved, then the distance function will not be smooth, developing singularities at centers of curvature. Theorem 6.6.3.ii proves that for $d \geq 2$ the only C^2 solutions of $|\nabla\psi| = 1$ everywhere defined on $\mathbb{R}^d$ are affine functions.

There are essentially two strategies for finding solutions of the eikonal equation. The first is to look for exact solutions. The simplest such are linear functions $\phi(t,x) = t\tau + x\xi$ when L_1 has constant coefficients and (τ, ξ) is characteristic. The second is to appeal to Hamlton-Jacobi Theory to obtain local solutions. To apply the results of Appendix 5.I, let $F(y,\eta) := \det\, L_1(y,\eta)$ so the eikonal equation reads $F(y, d\phi(y)) = 0$. Seek to determine ϕ from initial values $\phi(0, x) = g(x)$. To determine $\phi_t(0, x)$ from its values at a single point $(0, \underline{x})$ one uses the implicit function theorem which requires that

$$\frac{\partial F(0, \underline{y}, \tau, dg(x))}{\partial \tau}\Big|_{\tau=\phi_t(0,\underline{x})} \neq 0.$$

This is equivalent to τ being simple root of the polynomial equation $F(0, \underline{x}, \tau, dg(\underline{x})) = 0$. When this condition is satisfied, Hamilton-Jacobi Theory yields a local solution ϕ and $\dim \ker L_1(y, d\phi(y)) = 1$.

For the example of Maxwell's e equations above, the roots are double. However the double roots result from the fact that the factors τ and $\tau^2 - c^2|\xi|^2$ appear squared. To apply Hamilton-Jacobi Theory one solves (5.3.7) and not $(\phi_t^2 - c^2|\nabla_x \phi|^2)^2 = 0$.

This phenomenon of multiple roots which appear because of repeated factors is so common that we give a general treatment. Suppose that $F(0, \underline{x}, \phi_t(0, \underline{x}), dg(\underline{x})) = 0$ and that on a neighborhood of $(\underline{x}, \underline{\eta}) := (0, \underline{x}, \phi_t(0, \underline{x}), dg(\underline{x}))$ one has with integer $p > 1$,

$$F(y, \eta) \;=\; G(y, \eta)^p\, H(y, \eta),$$

with

$$\frac{\partial G(0, \underline{x}, \tau, dg(\underline{x}))}{\partial \tau}\Big|_{\tau=\phi_t(0,\underline{x})} \neq 0, \qquad \text{and} \qquad H(\underline{y}, \underline{x}) \neq 0.$$

In this case, one applies Hamilton-Jacobi Theory to the reduced equation

$$G(y, d\phi(y)) \;=\; 0,$$

which satisfies the simple root condition. The root has multiplicity $p > 1$ for F. This shows that $-\tau$ is an eigenvalue of multiplicity p of the hermitian matrix $A_0^{-1/2}\big(\sum A_j \xi_j\big)A_0^{-1/2}$. Therefore, $\dim \ker L_1(y, d\phi(y)) = p$.

All the above strategies, lead to phases ϕ satisfying the following hypothesis.

Constant rank hypothesis. *On an open connected subset* $\Omega \subset \mathbb{R}^{1+d}$, $\phi \in C^\infty(\Omega)$, $\partial_t \phi \neq 0$, *and* $\ker L_1(y, d\phi(y))$ *has strictly positive dimension independent of* y.

Example. If $L = L(\partial_y)$ has constant coefficients, and $\phi = y.\eta$ is linear with $\eta \in \operatorname{Char} L$, then $d\phi$ is constant so $\ker L_1(d\phi)$ is independent of y and the constant rank hypothesis is automatic. This is so even when η is a singular point of the characteristic variety.

Recall from (2.4.4) that $\pi(y, d\phi(y))$ denotes the orthogonal projection of $\mathbb{C}^N$ onto the kernel of $L_1(y, d\phi(y))$. Introduce the natural partial inverse, $Q(y)$ of the singular symmetric matrix valued function $L_1(y, d\phi(y))$ by

$$Q(y)\,\pi(y, d\phi(y)) \;=\; 0\,, \qquad Q(y)\,L_1(y, d\phi(y)) \;=\; I - \pi(y, d\phi(y))\,. \tag{5.3.8}$$

When the constant rank hypothesis is satisfied, Theorem 3.I.1 implies that these are smooth matrix valued functions.

Remark. The example of linear phase and constant coefficient L shows that while it is true that the points $\nabla_y \phi \in \operatorname{Char} L$ are smooth in y, the characteristic variety near the points $(y, d\phi(y))$ can be singular. Conical refraction in triaxial crstals is an important example (see [Ludwig], or [Joly-Métiver-Rauch, MR, Ann. Inst. Fourier]).

Equation (5.3.5) holds if and only if

$$a_0(y) \ \in \ \ker L_1(y, d\phi(y))\,.$$

Since $L_1(y, d\phi(y))$ is singular, it is not surjective. Thus the case $j = 1$ of (5.3.4) has information about a_0, namely

$$L(y, \partial)\, a_0 \ \in \ \text{range}\ L_1(y, d\phi(y))\,. \tag{5.3.9}$$

The last two displayed equations are sufficient to determine a_0 from its initial data, though this is by no means obvious. In the scalar analysis of §5.2, the analogous equations clearly determined the a_j from their initial data.

First check that the number of equations is equal to the number of unknowns. The polarization equation (5.3.11) shows that a takes values in a linear space of dimension equal to $k := \dim\big(\ker L_1(y, \phi(y)\big)$. Thus there k unknown functions. Equation (5.3.10) asserts that $L(y, \partial_y)\, a_0$ takes values in range $L_1(y, d\phi(y))$ which has codimension k. Thus this is equivalent to k partial differential equations for the k unknowns a_0.

For ease of reading, we suppose that ϕ is fixed and write $\pi(y)$ for $\pi(y, d\phi(y))$. Equation (5.3.5) is equivalent to

$$\pi(y)\, a_0(y) = a_0(y) \tag{5.3.10}$$

which is the analogue of equation (5.1.1). The condition (5.3.9) is satisfied if and only if,

$$\pi(y)\, L(y, \partial_y)\, a_0 = 0\,. \tag{5.3.11}$$

Equations (5.3.10) and (5.3.11) are our second formulation of the equations which determine a_0.

There are other equivalent forms of (5.3.9). In the science literature, the usual proceedure is to take linear combinations of the equations given by the rows of the system $L(y, \partial)a_0 + iL_1(y, d\phi(y))a_1 = 0$. The combinations which elimate the $L_1\, a_1$ terms are chosen. The constant rank hypothesis implies that the annihilator of $\text{rg}\, L_1(y, d\phi(y))$ is a smoothly varying subspace of dimension k. If $\ell_1(y), \ldots, \ell_k(y)$ form a basis, then (5.3.9) holds if and only if

$$\big\langle \ell_j(y)\,,\, L(y, \partial)a_0 \big\rangle \ = \ 0\,, \qquad 1 \le j \le k\,.$$

Equation (5.3.11) is of this form with the ℓ_j chosen to be k linearly independent rows of the matrix of the projector $\pi(y)$. More generally, if $K(y)$ is an $N \times N$ matrix satisfying for all y,

$$\text{rank}\, K(y) \ = \ k, \qquad K(y)\big(\text{rg}\, L(y, d\phi(y))\big) \ = \ 0,$$

then the equation (3.4.9) is equivalent to $K(y)\, L(y, \partial)\, a_0 = 0$ (see [MR] for more on this circle of ideas.

One consequence of the nonlinearity of the eikonal equation is that phases ϕ are usually only defined locally. As a result the amplitudes a_j are also only constructed locally, at best where the ϕ are defined. For the next result the local existence theorem for symmetric hyperbolic systems is used. In most concrete situations, the equations for the a_j are simple and sharper results true. This theme is investigated in §5.4.

Theorem 5.3.2. *Suppose that $\Omega \subset \{0 < t < T < \infty\}$ is a domain of determinacy for L as defined in the assumption at the start of §2.6 and that the phase $\phi \in C^\infty(\overline{\Omega})$ satisfies the the constant* *rank hypothesis and the eikonal equation on $\overline{\Omega}$. Given $g_j(x) \in C^\infty(\overline{\Omega}_0)$ satisfying $\pi(0,x)g_j = g_j$, there is one and only one sequence $a_j \in C^\infty(\overline{\Omega})$ satisfying (5.3.4) and the initial conditions*

$$\pi(0,x)\, a_j(0,x) = g_j(x) \qquad j = 0, 1, \cdots . \tag{5.3.13}$$

As initial data, what is needed is the projections $\pi(y)\, a_j(0,x)$. For $j = 0$ this is equal to a_0. For $j \geq 1$, it is only part of the values of $a_j|_{t=0}$. Those amplitudes are not in general polarized.

Proof of Theorem. Denote by (5.3.4j) the case j of (5.3.4). Each equation (5.3.4j) is equivalent to a pair of equations,

$$(5.3.4j) \qquad \Longleftrightarrow \qquad \pi(y)(5.3.4j) \quad \text{and} \quad \big(I - \pi(y)\big)(5.3.4j)\,.$$

At the same time note that since Q is an isomorphism on $\text{Range}\,(I - \pi(y))$,

$$\big(I - \pi(y)\big)(5.3.4j) \qquad \Longleftrightarrow \qquad Q(y)(5.3.4j)\,.$$

We will show that for each J, there are uniquely determined a_j for $j \leq J$ satisfying

$$(5.3.4j) \quad \text{and} \quad \pi(y)a_j|_{t=0} = g_j \quad \text{for} \quad j \leq J, \qquad \pi(y)(5.3.4(J+1)).$$

The proof is inductive. Suppose that $j \geq 0$ and that $a_{-1}, \ldots, a_{j-1}$ are determined and the profile equations up to $\pi(5.3.4j)$ are satisfied.
Multiplying (5.3.4j) by Q shows that

$$\big(I - \pi(y)\big)(5.3.4j) \qquad \Longleftrightarrow \qquad \big(I - \pi\big)a_j \;=\; -\,Q\,L(y,\partial_y)\,a_{j-1}\,. \tag{5.3.14}$$

Thus, $(I - \pi)a_j$ is determined from a_{j-1}.
Express

$$a_j \;=\; \pi\, a_j + (I - \pi)\, a_j \;=\; \pi\, a_j \;-\; Q\,L(y,\partial_y)\,a_{j-1}\,.$$

Inject this into $\pi(y)(5.3.4(j+1))$ to find

$$\pi(y)\,L(y,\partial_y)\,\pi(y)\,a_j \;=\; f_j\,, \qquad f_j \;:=\; \pi(y)\,L(y,\partial_y)\,Q\,L(y,\partial_y)\,a_{j-1}\,. \tag{5.3.15}$$

Reversing the steps shows that if (5.3.14) and (5.3.15) imply the equations $(I - \pi(y))(5.3.4j)$ and $\pi(y)(5.3.4(j+1))$.
Thus, to prove the iductive step it suffices to show that the equations (5.3.14) and (5.3.15) are uniquely solvable for $\pi(y)a_j$ for arbitrary initial data $\pi(y)a_j(0,y) = g_j$. That is the content of the next Lemma which completes the proof.

Lemma 5.3.3. *For any $f \in C^\infty(\overline{\Omega})$ and $g \in C^\infty(\overline{\Omega}_0)$ satisfying*

$$\pi(y)\,g = g\,, \qquad \pi(y)\,f \;=\; f\,,$$

there is a unique $w \in C^\infty(\overline{\Omega})$ satisfying

$$\pi(y)\,L(y,\partial_y)\,\pi(y)\,w \;=\; f\,, \qquad \pi(y)\,w \;=\; w\,, \qquad w\big|_{t=0} \;=\; g\,.$$

Proof of Lemma. For a solution, $\pi\, w = w$ so the differential equation implies that one $pi\, L\, \pi\, w = 0$. Since $(I-\pi)w = 0$ one has $((-\pi)\, L\, (I-\pi)w = 0$. Adding yields,

$$\pi\, L\, \pi\, w \;+\; (I-\pi)\, L\, (I-\pi)\, w \;=\; f\,. \tag{5.3.16}$$

The result is proved by showing that the differential operator $\widetilde{L} := \pi\, L\, \pi + \;(I-\pi)\, L\, (I-\pi)$ is symmetric hyperbolic and for all y, $\overline{\mathcal{T}}(y, L) \supset \overline{\mathcal{T}}(y, \widetilde{L})$. This last comparison implies that the propagation cones of $\widetilde{L}$ are contained in the propagtion cones of L. Therefore, $\overline{\Omega}$ is also a domain of determinacy for $\widetilde{L}$.

The coefficient matrices of $\widetilde{L}$ are,

$$\widetilde{A}_\mu \;:=\; \pi\, A_\mu \pi + (I-\pi)\, A_\mu\, (I-\pi).$$

They are symmetric since A_μ and π are. The coefficient of ∂_t is

$$\widetilde{A}_0 \;:=\; \pi\, A_0 \pi + (I-\pi)\, A_0\, (I-\pi)$$

Since A_0 is postive definite one estimates with $c > 0$,

$$\begin{aligned}\langle \widetilde{A}_0 v\,,\, v\rangle &= \langle \pi\, A_0\, \pi\, v\,,\, v\rangle + \langle (I-\pi)\, A_0\, (I-\pi)\, v\,,\, v\rangle \\ &= \langle A_0\, \pi\, v\,,\, \pi\, v\rangle + \langle A_0\, (I-\pi)v\,,\, (I-\pi)v\rangle \geq c\left(\|\pi\, v\|^2 + \|(I-\pi)v\|^2\right) = c\|v\|^2,\end{aligned}$$

proving that $\tilde{L}$ is symmetric hyperbolic on $\overline{\Omega}$

For the comparison of timelike cones, it suffices to remark that if $L_1(y,\eta) \geq 0$, then

$$\widetilde{L}_1(y,\eta) \;=\; \pi L_1(y,\eta)\pi \;+\; (I-\pi)L_1(y,\eta)(I-\pi) \;\geq\; 0$$

as the sum of nonnegative matrices.

Theorem 2.6.1 then implies that for a given initial data $w(0,x)$, (5.3.16) has a solution on $\overline{\Omega}$. This proves uniqueness of w.

To prove existence, we show that the function w so constructed solves the equations of the Lemma. Multiplying (5.3.16) by $\pi(y)$ shows that the solution w satisfies

$$\pi(y)\, L(y, \partial_y)\, \pi(y)\, w \;=\; f\,. \tag{5.3.18}$$

Thus, if w satisfies (5.3.16) then so does $v := \pi w$. Since $v = w$ at $t = 0$, it follows that v satisfies the same symmetric hyperbolic initial value problem as w so

$$\pi(y)\, w = w\,. \tag{5.3.19}$$

This together with (5.3.18) yields the equations of the Lemma completing the existence proof. ∎

The Lemma completes the proof of Theorem 5.3.2. ∎

Remark. An alternate approach to the Lemma is to consider g as a $\ker L_1(y, d\phi(y))$ valued function. The operator $g \mapsto \pi(y)\, L(y, \partial_y)\, g$ maps such functions to themselves. Choosing a smooth orthonormal basis $\mathbf{e}_j(y)$, $1 \leq j \leq k$ for $\ker L_1(y, d\phi(y))$ and expanding $w = \sum w_j(y)\mathbf{e}_j(y)$ yields a symmetric hyperbolic system for the w_j. Such bases exist locally. To avoid the choice of bases one can consider symmetric hyperbolic systems on hermitian vector bundles. The proof above avoids appealing to that abstract framework.

The next result proves that the asymptotic solutions differ by an infinitely small quantity from exact solutions.

Theorem 5.3.4. [Lax, 1957]. *Suppose that Ω, ϕ, and a_j are as in the Theorem 5.3.2 and that*

$$a(\epsilon, y) \ \sim \ \sum_{j=0}^{\infty} \epsilon^j \, a_j(y) \qquad \text{in} \quad C^\infty(\overline{\Omega}) \,.$$

Suppose that $u(\epsilon, y) \in C^\infty(\overline{\Omega})$ is the exact solution of the initial value problem

$$L(y, \partial_y) \, u(\epsilon, y) \ = \ 0 \,, \qquad u(\epsilon, 0, x) \ = \ a(\epsilon, 0, x) \, e^{i\phi(0,x)/\epsilon} \,. \tag{5.3.20}$$

Then

$$u(\epsilon, y) \ - \ e^{i\phi(y)/\epsilon} \, a(\epsilon, y) \ \sim \ 0, \qquad \text{in} \quad C^\infty(\overline{\Omega}) \,. \tag{5.3.21}$$

Remark. As in Theorem 4.3.3, neither the family of exact solution u^ϵ nor the family of approximation $e^{i\phi(y)/\epsilon} \, a(\epsilon, y)$ is smooth at $\epsilon = 0$.

Proof. For any $m, s \in \mathbb{N}$ there is a constant C so that

$$\| L(y, \partial_y) \big(u(\epsilon, y) - e^{i\phi(y)/\epsilon} \, a(\epsilon, y) \big) \|_{H^s(\overline{\Omega})} \ \le \ C \, \epsilon^m \,,$$

and

$$\| \, u(\epsilon, 0, x) - e^{i\phi(0,x)/\epsilon} \, a(0, x) \, \|_{H^s(\overline{\Omega}_0)} \ \le \ C \, \epsilon^m \,.$$

The basic linear H^s energy estimate from §2 implies that

$$\| u(\epsilon, y) - e^{i\phi(y)/\epsilon} \, a(\epsilon, y) \|_{H^s(\overline{\Omega})} \ \le \ C'(T, m, s) \, \epsilon^m \,.$$

Since this is true for all m, s the result follows from Sobolev's Embedding Theorem. ∎

For the study of nonlinear equations, it is important to understand the effect of oscillatory source terms. The case of nowhere characteristic phase is treated in Chapter 4. The case of an everywhere characteristic phase is analysed exactly as above. The result is the following.

Theorem 5.3.5. [Lax, 1957]. *Suppose that the domain of determinacy Ω and the real phase ϕ satisfying the eikonal equation are as above. Given smooth functions $b_j \in C^\infty(\overline{\Omega})$ there are uniquely determined amplitudes $\underline{a}_j \in C^\infty(\overline{\Omega})$ satisfying $\pi(y)\underline{a}_j(0, x) = 0$ and $\underline{a}_0(0, x) = 0$ and so that if*

$$\underline{a}(\epsilon, y) \ \sim \ \sum_{j=0}^{\infty} \epsilon^j \, \underline{a}_j(y) \quad \text{and} \quad b(\epsilon, y) \ \sim \ \sum_{j=0}^{\infty} \epsilon^j \, b_j(y)$$

then

$$L(y, \partial_y) \big(\sum \epsilon^j \, \underline{a}_j(y) \, e^{i\phi(y)/\epsilon} \big) \ - \ b(\epsilon, y) \, e^{i\phi(y)/\epsilon} \ \sim \ 0 \quad \text{in} \quad C^\infty(\Omega_T).$$

The principal amplitude is determined by the pair of equations

$$\pi(y)\,\underline{a}_0 = \underline{a}_0\,, \qquad \pi(y)\,L(y,\partial_y)\,\pi(y)\;\underline{a}_0 = b_0\,, \tag{5.3.22}$$

with the initial condition $\underline{a}_0(0,x) = 0$.

Remarks. i. This result shows that a source of size one with characteristic phase yields waves of size one. This contrasts with the case of noncharacteristic phases in Chapter 4 where the response is order ϵ.

ii. Once one knows that there is such an infinitely accurate approximation one usually studies only the leading term, which for ϵ small is dominant. One does NOT compute the correctors in practice. Their existence is crucial for accurate estimates of the error and its derivatives.

Exercise 5.3.1. *Prove Theorem 5.3.5.* **Discussion.** The proof of error estimates is exactly the same as Theorem 5.3.4.

§5.4 Rays and transport

§5.4.1. The smooth variety hypothesis

One of the key ideas in geometric optics is transport along rays. The equation

$$\pi\,L\,\pi\,a_0 \;=\; 0 \tag{5.4.1}$$

determines a_0 from its polarized initial data. The key and not obvious fact is that under the smooth variety hypothesis which is satisfied in the vast majority of applications, *the differential operator $\pi(y)\,L(y,\partial_y)\,\pi(y)$ has first order part which is just a directional derivative.*

Using the product rule for the derivative $\partial_\mu(\pi a_j)$ yields

$$\pi(y)\,L(y,\partial_y)\,\pi(y) \;=\; \sum_\mu \pi A_\mu \pi\,\partial_\mu \;+\; \sum_\mu \pi(A_\mu(\partial_\mu\pi) + B)\pi\,. \tag{5.4.2}$$

Each matrix $\pi(y)A_\mu(y)\pi(y)$ defines a linear transformation from $\ker L_1(y, d\phi(y))$ to itself. Where the variety is smooth it is true but not obvious that each of these transformations is a scalar multiple of $\pi(y)A_0(y)\pi(y)$ so the differential operator is essentially a directional derivative.

Example. When $\ker L_1(y, d\phi(y))$ is one dimensional, it is easy to see that one gets a directional derivative. In that case, $\pi(y)$ is a projector of rank 1, and the polarization $\pi\,a_0 = a_0$ determines a_0 up to a scalar multiple. Since $\ker L_1(y, d\phi(y))$ is one dimensional, there are uniquely determined scalars $v_\mu(y)$ such that

$$\pi(y)A_\mu(y)\pi(y) \;=\; v_\mu(y)\,\pi(y)\,.$$

Similarly there is a unique scalar valued $\gamma(y)$ such that

$$\pi(y)\sum_\mu \big(A_\mu(\partial_\mu\pi) + B\big)\,\pi(y) \;=\; \gamma(y)\,\pi(y)\,.$$

Define a vector field $V(y,\partial_y)$ by

$$V(y,\partial_y) \;:=\; \sum_{\mu=0}^{3} v_\mu(y)\,\frac{\partial}{\partial y_\mu}$$

The equation $\pi\, L\pi a_0 = 0$ is equivalent to the transport equation

$$V(y, \partial_y)\, a_0 + \gamma(y)\, a_0 \;=\; 0\,,$$

along the integral curves of V.

Example. Consider

$$L \;=\; \partial_t \;+\; \begin{pmatrix} c_1(y) & 0 \\ 0 & c_2(y) \end{pmatrix} \frac{\partial}{\partial x} \;+\; B\,, \qquad c_1 \;>\; c_2\,,$$

where $x \in \mathbb{R}^1$. The eikonal equation is

$$\big(\phi_t + c_1(y)\phi_x)\big)\,\big(\phi_t + c_2(y)\phi_x)\big) \;=\; 0\,.$$

Phases satisfy the linear eikonal equations $(\partial_t + c_j\partial_x)\phi = 0$. Consider $j = 1$. Then the projector π and principal profile are

$$\pi \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad a_0 \;=\; (a(y)\,,\,0)\,.$$

The transport equation is

$$\Big(\partial_t + c_1(y)\partial_x + B_{11}(y)\Big)a \;=\; 0\,.$$

A particularly interresting case is when $L = \partial_t + G$ with G antiselfadjoint, precisely

$$L \;=\; \partial_t \;+\; \begin{pmatrix} c_1(y) & 0 \\ 0 & c_2(y) \end{pmatrix} \frac{\partial}{\partial x} \;+\; \begin{pmatrix} \partial_x c_1(y) & 0 \\ 0 & \partial_x c_2(y) \end{pmatrix} \;+\; B\,, \qquad B = -B^*\,.$$

In this case solutions of $Lu = 0$ suitably small as $x \to \infty$ satisfy

$$\partial_t \int |u(t,x)|^2\; dx \;=\; 0\,.$$

The transport equation is

$$\Big(\partial_t + c_1(y)\partial_x + \partial_x c_1 + B_{11}\Big)a \;=\; 0\,, \qquad B_{11}(y) \in i\mathbb{R}\,.$$

Where $\partial_x c > 0$ (resp. < 0) the amplitude a decreases (resp. increases) along rays. Where $c_x > 0$ (resp. < 0) neighboring rays spread apart (resp. approach). The energy between neighboring rays is spread over a larger region in the first case so the amplitude decreases to compensate. In the second case the energy is compressed and the amplitude increases. These results are illustrated in the Figure 5.4.1. More general results are presented in the next sections.

Figure 5.4.1. Compression on the left and expansion on the right.

The operator $\pi(y)\, L(\partial)\, \pi(y)$ is a transport operator under the next hypothesis which holds in each of the examples above.

Smooth characteristic variety hypothesis. *The smooth characteristic variety hypothesis is satisfied at* $(\underline{y}, \underline{\eta}) = (\underline{\tau}, \underline{\xi}) \in \mathrm{Char\, L}$ *if there is a neighborhood of* $(\underline{y}, \underline{\eta})$ *so that in that neighborhood, the characteristic variety is a smooth graph* $\tau = \tau(y, \xi)$. †

Theorem 3.I.1 applied to $L_1(y, \tau(y,\xi), \xi)$ *which has* 0 *as an isolated eigenvalue, implies that* $\pi(y, \tau(y,\xi), \xi)$ *and* $Q(y, \tau(y,\xi), \xi)$ *are smooth functions of* y, ξ.

Examples. 1. When $\ker L_1(\underline{\tau}, \underline{\xi})$ is one dimensional, the eigenvalue τ of $H(\underline{y}, \underline{\xi})$ is simple. Thus for (y, ξ) near $(\underline{y}, \underline{\xi})$, there is a unique simple eigenvalue near $\underline{\tau}$ and the smooth variety hypothesis is satisfied.

2. If $L_1 = L_1(\partial)$ has constant coefficients, it was remarked in §2.4, that the stratification theorem of real algebraic geometry implies that the set of points where the smooth variety hypothesis is violated is at most a $d-1$ dimensional subvariety. In particular, the smooth variety hypothesis is satisfied for generic linear phases, $\phi(y) = y.\eta$, $\eta \in \mathrm{Char}\, L$.

3. The argument in **2.** extends to $L_1(y, \partial)$ with real analytic coefficients showing in that case that with the exception of a codimension one set in $\mathrm{Char}\, L$, the smooth variety hypothesis is satisfied.

Definitions. *If* (y, τ, ξ) *belongs to the characteristic variety and satisfies the smooth variety hypothesis, define the group velocity* $\mathbf{v}(y, \tau, \xi))$ *by*

$$\mathbf{v}(y, \tau, \xi).\partial_x \ := \ -\sum_{j=1}^{d} \frac{\partial \tau(y,\xi)}{\partial \xi_j} \frac{\partial}{\partial x_j}$$

If $\phi(t,x)$ *is a solution of the eikonal equation and the points* $(y, d_y\phi(y)) \in \mathrm{Char}\, L$ *satisfy the smooth variety hypothesis with associated function* $\tau(y,\xi)$, *define the associated transport operator by*

$$V(y, \partial_y) \ := \ \partial_t \ - \ \sum_{j=1}^{d} \frac{\partial \tau}{\partial \xi_j}(y, d\phi(y)) \, \frac{\partial}{\partial x_j} \ = \ \partial_t \ + \ \mathbf{v}(y, d\phi(y)).\partial_x \,. \qquad (5.4.3)$$

A geometric construction leading to in (5.4.3) was given in §2.4. The velocity also appeared in the nonstationary phase calculation in §1.3. To show that this same velocity is hidden in the leading profile equation requires an algebraic identity.

Proposition 5.4.1. *At characteristic points* (y, τ, ξ) *where the smooth variety hypothesis is satisfied, the following fundamental algebraic identities hold,*

$$\pi\big(y, \tau(y,\xi), \xi\big)\, A_j\, \pi\big(y, \tau(y,\xi), \xi\big) \ = \ -\frac{\partial \tau}{\partial \xi_j}\, \pi A_0\, \pi\big(y, \tau(y,\xi), \xi\big)\,, \qquad 1 \le j \le d\,.$$

Example. When $\ker L_1$ is not one dimensional, and $A_0 = I$, the operatores $\pi\, A_j \pi$ act as scalars on $\ker L_1(y, d\phi(y))$ in spite of their appearence as typical symmetric linear transformations on that space.

Proof. The variable y acts purely as a parameter and is suppressed. Consider the map

$$\xi_j \quad \mapsto \quad L_1\big(\tau(\xi), \xi\big)\ \pi\big(\tau(\xi), \xi\big)\,.$$

† It is sufficient to assume that the variety is locally a continuous graph. The smoothness then follows from Theorem 3.I.1.

The matrix on the right has $\lambda = 0$ as an isolated eigenvalue of constant multiplicity thanks to the smooth variety hypothesis.

The perturbation equation (3.I.2) implies that

$$\pi(\tau(\xi), \xi)\,\Big(\frac{\partial \tau(\xi)}{\partial \xi_j}\,A_0 + A_j\Big)\,\pi(\tau(\xi), \xi) = 0\,,$$

where the term in the middle is the derivative with respect to ξ_j of L_1. This identity is the desired relation. ∎

Using these identities yields

$$\pi(y)\,A_j(y)\,\pi(y)\,\frac{\partial}{\partial x_j} \;=\; \pi(y)\,A_0(y)\pi(y)\,\bigg(-\frac{\partial \tau(y, d\phi(y))}{\partial \xi_j}\bigg)\,\frac{\partial}{\partial x_j}\,.$$

Using this in (5.4.2) together with the definition of the group velocity $\mathbf{v}(y, d\phi(y))$ yields

$$\pi\,L\pi \;=\; \pi\,A_0\,\pi\Big(\partial_t \;+\; \mathbf{v}.\partial_x\Big) \;+\; \pi\Big(B(y) \;+\; \sum_\mu A_\mu \partial_\mu \pi\Big)\pi\,.$$

The matrix $\pi\big(B + \sum_\mu A_\mu \partial_\mu \pi\big)\pi$ annihilates $\ker \pi$ and maps the image of π to itself. Since $\pi\,A_0\,\pi$ is an isomorphism of the image of π to itself, there is a unique smooth matrix valued $\gamma(y)$ such that

$$\gamma(y)\,(I - \pi(y)) = (I - \pi(y))\,\gamma(y) = 0, \qquad \text{and} \qquad \pi\Big(B \;+\; \sum_\mu A_\mu \partial_\mu \pi\Big)\pi \;=\; \pi\,A_0\,\pi\,\gamma\,.$$

Therefore,

$$\pi\,L\,\pi \;=\; \pi(y)\,A_0(y)\,\pi(y)\Big(\partial_t \;+\; \mathbf{v}.\partial x \;+\; \gamma(y)\Big)\,,$$

and equation (5.4.1) is equivalent to the homogeneous transport equation,

$$\Big(\partial_t \;+\; \mathbf{v}.\partial x + \gamma(y)\Big)a_0 \;=\; 0\,. \tag{5.4.4}$$

Definition. *The integral curves of $\partial_t + \mathbf{v}.\partial_x$ are called* **rays** *associated to ϕ. Equation (5.4.4) is called the* **transport equation for** *a_0.*

Solving equation (5.4.4) amounts to solving ordinary differential equations. When the smooth variety hypothesis is satisfied, the operator $\pi\,L\,\pi$ is essentially a linear transport operator and the existence theorem Theorem 5.3.2 can be strengthened.

Theorem 5.4.2. *Suppose that ϕ satisfies the eikonal equation on a set Ω, the smooth variety hypothesis is satisfied at the points $(y, d\phi(y))$, and, for each point in Ω the backward ray can be continued in Ω till it reaches $t = 0$. Then for the asymptotic solutions of $L(a\,e^{i\phi/\epsilon}) \sim 0$ the determination of the a_j from the initial data $(\pi\,a_j)|_{t=0} \in C^\infty(\Omega \cap \{t = 0\})$ reduces to the solution of inhomogeneous tranport equations*

$$\Big(\partial_t \;+\; \mathbf{v}.\partial_x + \gamma(y)\Big)a_0 \;=\; f_j\,,$$

where $f_0 = 0$ and $f_j = \pi(y)\, f_j$ is determined from a_{j-1}. If for all j, the support of $\pi(y)\, a_j|_{t=0}$ is contained in a set E then the a_j are supported in the tube of rays with feet in E.

When the smooth variety hypothesis is satisfied let p denote the dimension of $\ker L_1(\underline{y},\underline{\eta})$). Then p is also the dimension for nearby characteristic points since $\pi(y, \tau(y,\xi), \xi)$ is smooth. Division of polynomials depending smoothly on y, ξ shows that

$$\det L(y,\eta) \;=\; (\tau - \tau(y,\xi))^p\, K(y,\xi), \qquad K \in C^\infty, \qquad K(\underline{y},\underline{\xi}) \neq 0.$$

Therefore phases can be determined from their initial data applying Hamilton-Jacobi Theory to the *reduced eikonal equation*

$$\phi_t(y) \;=\; \tau(y, \nabla_x \phi(t,x))\,. \tag{5.4.5}$$

as in the discussion before the Constant Rank Hypothesis with $G := (\tau - \tau(y,\xi))$.

Introduce the hamiltonian,

$$H(y,\eta) \;=\; H(t,x,\tau,\xi) \;:=\; \tau - \tau(y,\xi) \;=\; \tau - \tau(t,x,\xi)\,,$$

and its Hamlton vector field

$$\mathcal{X}_H \;:=\; \sum_{\mu=0}^{d} \Big(\frac{\partial H}{\partial \eta_\mu} \frac{\partial}{\partial y_\mu} - \frac{\partial H}{\partial y_\mu} \frac{\partial}{\partial \eta_\mu} \Big). \tag{5.4.6}$$

Hamilton-Jacobi theory shows that the graph of $d\phi$, that is $\{(y, d\phi(y))\}$, is generated from the initial points, $(0, x, \tau(0,x,\nabla_x\phi(0,x)), \nabla_x\phi(0,x))$, by flowing along the the integral curves of $\mathcal{X}_H$.

The function H is constant on integral curves. The integral curves along which $H = 0$ are curves in (y,η) space which lie in the characteristic variety and are called *bicharacteristics* or *null bicharacteristics*. The graph of $d\phi$ is foliated by a family of bicharacteristics parameterized by initial points over $\{t = 0\}$. The definitions of $\mathcal{X}_H$ V yield the following result.

Theorem 5.4.3. *Suppose that ϕ satisfies the eikonal equation and the points $\big(y, d\phi(y)\big)$ satisfy the smooth characteristic variety hypothesis so (5.4.5) holds. Then, the projection on spacetime of the bicharacteristics foliating the graph of $d\phi$ are exactly the rays. Equivalently,*

$$\sum_{\mu=0}^{d} \frac{\partial H}{\partial \eta_\mu}(y, d\phi(y))\, \frac{\partial}{\partial y_\mu} \;=\; \partial_t \;+\; \mathbf{v}.\partial_x\,. \tag{5.4.7}$$

Remark on numerics. If you find ϕ by solving the Hamilton-Jacobi equation using bicharacteristics, you will have computed integral curves of the vector field (5.4.6). The amplitude a_0 satisfy transport equations along the space time projections of these curves. The additional computational cost required to determine the a_0 is negligible. This is true theoretically and also when the theory is used for numerical simulations. This method has numerical defects when rays grow far apart, since then the phase ϕ and amplitudes are determined at a sparse set of points. There is a well developed computational art of inserting new rays to help overcome this weakness. The methods are called *ray tracing* algorithms.

§5.4.2. Transport for $L = L_1(\partial)$

When $L = L_1(\partial)$ has constant coefficients and no lower order terms the transport equation for phases with $(y, d\phi(y))$ satisfying the smooth variety hypothesis can be understood in purely geometric terms. To simplify the formulas we suppose that $A_0 = I$ which can always be achieved by the change of variable $u = A_0^{-1/2}\tilde{u}$.

Theorem 5.4.4. *Suppose that $L = L_1(\partial)$ has constant coefficients, no lower order terms, and $A_0 = I$. Suppose ϕ solves the eikonal equation and the points $(y, d\phi(y))$ satisfy the smooth characteristic variety hypothesis. Then when $\pi(y)w = w$, one has*

$$\pi(y)\, L(\partial)\, w \;=\; \Big(\partial_t + \mathbf{v}(y).\partial_x + \frac{1}{2}\operatorname{div}\mathbf{v}\Big) w\,, \tag{5.4.8}$$

where $\mathbf{v}$ denotes the group velocity determined by ϕ. The transport equation determining a_0 is

$$\Big(\partial_t + \mathbf{v}(y).\partial_x + \frac{1}{2}\operatorname{div}\mathbf{v}\Big) a_0 \;=\; 0\,. \tag{5.4.9}$$

The divergence of $\mathbf{v}$ involves second derivatives of τ. The calculation, from [Gues-Rauch, 2006], uses second order perturbation theory from Theorem 3.I.2.

Proof. In $\pi\, L(\partial)\, w$, write the spatial derivatives as

$$\pi\, A_j \partial_j w \;=\; \pi\, A_j \partial_j\, (\pi w) \;=\; \pi\, A_j\, \pi\, \partial_j w + \pi\, A_j (\partial_j \pi)\, \pi\, w\,.$$

The eikonal equation satisfied by ϕ is (5.4.5). Consider the eigenvalue $-\tau(\xi)$ and eigenprojection $\pi(\xi)$ of the matrix $A(\xi) := \sum_j A_j \xi_j$ as functions of the parameter ξ_j. Formula (3.I.2) implies that

$$\pi\, A_j\, \pi \;=\; -\frac{\partial \tau}{\partial \xi_j}\, \pi \;=\; \mathbf{v}_j \pi\,. \tag{5.4.10}$$

Consider $-\tau(s\xi)$ as an eigenvalue of $\sum_j A_j s\xi_j$. Since the second derivative of $\sum_j A_j s\xi_j$ with respect to s vanishes, formula (3.I.3) implies that

$$\begin{aligned}
-\pi\, \frac{d^2}{ds^2}\tau(s\xi) \;=\; \sum_{i,j} \pi\, \frac{\partial^2 \tau}{\partial \xi_i \partial \xi_j}\; \xi_i\; \xi_j \;&=\; -2\pi \frac{d}{ds}\Big(\sum_j A_j s\xi_j\Big) Q \frac{d}{ds}\Big(\sum_i A_i s\xi_i\Big)\, \pi \\
&=\; -2\pi \Big(\sum_j A_j \xi_j\Big) Q \Big(\sum_i A_i \xi_i\Big)\, \pi \\
&=\; -2 \sum_{i,j} \pi\, A_j Q A_i \xi_j \xi_i\, \pi\,.
\end{aligned}$$

Symmetrizing the right hand side yields

$$\frac{\partial^2 \tau}{\partial \xi_j \partial \xi_k}\, \pi \;=\; -\pi A_j Q A_k \pi \;-\; \pi A_k Q A_j \pi. \tag{5.4.11}$$

Next consider the eigenvalue $-\tau(\nabla_x \phi)$ and eigenprojections $\pi(y)$ of $M(y) := \sum_k A_k \partial_k \phi(y)$ as functions of the parameter x_j. The formula (3.I.4) yields

$$(\partial_j \pi)\, \pi \;=\; -Q\, \partial_j M\, \pi \;=\; -Q\, \sum_k A_k\, \pi\, \frac{\partial^2 \phi}{\partial x_k \partial x_j}\,.$$

Using (5.4.11) yields,

$$\pi\, A_j\, (\partial_j \pi)\pi \;=\; -\sum_k \pi A_j Q A_k \pi\, \frac{\partial^2\phi}{\partial x_k \partial x_j} \;=\; \sum_k \frac{1}{2}\, \frac{\partial^2\tau}{\partial\xi_j\partial\xi_k}\, \frac{\partial^2\phi}{\partial x_k\partial x_j}\; \pi\,.$$

Differentiating $\mathbf{v} = -\nabla_\xi \tau\big(\nabla_x\phi(y)\big)$ shows that

$$\mathrm{div}\, \mathbf{v} \;=\; -\sum_j \frac{\partial}{\partial x_j}\, \frac{\partial\tau(\nabla_x\phi)}{\partial\xi_j} \;=\; \sum_{j,k} \frac{\partial^2\tau}{\partial\xi_j\partial\xi_k}\, \frac{\partial^2\phi}{\partial x_k\partial x_j} \;=\; .$$

Therefore,

$$\sum_j \pi\, A_j\, (\partial_j\pi)\pi \;=\; \frac{1}{2}\big(\mathrm{div}\, \mathbf{v}\big)\, \pi\,. \tag{5.4.12}$$

Combining (5.4.10), (5.4.11), (5.4.12) with the first identity of the proof yields when $w = \pi\, w$,

$$\pi\, L(\partial) w \;=\; \pi\, \big(\partial_t + \mathbf{v}.\partial_x\big) w + \frac{1}{2}\, \big(\mathrm{div}\, \mathbf{v}\big) w\,.$$

Since $\nabla_x\phi$ is constant along integral curves of $\partial_t + \mathbf{v}.\partial_x$ it follows that $\pi(y)$ is also constant so

$$\pi\, \big(\partial_t + \mathbf{v}.\partial_x\big) w \;=\; \big(\partial_t + \mathbf{v}.\partial_x\big)\, \pi w \;=\; \big(\partial_t + \mathbf{v}.\partial_x\big) w\,,$$

and the proof of the Theorem is complete. ∎

Exercise 5.4.1. *The corresponding computation in the scalar case is simpler. For a homogeneous second order scalar strictly hyperbolic operator without lower order terms, modify the computation of §5.2 to treat the case of nonlinear phases $\phi(y)$. In particular find a formula for the associated group velocity $\mathbf{v}$ and show that the leading amplitude a_0 satisfies (5.4.9).*

Both ϕ and $\nabla_x\phi$ are constant on the ray which start at point $(0, x)$ and has velocity given by $\big(1, -\tau_\xi(\nabla_x\phi_0(x))\big) = (1, \mathbf{v})$. This ray has equation,

$$t \quad\mapsto\quad \Big(t\,,\, x + t\, \mathbf{v}\big(\nabla_x\phi(0, x)\big)\Big) \;:=\; \big(t, \Phi(t, x)\big)\,. \tag{5.4.13}$$

The rays $t \mapsto \big(t, \Phi(t, x)\big)$ are integral curves of the vector field $\partial_t + \mathbf{v}.\partial_x$. Equivalently,

$$\frac{d}{dt}\, \big(t, \Phi(t, x)\big) \;=\; \Big(1\,,\, \mathbf{v}(\Phi(t, x))\Big)\,, \qquad \Phi(0, x) = x\,.$$

This equation shows that Φ is the flow of the time dependent vector field $\mathbf{v}$,

$$\frac{d}{dt}\, \Phi(t, x) \;=\; \mathbf{v}(\Phi(t, x))\,, \qquad \Phi(0, x) = x\,.$$

The Jacobian determinant,

$$J(t, x) \;:=\; \det\big(D_x\Phi(t, x)\big),$$

describes the infinitesimal deformation of d dimensional volumes as in (5.4.20) below.

Differentiating with respect to x yields the evolution of $D_x\Phi(t,x)$,

$$\frac{d}{dt}\, D_x\Phi(t,x) = \big(D_x\mathbf{v}\big)(\Phi(t,x))\ D_x\Phi(t,x)\,.$$

For x fixed this is an equation of the form

$$\frac{dM}{dt} \;=\; A(t)\,M(t),$$

where A and M are smooth $N\times N$ matrix valued functions. It follows that

$$\frac{d\det M}{dt} \;=\; \big(\mathrm{trace}A(t)\big)\ \det M\,.$$

This follows from the following three estimates as $\Delta t\to 0$,

$$\begin{aligned}
M(t+\Delta t) \;&=\; (I + A(t)\Delta t)M(t) \;+\; O((\Delta t)^2),\\
\det M(t+\Delta t) \;&=\; \det(I+A(t)\Delta t)\ \det M(t) + \; O((\Delta t)^2)\,,\\
\det(I+A(t)\Delta t) \;&=\; 1 \;+\; \mathrm{trace}A(t)\,\Delta t \;+\; O((\Delta t)^2)\,.
\end{aligned}$$

Applying the formula for $(\det M)'$ yields,

$$\frac{d}{dt}\,J(t,x) \;=\; \mathrm{trace}\big(D_x\mathbf{v}\big(\Phi(t,x)\big)\ J \;=\; \big(\mathrm{div}\,\mathbf{v}(\Phi(t,x)\big)\,J\,. \tag{5.4.14}$$

Corollary 5.4.5 *The amplitude $a_0=\pi(y)\,a_0$ satisfies the transport equation (5.4.9) if and only if the function*

$$a_0(t,\Phi(t,x))\,\sqrt{J(t,x)} \tag{5.4.15}$$

does not depend on t.

Proof. Equation (2.8) implies that $\sqrt{J(t,x)}$ satisfies

$$\partial_t\sqrt{J(t,x)} \;=\; \frac{1}{2\,\sqrt{J(t,x)}}\ \partial_t J \;=\; \frac12\ \big(\mathrm{div}\,\mathbf{v})(\Phi(t,x)\big)\ \sqrt{J(t,x)}\,.$$

Therefore

$$\begin{aligned}
&\partial_t\Big(a_0(t,\Phi(t,x))\,\sqrt{J(t,x)}\Big)\\
&\qquad = \big(\partial_t\sqrt{J(t,x)}\big)\,a_0(t,\Phi(t,x)) \;+\; \sqrt{J(t,x)}\,\Big(\partial_t a_0 + \mathbf{v}.\partial_x a_0\Big)\big(t,\Phi(t,x)\big)\,.
\end{aligned}$$

Using the formula for $\partial_t\sqrt{J(t,x)}$, yields

$$\partial_t\Big(a_0(t,\Phi(t,x))\,\sqrt{J(t,x)}\Big) \;=\; \sqrt{J(t,x)}\,\Big(\partial_t a_0 + \mathbf{v}.\partial_x a_0 + \frac12\big(\mathrm{div}\,\mathbf{v}\big)a_0\Big)\big(t,\Phi(t,x)\big)\,. \qquad \blacksquare$$

For fixed t denote by $X=\Phi(t,x)$ the point on the ray whose initial point is x. Then the infinitesimal volumes satisfy

$$dX \;=\; \left|\det\frac{\partial X}{\partial x}\right| dx \;=\; J(t,x)\,dx\,. \tag{5.4.20}$$

Equation (5.4.15) implies that

$$|a_0(t,X)|^2\, J(t,x) = |a_0(0,x)|^2\,, \qquad \text{equivalently} \qquad |a_0(t,X)|^2\, dX \;=\; |a_0(0,x)^2|\, dx\,.$$

This is an *infintesimal conservation of energy law.*

If $\omega \subset \{t=0\}$ is a nice bounded open set the family of rays starting in ω is called a *bundle* or *tube of rays.* It is denoted $\mathcal{T}$ and its section at time t is denoted $\omega(t)$. In particular, $\omega = \omega(0)$. Then,

$$\int_\omega |a_0(0,x)^2|\; dx \;=\; \int_{\omega(t)} |a_0(t,X)|^2\; dX\,. \tag{5.4.21}$$

Since $A_0 = I$, the energy density for solutions of $Lu = 0$ is $\langle u, A_0 u\rangle = \langle u, u\rangle$. Equation (5.4.21) shows that to leading order, the energy in any tube of rays is conserved.

Example. Linear phases. Suppose that $\phi(y) = t\tau + x.\xi$ is linear and satisfies the smooth variety hypothesis. The group velocity $\mathbf{v} = -\nabla_\xi \tau(\xi)$ is then constant and the rays are lines in space time with this velocity. The divergence of $\mathbf{v}$ vanishes, so a_0 is constant on rays, so $a_0(t,x) = g(x - t\,\mathbf{v})$ where $g(x) := a_0(0,x)$ is the initial value of a_0. The leading approximation is

$$e^{i(t\tau + x.\xi)/\epsilon}\, g(x - t\mathbf{v})\,, \qquad \pi\, g = g\,.$$

This generalizes the result obtained in §1.2.

The transport equation and its solution in the last two results depend only the phase ϕ and its associated group velocity. Two constant coefficient homogeneous systems leading to the same eikonal equation (5.4.5) lead to the same profile equation. For example Maxwell's equations, Dirac equations, and the wave equation all have the same eikonal equation $\partial_t \phi^2 \;=\; |\nabla_x \phi|^2$ so their principal profiles satisfy the same transport equations. A third example is the operator

$$L \;:=\; \partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x_1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x_2}\,. \tag{5.4.16}$$

The characteristic polynomial is $\det L_1(\tau,\xi) = \tau^2 - |\xi|^2$ and the eikonal equation is $\phi_t^2 = |\nabla_x \phi|^2$. The computation that follows is performed in dimension d. The case $d = 3$ applies to Maxwell's equations. The computation for any d applies to any Lorentz invariant field equation without lower order terms, e.g. the wave equation.

Example. Outgoing spherical solutions of $\phi_t^2 = |\nabla_x \phi|^2$. A particular solution of $\phi_t^2 = |\nabla_x \phi|^2$ for $|x| \neq 0$ is,

$$\phi(y) \;:=\; t - |x|\,, \qquad \phi_t \;=\; |\nabla_x \phi|\,. \tag{5.4.17}$$

With $\tau(\xi) = |\xi|$, the group velocity is

$$\mathbf{v} \;=\; -\nabla_\xi \tau(\nabla_x \phi) \;=\; -\left.\frac{\xi}{|\xi|}\right|_{\xi = \nabla_x \phi} \;=\; \frac{x}{|x|}\,.$$

Since

$$\partial_j \frac{x_j}{|x|} \;=\; \frac{1}{|x|} + x_j \partial_j \frac{1}{|x|}\,, \qquad \operatorname{div} \mathbf{v} \;=\; \frac{d}{|x|} \;+\; r\partial_r(r^{-1}) \;=\; \frac{d}{|x|} \;-\; \frac{r}{r^2} = \frac{d-1}{|x|}\,.$$

The rays move radially away from the origin with speed equal to one. The flow is given by

$$\Phi(t,x) \;=\; x \;+\; t\,\frac{x}{|x|}\,.$$

The annulus $\rho < r < \rho + \delta\rho$ is mapped to the annulus $\rho + t < r < \rho + t + \delta\rho$.

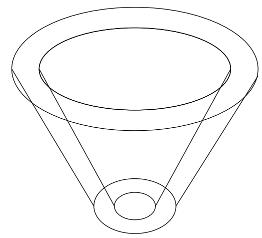


Figure 5.4.1. Outgoing annulus has growing volume

Considering $\delta\rho << 1$ shows that the volume is amplified by $\big((\rho{+}t)/\rho\big)^{d-1}$. Therefore the Jacobian, which depends only on $|x|$, is given by

$$J(t,x) \;=\; \left(\frac{|x|+t}{|x|}\right)^{d-1}. \tag{5.4.18}$$

Given initial values $a_0(0,x) = g(x) \in C_0^\infty(\mathbb{R}^d \setminus 0)$ the amplitude a_0 is defined for all $t \geq 0$ and has support in the set of outgoing rays with feet in the initial support. Equation (5.4.15) yields

$$a_0(t, x + tx/|x|)\; J(t,x)^{1/2} \;=\; a_0(0,x)\,, \quad \text{so} \quad a_0(t,x) \;=\; g(x - tx/|x|)\, \left(\frac{|x|}{|x|+t}\right)^{(d-1)/2}.$$

The leading term in the geometric optics approximation is

$$e^{i(t-|x|)/\epsilon}\; g(x - tx/|x|)\, \left(\frac{|x|}{|x|+t}\right)^{(d-1)/2}. \tag{5.4.19}$$

Exercise 5.4.2. *Compute formula (5.4.15) for the Jacobian directly using the definition of $J(t,x)$ and/or (5.4.14).*

Example. Incoming spherical solutions of $\phi_t^2 = |\nabla_x \phi|^2$. The preceding example has an important sibling. If one considers the outgoing example for times $t < 0$, the level sets $\phi = \text{const.}$ are incoming spherical shells, which focus to a point. If the closest point to the origin in the support of $a_0(t,x)$ is at distance r, then the equation for the amplitude becomes singular at $t = -r$.

Equivalently, had we considered the phase $\phi(t,x) = t + |x|$, the group velocity would have been $\mathbf{v} = -x/|x|$ and the wavefonts would be incominging.

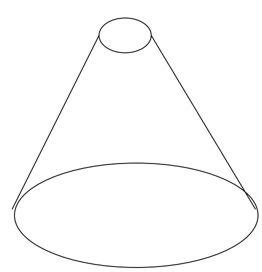


Figure 5.4.2. Incoming or focussing wavefronts.

The leading approximation of geometric optics for such focussing spherical wavefronts is,

$$e^{i(t+|x|)/\epsilon}\; g(x+tx/|x|)\; \left(\frac{|x|}{|x|-t}\right)^{(d-1)/2}. \tag{5.4.21}$$

The amplitudes become infinitely large where $x = t$. If you follow a ray approaching the origin, the amplitudes grow to compensate for the volume compression. When a spherical front focuses to a point, the amplitude explodes. For ϵ fixed, the initial data are smooth and the exact solution of the initial value problem is smooth. BUT, the approximation of geometric optics becomes infinitely large at focal points. The conclusion is that in a small neighborhood of the focal point, the approximation is inaccurate. What is surprising is that after the focus one finds that, in the linear case, the approximation becomes accurate again, with the phase changed by an additive constant called the Guoy shift (see chapter 12.2 of [Hö2]) after the physicist whose two mirror experiment verified the phenomenon for $d = 3$.

The geometric optics approximation is valid until the first ray along which a_0 is nonzero touches the origin. For example so long as the rays lie in $|x| \ge \delta > 0$. Choosing δ small shows that solutions do grow as they focus. They just do not grow infinitely large.

Exercise 5.4.3. *Prove that the solutions of the the wave equation in $d = 3$ with oscillatory radial initial data*

$$u^\epsilon(0,r) = a(r)\, e^{ir/\epsilon}, \qquad u^\epsilon_t(0,r) = \epsilon^{-1}\, b(r)\, e^{ir/\epsilon}$$

with smooth compactly supported radial a, b vanishing for $r \le R$, has maximum value that grows no faster than $1/\epsilon$ as $\epsilon \to 0$. **Hint.** Use the formula for the general radial solution of the $d = 3$ wave equation,

$$u \;=\; \frac{f(t+r) - f(t-r)}{r} \quad \text{for} \quad r \neq 0, \qquad u(t,0) = 2f'(t).$$

Find f in terms of a, b. **Discussion. i.** Each solution u^ϵ is a sum of an outgoing and an incoming spherical solution. For $\delta > 0$ and $0 \le t \le R - \delta$ the solution is supported in $r \ge \delta$ and the approximations of geometric optics are accurate. When $t = R$ the incoming wave can arrive at the origin, the phase loses smoothness, and the approximation breaks down. **ii.** One $0 \le t \le R - \delta$, the family $(\epsilon\partial)^\alpha u^\epsilon$ is bounded as $\epsilon \to 0$. The exercise shows that for typical a, b, this is not true $[0, R+1] \times \{|x| \le 1\}$.

§5.4.3. Energy transport with variable coefficients

The energy identity (2.3.1) implies that when $Lu = 0$,

$$\frac{d}{dt}\int \big\langle A_0(y)u(t,x), u(t,x)\big\rangle\; dx + \int \big\langle (B + B^* - \sum \frac{\partial A_\mu}{\partial y_\mu}) u(t,x), u(t,x)\big\rangle\; dx = 0.$$

Denote,

$$Z(y) \;:=\; B + B^* - \sum \partial_\mu A_\mu\,.$$

Definition. *A symmetric hyperbolic system is* **conservative** *when*

$$B + B^* - \sum \frac{\partial A_\mu}{\partial y_\mu} \;\equiv\; 0\,. \tag{5.4.22}$$

Since Cauchy data at time t are abitrary, one has the following equivalence.

Proposition 5.4.6. *A system is conservative if and only if* $\int_{\mathbb{R}^d} \langle A_0 u(t,x), u(t,x)\rangle\, dx$ *is independent of time for all solutions of* $Lu = 0$ *whose Cauchy data are compactly supported in* x.

Theorem 5.4.7. *Suppose that* L *is conservative, and, the smooth characteristic variety hypothesis is satisfied at all points* $\big(y, d\phi(y)\big)$ *over a tube of rays* $\mathcal{T}$ *with sections* $\omega(t)$. *If* $a(\epsilon, y)e^{i\phi(y)/\epsilon}$ *is an asymptotic solution of* $Lu \sim 0$, *then at leading order, the energy in the tube is conserved, that is*

$$\int_{\omega(t)} \big\langle A_0(y)\, a_0(t,x), a_0(t,x)\big\rangle\, dx \tag{5.4.23}$$

is independent of t.

Proof. For $0 < \delta << 1$, choose a cutoff function $0 \le \chi_\delta(x) \le 1$ such that χ is equal to one on ω_0 and $\chi(x) = 0$ when $\mathrm{dist}\,(x, \omega_0) > \delta$. Construct a Lax solution $\tilde{a}(\epsilon, y)e^{i\phi(y)/\epsilon}$ with

$$\tilde{a}_0(0,x) \;=\; \chi_\delta(x)\, a_0(x)\,. \tag{5.4.24}$$

Then Lax's Theorem together with conservation of energy implies that for all m and t

$$\int_{\mathbb{R}^d} \big\langle A_0\, \tilde{a}(\epsilon,t,x), \tilde{a}(\epsilon,t,x)\big\rangle\, dx \;-\; \int_{\mathbb{R}^d} \big\langle A_0\, \tilde{a}(0,x), \tilde{a}(0,x)\big\rangle\, dx \;=\; O(\epsilon^m)\,.$$

In addition the quantity on the left is controlled by its principal term, so

$$\int_{\mathbb{R}^d} \big\langle A_0\, \tilde{a}_0(t,x), \tilde{a}_0(t,x)\big\rangle\, dx \;-\; \int_{\mathbb{R}^d} \big\langle A_0\, \tilde{a}_0(0,x), \tilde{a}_0(0,x)\big\rangle\, dx \;=\; O(\epsilon)\,. \tag{5.4.25}$$

Since the left hand side is independent of ϵ it must vanish identically.
The amplitudes $\tilde{a}$ are uniformly bounded for bounded t and for $\delta < 1$. They differ from a on a set of measure $O(\delta)$. Therefore, for all t

$$\int_{\mathbb{R}^d} \big\langle A_0\, \tilde{a}_0(t,x), \tilde{a}_0(t,x)\big\rangle\, dx \;-\; \int_{\omega(t)} \big\langle A_0\, a_0(t,x), a_0(t,x)\big\rangle\, dx \;=\; O(\delta)\,. \tag{5.4.26}$$

The result follows from (5.4.24) and (5.4.25) by letting δ tend to zero. ∎

Consider $\omega(0)$ shrinking to a point $\underline{x}$ so the tube converges to the ray through $\underline{x}$. Then

$$\frac{\mathrm{vol}\,(\omega(t))}{\mathrm{vol}\,(\omega(0))} \;\to\; J(t, \underline{x}) \tag{5.4.27}$$

where $J(t, \underline{x})$ is the jacobian $\det \frac{d\Psi}{dx}$ of the flow $\Psi(t,x)$ generated by the vector field V.
The law of conservation of energy applied to a tube of diameter δ implies that

$$\begin{aligned}\mathrm{vol}\,(\omega(t))\, \big\langle A_0(\psi(t,\underline{x}))\, a_0(\psi(t,\underline{x}))\,, a_0(\psi(t,\underline{x}))\,\big\rangle \;=&\\ \mathrm{vol}\,(\omega(0))\, \big\langle A_0(\psi(0,\underline{x}))\, a_0(\psi(0,\underline{x}))\,,\, a_0(\psi(0,\underline{x}))\big\rangle\, &(1 + O(\delta))\,.\end{aligned}$$

Dividing by vol $(\omega(0))$ and passing to the limit $\delta \to 0$ implies that the quantity

$$\big\langle A_0(\psi(t,\underline{x}))\, a_0(\psi(t,\underline{x}))\,,\, a_0(\psi(t,\underline{x}))\, \big\rangle J(t,\underline{x})$$

is independent of t proving the next result.

Corollary 5.4.8. *Suppose that the the smooth characteristic variety hypothesis is satisfied at all points $\big(y, d\phi(y)\big)$ over a tube of rays $\mathcal{T}$, and that the system L is conservative. If $a(\epsilon, y)e^{i\phi(y)/\epsilon}$ is an asymptotic solution of $Lu \sim 0$, then the quantity*

$$\big\langle A_0(y)\, a_0(y)\,,\, a_0(y)\, \big\rangle\, J(y)$$

is constant on the rays associated to ϕ.

Remark. This shows that for converative problems the size of the leading amplitude a_0 is determined from its initial size entirely by conservation and volume deformation. This is weaker than the results for $L = L_1(\partial)$ where volume deformation alone determined the exact values of a_0.

The above results show that there is negligible energy flux into or out of the tube of rays. We give an alternate proof which allows us to generalize the results to nonconservative problems. The point of departure is the energy law (2.3.1). For a smooth w, the energy flux per unit area across an element of hypersurface $d\sigma$ with unit outward conormal ν is given by

$$\Big\langle \sum_\mu \nu_\mu\, A_\mu\, w\,,\, w \Big\rangle\,.$$

Proposition 5.4.9. *Suppose that ϕ satisfies the eikonal equation and $\big(y, d\phi(y)\big)$ satisfies the smooth variety hypothesis. Then transport is along $(1, \mathbf{v}) = (1, -\nabla_\xi \tau(y, \nabla_x \phi(y))$. If the conormal to an infinitesimal hypersuface element $d\sigma$ is orthogonal to $(1, \mathbf{v})$, that is,*

$$\nu_0 \;+\; \sum_j \nu_j\, \mathbf{v}_j \;=\; 0\,, \tag{5.4.28}$$

then, for any polarized $w = \pi(y, d\phi(y))\, w$, the flux through $d\sigma$ vanishes,

$$\Big\langle \sum_\mu \nu_\mu\, A_\mu\, w\,,\, w \Big\rangle \;=\; 0\,. \tag{5.4.29}$$

Proof. Using the polarization, the flux is equal to

$$\Big\langle \sum_\mu \nu_\mu\, A_\mu\, w\,,\, w \Big\rangle \;=\; \Big\langle \sum_\mu \nu_\mu\, A_\mu\, \pi\, w\,,\, \pi\, w \Big\rangle \;=\; \Big\langle \sum_\mu \nu_\mu\, \pi\, A_\mu\, \pi\, w\,,\, w \Big\rangle\,.$$

The identity of Theorem 5.4.1 implies that this is equal to

$$\Big\langle \big(\nu_0 + \sum_j \nu_j\, \mathbf{v}_j\big)\, \pi\, A_0\, \pi\, w\,,\, w \Big\rangle\,,$$

which vanishes thanks to (5.4.28). ∎

Corollary 5.4.10. *If ϕ satisfies the eikonal equation and the smooth variety hypothesis on a tube of rays, and $w = \pi\, w$ satisfies the the profile equation $\pi\, L\, \pi\, w = 0$, then along each ray $(t, \Phi(t,x))$ of $\mathcal{T}$ the energy density satisfies*

$$\frac{\partial}{\partial t}\Big(\Big\langle A_0\, w\,,\, w\Big\rangle_{(t,\Phi(t,x))}\ J(t,x)\Big)\ +\ \Big\langle Z\, w\,,\, w\Big\rangle_{(t,\Phi(t,x))}\ J(t,x)\ =\ 0\,. \tag{5.4.30}$$

Remark. In the conservative case, $Z = 0$, one recovers Corollary 5.4.9.

Proof. Using the polarization and transport equation yields

$$\Big\langle L\, w\,,\, w\Big\rangle\ =\ \Big\langle L\, \pi\, w\,,\, \pi\, w\Big\rangle\ =\ \Big\langle \pi\, L\, \pi\, w\,,\, w\Big\rangle\ =\ 0\,.$$

The energy identity (2.3.1) yields the conservation law

$$\sum_\mu \partial_\mu \Big\langle A_\mu\, w\,,\, w\Big\rangle\ +\ \Big\langle Z\, w\,,\, w\Big\rangle\ =\ 0\,.$$

Integrate over the tube from $t = 0$ to t. The lateral boundaries of the tube are foliated by rays. Therefore the conormal to the lateral boundaries are orthogonal to the transport direction. Proposition 5.3.9 shows that the flux through the lateral boundaries vanishes. Therefore,

$$\int_{\omega(t)} \Big\langle A_0\, w\,,\, w\Big\rangle\ dx\ +\ \int_0^t \int_{\omega(s)} \Big\langle Z\, w\,,\, w\Big\rangle\ dx\, ds\ =\ \int_{\omega(0)} \Big\langle A_0\, w\,,\, w\Big\rangle\ dx\,.$$

Dividing by the volume of $\omega(0)$ and shrinking $\omega(0)$ to the single point x, the tube contracts to the ray $(t, \Phi(t,x))$ and one finds using (5.4.27) that

$$\Big\langle A_0\, w\,,\, w\Big\rangle_{(t,\Phi(t,x))}\ J(t,x)\ +\ \int_0^t \Big\langle Z\, w\,,\, w\Big\rangle_{(s,\Phi(s,x))}\ J(s,x)\ ds\ =\ \Big\langle A_0\, w\,,\, w\Big\rangle_{(0,x)}\,.$$

This is equivalent to (5.4.30). ∎

§5.5. The Lax parametrix and propagation of singularities

The seminal paper [Lax, 1957] made several crucial advances. It systematized the formal aspects of the high frequency asymptotic solutions in the strictly hyperbolic case and showed how to prove their accuracy using energy estimates. Taking a very large step toward the creation of Fourier Integral Operators, it used these solutions to solve the the Cauchy problem with distribution initial data up to a smooth error, at least for small time. The necessity of small time comes from the fact that the nonlinear eikonal equations are solvable only locally in time.

The locality in time was removed as a hypothesis by Ludwig [Lu] by piecing together the local solutions and making a nonstationary phase argument which is a special case of general results of Hörmander on the compostion of Fourier Integral Operators.

In the late sixties, Hörmander introduced the wavefront set. This refined the notion of singular support motivated by work of Sato in the analytic category. In this section we show that using the notion of wavefront set, the local construction of Lax gives the global result of Ludwig and Hörmander.

§5.5.1. The Lax parametrix

The point of departure is a representation of the solution of the initial value problem,

$$Lu \;=\; 0\,,\qquad u(0,x) \;=\; f(x) \;\subset\; \cup_s H^s(\mathbb{R}^d)\,, \tag{5.5.1}$$

using the Fourier integral representation of the initial data,

$$f(x) \;=\; (2\pi)^{-d/2}\int e^{ix\xi}\;\hat f(\xi)\;d\xi\,. \tag{5.5.2}$$

From the point of view of singularities, only large ξ are important. The solution of initial value problems with data $e^{ix\xi}\hat f(\xi)$ with $|\xi| >> 1$ are the key ingredients.

As in the treatment of elliptic regularity, introduce $\omega := \xi/|\xi|$ and $\epsilon = 1/|\xi|$. The initial data of interest are $e^{ix.\omega/\epsilon}\,\hat f(\xi)$. This is a family of short wavelength data parameterized by $\{|\omega| = 1\}$.

Convention. *By a linear change of dependent variable we may, without loss of generality assume that $A_0 = I$ for the remainder of §5.5.*

Hypothesis. *Suppose that the smooth characteristic variety hypothesis holds at every point of the characteristic variety. In addition, assume that the distance between the sheets of the variety is bounded below by $C|\eta|$ uniformly in y.*

Remarks. i. Since there is finite speed of propagation, the uniform separation hypothesis as $|y| \to \infty$ is just for convenience and simplicity of statements.

ii. If one assumes the smooth variety characteristic hypothesis only along a neighborhood of a bicharacteristic, there is an analogous construction microlocally along that curve.

This hypothesis implies that the eigenvalues $\lambda_\nu(t,x,\xi)$ of $-\sum_j \xi_j A_j$ are defined and smooth in $\xi \neq 0$ and are uniquely determined by the ordering $\lambda_\nu < \lambda_{\nu+1}$, and,

$$\mathrm{Char}\, L \;=\; \cup_\nu \big\{\tau = \lambda_\nu(t,x,\xi)\big\}\,.$$

Define

$$\pi_\nu(t,x,\xi) \;:=\; \pi_\mu(t,x,\lambda_\nu(t,x,\xi),\xi)\,, \tag{5.5.3}$$

to be the projection along the range and onto the kernel of $L(t,x,\lambda_\nu(t,x,\xi),\xi)$. Since $A_0 = I$ these are also the orthogonal projections onto the eigenspace associated to λ_ν. The rank of $\pi_\nu(t,x,\lambda_\nu(t,x,\xi),\xi)$ is constant on each connected sheet of the variety. One says that L has **constant multiplicity**. In addition,

$$\sum_\nu \pi_\nu(t,x,\xi) \;=\; I\,.$$

Define $\phi_\nu(t,t',x,\xi)$ as the solution of the eikonal equation

$$\partial_t\phi_\nu = \lambda_\nu(t,x,\nabla_x\phi_\mu)\,,\qquad \phi_\nu(t,t',x,\xi)\big|_{t=t'} \;=\; x.\xi\,. \tag{5.5.4}$$

Since ϕ_ν is positive homogeneous of degree one in ξ, it suffices to consider $|\xi| = 1$. Choose $\underline{T} > 0$ so that for each $\nu, t', |\xi| = 1$ the solution of the eikonal equation (5.5.4) exists and is smooth on $]t' - 2\underline{T}, t' + 2\underline{T}[\times\mathbb{R}^d$.

Begin with the choice $t'=0$ and for ease of reading, suppress the t' dependence of ϕ_ν. Seek matrix valued asymptotic solutions

$$\mathbf{U}_\nu \;=\; e^{i\phi_\nu(y,\omega)/\epsilon}\,\mathbf{a}_\nu(\epsilon,y,\omega)\,,\qquad \mathbf{a}_\nu(\epsilon,y,\xi)\;\sim\;\sum_{j=0}^{\infty}\epsilon^j\,\mathbf{a}_{\nu,j}(t,x,\omega). \tag{5.5.5}$$

For the matrix valued solution $\mathbf{U}_\nu$, one either repeats the derivation of the equations for the profiles, or reasons column by column. From either point of view it is no harder to consider matrix valued asymptotic solutions than the vector valued case.

Seek $\mathbf{U}_\nu$ so that

$$L\,\mathbf{U}_\nu\;\sim\;0\,,\qquad \sum_{\nu,j}\epsilon^j\,\mathbf{a}_{\nu,\mathbf{j}}(\mathbf{0},\mathbf{x})\;\sim\;\mathbf{I} \tag{5.5.6}$$

The Lax parametrix for the initial value problem (5.5.1) is then

$$u_{\rm approx}\;:=\;\sum_\nu u_\nu\,,\quad u_\nu\;:=\;\int \mathbf{a}_\nu(\epsilon,y,\xi)\;e^{i\phi_\nu(t,x,\omega)/\epsilon}\;\chi(\xi)\;\hat f(\xi)\;d\xi\,. \tag{5.5.7}$$

The expression emphasizes the parameter ϵ and the origin in the short wavelength asymptotics.

We next determine smooth initial values of $\pi_\nu(0,x)\,\mathbf{a}_{\nu,j}(0,x)$ to achieve the second relation in (5.5.6). The leading symbols $\mathbf{a}_{\nu,0}$ must satisfy

$$\sum_\nu \mathbf{a}_{\nu,0}\;=\;I\,,\qquad \pi_\nu\,\mathbf{a}_{\nu,0}=\mathbf{a}_{\nu,0}\,.$$

Multiplying the first by π_ν shows that these two equations uniquely determine

$$\mathbf{a}_{\nu,0}(0,x,\omega)\;=\;\pi_\nu(0,x,\omega)\,.$$

Transport equations

$$\Big(\partial_t+\mathbf{v}(y,\omega).\partial_x+\pi_\nu\big(B+(L\,\pi)\big)\pi_\nu\Big)\mathbf{a}_{\nu,0}\;=\;0\,.\qquad \pi_\nu=\pi_\nu(y,\nabla_x\phi(y,\omega))$$

then determine $\mathbf{a}_{\nu,0,\omega}$. Since π_ν and $\mathbf{v}_\nu$ are smooth and homogeneous in $\xi\neq 0$, it follows that the tranport equations are solvable with uniform estimates on $\{|t|<2\underline{T}\}\times\mathbb{R}^d\times\{|\omega|=1\}$.

The components $(I-\pi_\nu)\mathbf{a}_{\nu,1}$ are determined from $\mathbf{a}_{\nu,0}$ by,

$$(I-\pi_\nu)\,\mathbf{a}_{\nu,1}\;=\;-Q_\nu\,L(y,\partial_y)\,\mathbf{a}_{\nu,0}\,,$$

where $Q_\nu(y)$ is the partial inverse of $L(y,\lambda_\nu(y,\xi),\xi)|_{\xi=\nabla_x\phi(y)}$.

To achieve (5.5.6), the symbols $\mathbf{a}_{\nu,1}$ must satisfy

$$\sum_\nu \mathbf{a}_{\nu,1}\big|_{t=0}=0\,.$$

Decomposing $\mathbf{a}_{\nu,1}=\pi_\nu\mathbf{a}_{\nu,1}+(I-\pi_\nu)\mathbf{a}_{\nu,1}$ yields

$$\Big(\sum_\nu\pi_\nu\mathbf{a}_{\nu,1}\;-\;\sum_\nu Q_\nu\;L\;\mathbf{a}_{\nu,0}\Big)_{t=0}\;=\;0\,.$$

Multiplying by π_ν shows that this uniquely determines,

$$\pi_\nu \mathbf{a}_{\nu,1}(0,x,\omega) \;=\; \sum_{\nu'\neq\nu} \pi_\nu \; Q_{\nu'} \; L \; \mathbf{a}_{\nu,0}(0,x,\omega)\,.$$

Then transport equations determine $\pi_\nu \, \mathbf{a}_{\nu,1}$ in $\{|t| \le 2\underline{T}\}$. Once these are known, algebraic equations determine $(I-\pi_\nu)\mathbf{a}_{\nu,2}$, and so on.

Choose $\chi(\xi)$ smooth with support in $|\xi|>1$ with $\chi(\xi)=1$ for $|\xi|\ge 2$ and

$$\mathbf{a}_\nu(\epsilon,y,\xi) \;\sim\; \sum_j \mathbf{a}_{\nu,j}(y,\omega)\,\epsilon^j \quad \text{in} \quad \{|\xi|\ge 1\}\,, \qquad \epsilon := |\xi|^{-1}\,.$$

The j^{th} term is $\mathbf{a}_{\nu,j}(y,\xi/|\xi|)|\xi|^{-j}$ and is homogeneous of degree $-j$ in ξ. Since $\epsilon = 1/|\xi|$ the ϵ dependence in $\mathbf{a}_\nu$ can be omitted and we have

$$\mathbf{a}_\nu(y,\xi) \;\sim\; \sum_j \mathbf{a}_{\nu,j}(y,\omega)\,|\xi|^{-j} \quad \text{in} \quad \{|\xi|\ge 1\}\,.$$

Since the π_ν and $\mathbf{a}_\nu$ are bounded on $[-2T,2T]\times\mathbb{R}^d$, the integral is absolutely convergent as soon as $\hat f \in L^1(\mathbb{R}^d)$. The formula

$$u_\nu(t,x) \;:=\; \int \mathbf{a}_\nu(t,x,\xi)\; e^{i(\phi_\nu(t,0,x,\xi)-w.\xi)}\;\chi(\xi)\; f(w)\; dw\,d\xi\,, \tag{5.5.8}$$

for of u_ν in terms of f is not absolutely convergent. The linear map $f \;\mapsto\; u$ has expression $\int K(t,x,w)\,f(w)\,dw$ with distribution kernel $K(t,x,w)$ given by

$$K(t,x,w) \;:=\; \int \mathbf{a}_\nu(t,x,\xi)\; e^{i(\phi_\nu(t,0,x,\xi)-w.\xi)}\;\chi(\xi)\; d\xi\,. \tag{5.5.9}$$

This oscillatory integral has integrand which is not L^1. The next section introduces the technique used to analyse expressions such as (5.5.8) and (5.5.9).

§**5.5.2. Oscillatory integrals and Fourier Integral Operators**

In this section, the method of nonstationary phase introduced in §1.3, is used to study oscillatory integrals. We present the key definitions and two fundamental results (For the full theory of Fourier integral operators the I recommend [Hörmander 1971] and [Duistermaat].) The two theorems show that (5.5.8) and (5.5.9) define well defined distribution for any $f\in\mathcal{E}'(\mathbb{R}^d)$ compute their wave front sets in terms of the wavefront set of f.

The method of oscillatory integrals is a generalization of the definition of the Fourier transform of distributions. For example, the expression

$$\delta(x) \;=\; (2\pi)^{-d}\int\; e^{-ix.\xi}\; d\xi$$

is a familiar oscillatory integral. The interpretation is that for cutoff $\gamma\in C_0^\infty(\mathbb{R}^d)$ which is equal to one on a neighborhood of the origin,

$$\lim_{\epsilon\to 0}\;(2\pi)^{-d}\int \gamma(\epsilon\xi)\; e^{-ix.\xi}\; d\xi \;=\; \delta(x)\,,$$

in the sense of distributions. Equivalently, for test functions $\psi(x)$,

$$\lim_{\epsilon\to 0}\ (2\pi)^{-d}\int \psi(x)\ \gamma(\epsilon\xi)\ e^{-ix.\xi}\ d\xi\, dx\ =\ \psi(0)\,.$$

In (5.5.8) one can fix t in which case the output $u_\mu(t)$ is a distribution on $\mathbb{R}^d_x$ or one can leave t as variable in which case the output in a distribution on $\mathbb{R}^{1+d}_{t,x}$. In the first case the kernel is a distribution in x, w depending on a parameter t while in the second case the kernel is a distribution in t, x, w. In (5.5.8) the phase, amplitude, and oscillatory integral are,

$$\phi(y,\xi)\ :=\ \phi_\mu(t,0,x,\xi) - w.\xi\,, \qquad a(y,\xi) = \mathbf{U}_\mu\,\pi_\mu\,\chi\,, \qquad \int a(y,\xi)\ e^{i\phi(y,\xi)}\ d\xi\,.$$

The phase is homogeneous of degree 1 in ξ. Showing that u_μ defines a well defined distribution requires showing that the expression

$$\int \psi(y)\ a(y,\xi)\ e^{i\phi(y,\xi)}\ d\xi\, dy$$

is a continuous linear function of the test function ψ. As in the case of the δ function the analysis is by cutting off in ξ and analysing the limit.

A typical symbol a behaves like a function homogeneous in ξ perhaps of positive degree. The integrals and even worse, their y derivatives are not absolutely convergent. The integral is finite only because of cancellations. Conditionally convergent integrals are often dangerous for analysis. The theory of oscillatory integrals gives a notable exception.

Definition. *If Ω is an open subset of $\mathbb{R}^M$ then the symbol class $S^m(\Omega\times\mathbb{R}^N)$ consists of functions $a(y,\xi)\in C^\infty(\Omega\times\mathbb{R}^N)$ such that*

$$\forall\omega\subset\subset\Omega,\ \ \forall(\alpha,\beta)\in\mathbb{N}^{M+N},\ \ \exists C,\ \ \forall(y,\xi)\in\omega\times\mathbb{R}^N,\quad \left|\partial_y^\alpha\,\partial_\xi^\beta\, a(y,\xi)\right|\ \le\ C\,\langle\xi\rangle^{m-|\beta|}\,.$$

Examples. 1. Functions a which are everywhere smooth and also positively homogeneous of degree m in ξ for $|\xi|\ge 1$.

2. Finite sums of functions as in **1** with degrees of homogeneity less than or equal to m.

3. The function $\mathbf{a}_\mu(y,\xi)\,\chi(\xi)$ with

$$\mathbf{a}_\mu\ \sim\ \sum_j |\xi|^{-j}\,\mathbf{a}_{\mu,j}(y,\xi/|\xi|)\,.$$

belongs to $S^0(([0,T]\times\mathbb{R}^d)\times\mathbb{R}^d)$ with symbol estimates, $|\partial_y^\alpha\,\partial_\xi^\beta\, a|\le C\langle\xi\rangle^{m-|\beta|}$ uniform on $y\in[0,T]\times\mathbb{R}^d$.

4. The function $\mathbf{a}_\mu$ is an asymptotic sum of homogeneous terms. Such symbols are called **polyhomogeneous** and are the most common examples. To avoid singularities at $\xi=0$ they must be multiplied by a cutoff like $\chi(\xi)$ to placed them in the classes S^m. For a polyhomogeneous symbol

$$a(y,\xi)\ \sim\ \sum_{j\le m} a_j(y,\xi)$$

with a_j homogeneous of degree j, one has $\chi(\xi)\, a \in S^m$.

Exercise 5.5.1. *Prove the assertions in the example.*

The results of the next exercise are used in defining oscillatory integrals by passing to the limit in symbols compactly supported in ξ.

Exercise 5.5.2. *Prove the following assertions.* **i.** *S^m is a Fréchet space.* **ii.** *If $\gamma \in C_0^\infty(\mathbb{R}^d_\xi)$ and $\gamma = 1$ on a neighborhood of $\xi = 0$, then the symbols $\gamma(\epsilon\xi)\, a,\ 0 < \epsilon \le 1$ are bounded in S^m.* **iii.** *For any $\mu < m$, $\gamma(\epsilon\xi)\, a$ converges to a in S^μ as $\epsilon \to 0$.*

Definition. *A real valued $\phi(y,\xi) \in C^\infty\big(Y \times (\mathbb{R}^N \setminus 0)\big)$ is a* **nondegenerate phase function** *if it is positive homogeneous of degree one in ξ and has nowhere vanishing gradient $\nabla_{y,\xi}\phi$.*

Examples. The phases $-x.\xi$ and the phase in the Lax parametrix (5.5.8).

Proposition 5.5.1. *If ϕ is a nondegenerate phase function then the map*

$$C_0^\infty \ni \psi \quad \mapsto \quad \int \psi(y)\ a(y,\xi)\ e^{i\phi(y,\xi)}\ d\xi\, dy$$

is well defined for smooth a with compact support in ξ. It extends uniquely by continuity to $a \in S^m(\Omega \times \mathbb{R}^N)$ for any m. For such symbols, it defines a distribution of order $k \in \mathbb{N}$ provided $k < -m - N$.

Proof. For a smooth $\chi(\xi)$ vanishing on a neighborhood of $\xi = 0$ and identically equal to one outside a compact set, write $a = \chi a + (1-\chi)a$. The second term is compactly supported and the associated map defines a distribution of order $-\infty$. So, it suffices to construct the extension for symbols which vanish on a neighborhood of $\xi = 0$. That is done as follows.

Since ϕ is nondegenerate we can introduce the first order differential operator in $\xi \ne 0$,

$$\mathcal{L} \ :=\ \frac{-i}{|\nabla_y\phi|^2 + |\xi|^2\, |\nabla_\xi\phi|^2}\ \Big(\nabla_y\phi.\partial_y \ +\ |\xi|^2\nabla_\xi\phi.\partial_\xi\Big), \quad \text{so that,} \quad \mathcal{L}e^{i\phi} = e^{i\phi}\,.$$

The denominator is nonvanishing and homogeneous of degree 2 in ξ. Therefore the coefficient of ∂_y (resp. ∂_ξ) is homogeneous of degree -1 (resp. 0) in ξ. Away from $\xi = 0$, the coefficient of ∂_y belongs to S^{-1} and the coefficient of ∂_ξ belongs to S^0.

If a is compactly supported and vanishes on a neighborhood of $\xi = 0$ one has for any k,

$$\int \psi(y)\ a(y,\xi)\ e^{i\phi(y,\xi)}\ d\xi\, dy \ =\ \int \psi(y)\ a(y,\xi)\ \mathcal{L}^k\, e^{i\phi(y,\xi)}\ d\xi\, dy\,.$$

Denote by $\mathcal{L}^\dagger$ the transpose. Integrating by parts yields

$$\int \psi(y)\ a(y,\xi)\ e^{i\phi(y,\xi)}\ d\xi\, dy \ =\ \int e^{i\phi(y,\xi)}\ (\mathcal{L}^\dagger)^k\big(\psi(y)\, a(y,\xi)\big)\ d\xi\, dy\,.$$

Since $\mathcal{L} = S^{-1}\partial_x + S^0\partial_\xi$ one has away from $\xi = 0$,

$$\mathcal{L}^\dagger \ =\ S^{-1}\partial_x \ +\ S^0\partial_\xi \ +\ (\partial_x S^{-1}) \ +\ (\partial_\xi S^0) \ =\ S^{-1}\partial_x \ +\ S^0\partial_\xi \ +\ S^{-1}\,.$$

Each summand maps S^r to S^{r-1}. For $a \in S^m$ vanishing on a neighborhood of $\xi = 0$, it follows that $(\mathcal{L}^\dagger)^k(\psi\, a) \in S^{m-k}$. The integrand in $\int e^{i\phi}(\mathcal{L}^\dagger)^k(\psi\, a) d\xi dy$ is $O(\langle\xi\rangle^{m-k})$. When k is so large that $m - k < -N$ it is absolutely integrable. The integral for those values of k yields the extension by continuity for symbols $a \in S^m$ vanishing on a neighborhood of $\xi = 0$.

For those k, the integral is bounded by $C(\omega, k)\|\psi\|_{C^k}$ proving that the distribution has order k. ∎

Exercise 5.5.3. *Use Proposition 5.5.1 to estimate the order of* $\delta(x) = (2\pi)^{-d} \int e^{-ix\xi}\, d\xi$.

Remarks. 1. If $\gamma(\xi) \in C_0^\infty(\mathbb{R}^N)$ is identically equal to one on a neighborhood of $\xi = 0$, Proposition 5.5.1 shows that,

$$\lim_{\epsilon \to 0} \int \gamma(\epsilon\xi)\ e^{i\phi(y,\xi)}\ a(y,\xi)\ d\xi$$

exists in the topology of distributions of order k. Following the lead of Hörmander and the tradition of the Fourier transform, the oscillatory integral is simply written as $\int e^{i\phi}\, a\, d\xi$ as if it were an integral.

2. In case m is very negative, the formula yields negative values of k. This corresponds to $u \in C^{-k} = C^{|k|}$. That conclusion is correct and is justified by direct differentiation under the integral sign defining the oscillatory integral. In particular, if $a \in \cap_m S^m$ then the distribution is a C^∞ function. Therefore, if $a \sim 0$ is polyhomogeneous, the associated distribution is smooth.

Exercise 5.5.4. *Prove* **2.**

3. The proposition allows one to manipulate oscillatory integrals as if they were integrals. For example, the integration by parts formula $\int \psi a e^{i\phi} d\xi dy = \int e^{i\phi}(\mathcal{L}^\dagger)^k(\psi a) d\xi dy$ is true for all $a \in S^m$ since the two sides are continuous and are equal on symbols with compact support.

4. If the test function ψ and symbol a depend in a continuous fashion on a parameter then the associated oscillatory integral also depends continuously on the parameter since it is the uniform limit of the cutoff oscillatory integrals. Similarly, one justifies differentiation under the (oscillatory) integral sign.

The next definition uses the notion of the **distribution kernel of a linear map**. Formally the operator with distribution kernel $A(y_1, y_2)$ is given by,

$$(Au)(y_2) \ = \ \int A(y_1, y_2)\ u(y_1)\ dy_1\,.$$

The precise version is for $u \in C_0^\infty$ and test function $\zeta(y_2) \in C_0^\infty$,

$$\langle \zeta(y_2)\,,\, Au \rangle \ := \ \big\langle \zeta(y_2)\, u(y_1)\,,\, A \big\rangle\,.$$

Any distribution A defines a continous linear map $A : C_0^\infty(\Omega_1) \to \mathcal{D}'(\Omega_2)$.[†]

Definition. *If* $\Omega_j \in \mathbb{R}^{N_j}$ *are open subsets, a continuous linear map* $A : C_0^\infty(\Omega_1) \to \mathcal{D}'(\Omega_2)$ *is a* **Fourier Integral Operator** *when it is given by a kernel,* $A \in \mathcal{D}'(\Omega_1 \times \Omega_2)$*, defined by on oscillatory integral. Equivalently, the operator is given by a formula*

$$Au(y_2) \ = \ \int a(y_1, y_2, \xi)\ e^{i\phi(y_1,y_2,\xi)}\ u(y_1)\ d\xi\, dy_1\,, \tag{5.5.10}$$

[†] The converse, called the Schwartz Kernel Theorem is true and is not needed below.

with amplitude $a(y,\xi) \in S^m$ for some m and nondegenerate phase $\phi(y,\xi)$.

Examples. 1. With $\Omega_1 = \Omega_2$ and $D = \frac{1}{i}\partial$, the identity map and the differential operator $a_\alpha(y)D^\alpha$ have kernels

$$\delta(y_2 - y_1) \;=\; (2\pi)^{-d} \int e^{i(y_2-y_1).\xi}\, d\xi\,, \qquad (2\pi)^{-d} \int\; a_\alpha(y_2)\, \xi^\alpha e^{i(y_2-y_1).\xi}\, d\xi\,,$$

with $\phi = (y_2 - y_1).\xi$. For a general differential operator, $a(y,D)$, one inserts the the symbol of $a(y,\xi)$. Pseudodiferential operators, about which we say very little, are also Fourier Integral Operators with this same phase. The pseudodifferential operator with symbol $a(y,\xi) \in S^m$ has kernel

$$(2\pi)^{-d} \int\; a(y_2,\xi)\; e^{i(y_2-y_1).\xi}\; d\xi,$$

so,

$$a(y,D)u \;=\; \int a(y,\xi)\; e^{iy.\xi}\; \hat{u}(\xi)\; d\xi.$$

2. The operator appearing in the Lax parametrix has the special structure

$$\int a(y_2,\xi)\; e^{i(\phi(y_2,\xi)-y_1\xi)}\; u(y_1)\; dy_1\, d\xi\,, \qquad \nabla_{y_2,\xi}\phi \neq 0\,, \tag{5.5.11}$$

The phase is $\Phi(y_1,y_2,\xi) := \phi(y_2,\xi) - y_1.\xi$.

Proposition 5.5.2. i. *If A is a Fourier Integral Operator with phase ϕ and amplitude a, then A maps $C_0^\infty(\Omega_1) \to C^\infty(\Omega_2)$ when $\nabla_{y_1,\xi}\phi$ is nowhere zero on the $\{\xi \neq 0\} \cap \operatorname{supp} a$.*
ii. *The operator extends uniquely to a continuous linear map from $\mathcal{E}'(\Omega_1) \to \mathcal{D}'(\Omega_2)$ provided that the partial gradient $\nabla_{y_2,\xi}\phi$ is nowhere zero on $\{\xi \neq 0\} \cap \operatorname{supp} a$.*

Examples. All of the preceding examples satisfy **i** and **ii.**

Proof of Propostion. i. One has

$$\langle Au\,,\, \psi\rangle \;=\; \int \psi(y_2)\; u(y_1)\; e^{i\phi(y_1,y_2,\xi)}\; a(y_1,y_2,\xi)\; d\xi\, dy_1\, dy_2. \tag{5.5.12}$$

Under the hypotheses the $d\xi\, dy_1$ integral is an oscillatory integral. The nondegenerate phase and the amplitude depend smoothy on y_2. Remark **4** shows that

$$\int\; u(y_1)\; e^{i\phi(y_1,y_2,\xi)}\; a(y_1,y_2,\xi)\; d\xi\, dy_1 \;\in\; C^\infty(\Omega_2),$$

That this smooth function is equal to Au follows by $\gamma(\epsilon\xi)$ truncation and passage to the limit.
ii. To show that the operator extends to a continuous linear map from $\mathcal{E}'(\Omega_1) \to \mathcal{D}'(\Omega_2)$ one must show that

$$\int \psi(y_2)\; a(y_1,y_2,\xi)\; e^{i\phi(y_1,y_2,\xi)}\; u(y_1)\; d\xi\, dy_1\, dy_2\,,$$

is a continuous functional of $\psi \in C_0^\infty(\Omega_2)$ and $u \in \mathcal{E}'(\Omega_1)$. This is so if (and only if) for every test function $\psi \in C_0^\infty(\Omega_2)$ one has

$$\int \psi(y_2)\; a(y_1,y_2,\xi)\; e^{i\phi(y_1,y_2,\xi)}\; d\xi\, dy_2 \;\in\; C^\infty(\Omega_1)\,. \tag{5.5.13}$$

The hypothesis of **ii** guarantees that the phase in (5.5.13) is nondegenerate for the $d\xi\, dy_2$ integration. This implies (5.5.13) and **ii** follows. ∎

The next result estimates the action on wave front sets of the class of Fourier Integral Operators (5.5.11) which arise in the Lax construction. The idea of treating only this class is inspired by [Taylor 1981]. For more general results see [Hörmander 1971, Duistermaat, Gabor].

Proposition 5.5.3. *For a Fourier Integral Operator of the form (5.5.11) with phase $\phi(y_2,\xi) - y_1.\xi$, $\nabla_{y_2,\xi}\phi \neq 0$, and, $u \in \mathcal{E}'(\Omega_1)$,*

$$WF(Au) \subset \Big\{ (y_2, \nabla_{y_2}\phi(y_2,\xi)) \,:\, \exists (y_2,\xi) \in \operatorname{supp} a,\ \big(\nabla_\xi \phi(y_2,\xi)\,,\, \xi\big) \in WF\, u \Big\}. \tag{5.5.14}$$

Examples. 1. For t fixed consider solutions of the wave equation which are superpositions of the plane waves $e^{i(-|\xi|t + x.\xi)} := e^{i\phi(t,x,\xi)}$,

$$\int e^{i(\phi(t,x,\xi) - w.\xi)}\ \chi(\xi)\ f(w)\ dw\, d\xi\,, \qquad \phi(t,x,\xi)\ =\ -t|\xi| + x.\xi\,.$$

This has form (5.5.11). The variable w plays the role of y_1 and x the role of y_2. We consider t as a parameter. One has,

$$\phi_\xi = x - t\frac{\xi}{|\xi|}\,, \qquad \text{and,} \qquad \phi_x = \xi\,.$$

A point $(x - t\xi/|\xi|, \xi) \in WF\, f$ may produce a singularity (x,ξ) in $u(t)$. The frequency is fixed. The position has moved t units in the direction $\xi/|\xi|$. Singularities associated with the plane waves $e^{i(-t|\xi| + x.\xi)}$ propagate at the group velocity $\xi/|\xi|$. The opposite sign choice, $+t|\xi$, in the phase yields the velocity $-\xi/|\xi|$.

2. The identity map and pseudodifferential operators are Fourier Integral operators with phase $(y_2 - y_1).\xi$. The Proposition shows that for this phase, $WF(Au) \subset WF(u)$ which is the *pseudolocal property* pseudodifferential operators.

Remark. Statement (5.5.14) asserts that $WF(Au)$ is contained in the set of (y_2, η_2) such that there is a point $(y,\xi) \in WF(u)$ so that

$$(y_2,\xi) \in \operatorname{supp} a, \qquad \eta_2 = \nabla_{y_2}\phi(y_2,\xi), \qquad \text{and,} \qquad y = \nabla_\xi \phi(y_2,\xi)\,. \tag{5.5.15}$$

The last two equations assert that $\nabla_{y_2,\xi}\big(\phi(y_2,\xi) - y_2.\eta_2 - y_1.\xi\big) = 0$.

Proof of Proposition. It suffices to show that if $(\underline{y}_2, \underline{\eta}_2)$ is not in the set on the right of (5.5.14) then it does not belong to $WF(Au)$. To show that $(\underline{y}_2, \underline{\eta}_2) \notin WF(Au)$, it suffices to show that for $\zeta \in C^\infty$ supported near $\underline{y}_2$ and η_2 in a small conic neighborhood of $\underline{\eta}_2$

$$\int \zeta(y_2)\ e^{-iy_2.\eta_2}\ a(y_2,\xi)\ e^{i\phi(y_2,\xi)}\ e^{-iy_1,\xi}\ u(y_1)\ dy_1\, d\xi\, dy_2$$

is rapidly decreasing as $\eta_2 \to \infty$.

Choose finite smooth partitions of unity $\psi_\nu(y_1)$ of $\operatorname{supp} u$, and, $\theta_\mu(\xi)$ of $\mathbb{R}^N \backslash 0$ with θ_μ homogeneous of degree zero in ξ. It suffices to consider individual summands,

$$\int \zeta(y_2)\ e^{-iy_2.\eta_2}\ a(y_2,\xi)\ e^{i\phi(y_2,\xi)}\ e^{-iy_1,\xi}\ \psi_\nu(y_1)\ \theta_\mu(\xi)\ u(y_1)\ dy_1\, d\xi\, dy_2\,. \tag{5.5.16}$$

If the support of $\psi_\nu\,\theta_\mu$ does not meet $WF\,u$ then using Proposition 4.6.2 one finds that

$$b(\xi) \;:=\; \int e^{-iy_1,\xi}\;\psi_\nu(y_1)\;\theta_\mu(\xi)\;u(y_1)\;dy_1$$

is rapidly decreasing. Then differentiation under the integral shows that

$$g(y_2) \;:=\; \int a(y_2,\xi)\;b(\xi)\;e^{-i\phi}\;d\xi \;\in\; C^\infty(\Omega_2)\,.$$

Then

$$(5.5.16) \;=\; \int \zeta(y_2)\;g(y_2)\;e^{-iy_2\eta_2}\;dy_2 \;=\; \mathcal{F}\big(\zeta(y_2)g(y_2)\big) \;\in\; \mathcal{S}(\mathbb{R}^{N_2}_{\eta_2})$$

is rapidly decreasing.

Therefore, it suffices to consider μ,ν so that the support of $\psi_\nu(y_1)\,\theta_\mu(\xi)$ belongs to a small conic neighborhood of a point $(\underline{y}_1,\underline{\xi})\in WFu$.

The remark after the Proposition shows that when $(\underline{y}_2,\underline{\eta}_2)$ is not in the set on the right of (5.5.14), the integral (5.5.16) for the important μ,ν has phase satisfying,

$$\nabla_{y_2,\xi}\Phi(y_1,y_2,\xi,\eta) \;:=\; \nabla_{y_2,\xi}\Big(\phi(y_2,\xi)-y_2.\eta_2-y_1.\xi\Big) \;\neq\; 0$$

on the support of the integrand in,

$$\int \zeta(y_2)\;a(y_2,\xi)\;e^{i\Phi}\;\psi_\nu(y_1)\;\theta_\mu(\xi)\;d\xi\,dy_2\,dy_1\,.$$

By homogenity,

$$|\nabla_{y_2}\Phi|^2 \;+\; |\xi|^2\,|\nabla_\xi\Phi|^2 \;\geq\; C\,(|\xi|^2+|\eta|^2) \tag{5.5.17}$$

in the support of the integrand. Introduce

$$\mathcal{L} \;:=\; \frac{-i}{|\nabla_{y_2}\Phi|^2 \;+\; |\xi|^2\,|\nabla_\xi\Phi|^2}\Big(\nabla_{y_2}\Phi.\partial_{y_2}+|\xi|^2\nabla_\xi\Phi.\partial_\xi\Big)\,,\qquad \text{so,}\qquad \mathcal{L}e^{i\Phi}=e^{i\Phi}\,.$$

Then

$$\int \zeta(y_2)\;a(y_2,\xi)\;e^{i\Phi}\;\psi_\nu(y_1)\;\theta_\mu(\xi)\;d\xi\,dy_2\,dy_1 \;=\; \int e^{i\Phi}\;(\mathcal{L}^\dagger)^k\big[\zeta(y_2)\;a(y_2,\xi)\;\psi_\nu(y_1)\;\theta_\mu(\xi)\big]\;d\xi\,dy_2\,dy_1$$

since the identity is true for compactly supported a and so extends by continuity for arbitrary a. Use $a\in S^m$ and (5.5.17) to see that each application of $\mathcal{L}^\dagger$ gains a factor of $(|\xi|^2+|\eta|^2)^{-1/2}$. Only one term requires a remark, namely

$$\frac{O(|\xi|^2)\;\partial_\xi S^\mu}{O(|\xi|^2+|\eta|^2)} \;\sim\; \frac{O(|\xi|^2)\;O(1/|\xi|)\;S^\mu}{O(|\xi|^2+|\eta|^2)} \;\leq\; \frac{S^\mu}{O\big((|\xi|^2+|\eta|^2)^{1/2}\big)}\,.$$

Choosing k large, it follows that the integral on the right is no larger than,

$$C_k\int_{|y_1,y_2|\leq R}\int_{|\xi|\geq r}\frac{|\xi|^m}{(|\xi|^2+|\eta|^2)^{k/2}}\;d\xi\,dy_2\,dy_1\,,$$

implying the desired rapid decrease. ∎

§5.5.3. Small time propagation of singularities

Small Time Propagation of Singlularities Theorem 5.5.4. (Lax-Hörmander). *There is a* $0 < T_1 \le \underline{T}$ *so that* $u_{\text{approx}} = \sum u_\mu$ *from (5.5.7) satisfies,*

i. *If* $f \in \cup_s H^s(\mathbb{R}^d)$ *and* u *the solution of (5.5.1), then* $u - u_{\text{approx}} \in C^\infty[0, T_1] \times \mathbb{R}^d)$.

ii. *For* $t \in [0, T_1]$ *the wavefront set of* $u_\mu(t)$ *in* $T^*(\mathbb{R}^d_x)$ *is the image of the wavefront set of* $u_\mu(0)$ *by the symplectic map on* $T^*(\mathbb{R}^d_x)$ *which is the time* t *flow of the hamiltonian field with time dependent hamiltonian* $-\lambda_\mu(t, x, \xi)$.

iii. *The wavefront set of* u_μ *in* $T^*(\mathbb{R}^{1+d})$ *is contained in the set* $\tau = \lambda_\mu(t, x, \xi)$ *and is invariant under the hamiltonian flow of* $\tau - \lambda_\mu(t, x, \xi)$. $WF\, u_\mu$ *contains only integral curves which pass over the wavefront set of* f.

Remarks. 1. By time reversal symmetry one has an analgous result on $-T_1 \le t \le 0$.

2. If one had started at a time T_0 one would obtain an analgous result on $|t - T_0| \le T_1$. It is important to note that one can find T_1 independent of T_0. The interval is determined by the fact that the eikonal equation is solvable and the maps C_μ stay close to the identity. These intervals cannot shrink because of the uniform bounds on the derivatives of the coefficients of L together with the uniform smooth variety hypothesis at the beginning of §5.5. If one were making such hypotheses only locally one would obtain a T_1 uniformly bounded on compact sets.

Proof. i. It suffices to verify that,

$$L(u - u_{\text{approx}}) \in C^\infty\big([0, \underline{T}] \times \mathbb{R}^d\big)\,, \qquad \text{and} \qquad u\big|_{t=0} - u_{\text{approx}}\big|_{t=0} \ \in\ C^\infty(\mathbb{R}^d)\,.$$

These are consequences of (5.5.6) We show how the first part of (5.5.6) yields the first of the two conclusions.

Since $Lu = 0$, it is sufficient to show that $Lu_\mu \in C^\infty([0, \underline{T}] \times \mathbb{R}^d)$ for each μ. Differentiating under the oscillatory integral sign in (5.5.8) shows that

$$Lu_\mu \ :=\ \int L\big(\mathbf{a}_\mu(t, x, \xi)\ e^{i(\phi_\mu(t,0,x,\xi) - w.\xi)}\big)\ \chi(\xi)\ f(w)\ dw\, d\xi\,.$$

This identity would be true if $\mathbf{a}_\mu$ had compact support so extends to general $\mathbf{a}_\mu$ by continuity. By construction,

$$L\big(\mathbf{a}_\mu(t, x, \xi)\ e^{i(\phi_\mu(t,0,x,\xi))}\big)\ =\ b(t, x, \xi)\, e^{i(\phi_\mu(t,0,x,\xi)}, \qquad b \in S^{-\infty}(([0, \underline{T}] \times \mathbb{R}^d_x) \times \mathbb{R}^d_\xi).$$

Therefore

$$Lu_\mu \ =\ \int b(t, x, \xi)\ e^{i(\phi_\mu(t,0,x,\xi) - w.\xi)}\ \chi(\xi)\ f(w)\ dw\, d\xi\,.$$

Since the amplitude b is of order $-\infty$, this oscillatory integral is smooth by Proposition 5.5.1. ∎

Exercise 5.5.5. *Prove that* $u(0, x) - u_{\text{approx}}(0, x) \in C^\infty(\mathbb{R}^d)$.

To prove **ii**, analyse the oscillatory integral defining $u_\mu(t)$,

$$\int \mathbf{a}_\mu(t, x, \xi)\ e^{i[\phi_\mu(t,0,x,\xi) - w.\xi]}\ \chi(\xi)\ f(w)\ dw\, d\xi\,. \tag{5.5.18}$$

For ease of reading, omit the subscript μ and the $s = 0$ argument from ϕ. The proposition estimating $WF(Au)$ implies that

$$WF\,u(t) \ \subset\ \Big\{\big(x, \nabla_x\phi(t,x,\xi)\big)\ :\ \big(\nabla_\xi\phi(t,x,\xi),\xi\big) \in WF\,u_\mu(0)\Big\}.$$

The mapping transforming the initial wave front to that at time t is given by

$$\big(\nabla_\xi\phi(t,x,\xi)\,,\,\xi\big) \quad\mapsto\quad \big(x\,,\,\nabla_x\phi(t,x,\xi)\big)\,. \tag{5.5.19}$$

At $t = 0$, $\phi(0, x, \xi) = x.\xi$. Thus at $t = 0$, the transformation is equal to the identity. It follows from uniform smoothness and the implicit function theorem that there is a $\underline{T}_1 > 0$ so that for $0 \le t \le \underline{T}_1$ each of the maps

$$(x,\xi)\ \mapsto (x, \nabla_\xi\phi(t,x,\xi)) \qquad \text{and} \qquad (x,\xi)\ \mapsto\ (\nabla_x\phi(t,x,\xi),\xi)\,,$$

is an invertible map close to the identity. It follows that for the same t, (5.5.19) defines a diffeomorphism of $T^*(\mathbb{R}^d)$ to itself which is close to the identity.

Lemma 5.5.5. *Denote by $C_\mu(t)$ the flow on $T^*(\mathbb{R}^d\times)\setminus 0$ of the time dependent hamilton field with hamiltonian $-\lambda_\mu(t,x,\xi)$. Then for $t \in [0, T_1]$, $C_\mu(t)$ is equal to the diffeomorphism defined by (5.5.19).*

Proof of Lemma. Suppress the subscripts μ. Denote by $(x(t), \xi(t))$ the curve traced by the diffeomorphism (5.5.19) so that $(x(0), \xi(0)) = (x_0, \xi_0)$. The formula (5.5.19) means that to compute $(x(t), \xi(t))$ one must find a pair (x, ξ) so that

$$\big(\nabla_\xi\phi(t,x,\xi)\,,\,\xi\big)\ =\ (x_0,\xi_0)\,, \qquad \text{then} \qquad \big(x\,,\,\nabla_x\phi(t,x,\xi)\big)\ =\ (x(t),\xi(t))\,.$$

Equivalently, $\big(x(t), \xi(t)\big)$ is determined by,

$$x_0\ =\ \nabla_\xi\phi(t,x(t),\xi_0), \qquad \xi(t)\ =\ \nabla_x\phi(t,x(t),\xi_0)\,. \tag{5.5.20}$$

It suffices to show that

$$x' = \nabla_\xi(-\lambda(t,x(t),\xi(t)))\,, \qquad \xi' = -\nabla_x(-\lambda(t,x(t),\xi(t)))\,. \tag{5.5.21}$$

Determine x' by differentiating the first equation in (5.5.20) with respect to t to find,

$$0\ =\ \nabla_\xi\,\phi_t + \nabla^2_{x\xi}\phi\big|_{(t,x(t),\xi_0)}\,x'\,. \tag{5.5.22}$$

Differentiating $\phi_t(t, x, \xi) = \lambda(y, \nabla_x\phi)$ with respect to ξ and evaluating at $(t, x(t), \xi_0)$ yields

$$0\ =\ \nabla_\xi\,\phi_t\ -\ \nabla_\xi\lambda(t,x(t),\xi(t))\,\nabla^2_{\xi x}\phi\big|_{(t,x(t),\xi_0)}\,. \tag{5.5.23}$$

The linear equation (5.5.22) satisfied by $x'(t)$ is identical to the equation (5.5.21) satisfied by $-\nabla_\xi\lambda(t, x(t), \xi(t))$. By the choice of T_1 the matrices $\nabla^2_{x\xi}\phi$ are invertible and close to the identity. Therefore, using (5.5.20),

$$x'\ =\ -\nabla_\xi\lambda(t,x(t),\xi(t))\,,$$

verifying half of (5.5.21).

Determine ξ' by differentiating the second equation in (5.5.20) with respect to time to find,

$$\xi' \;=\; \nabla_x \phi_t \;+\; \nabla^2_{xx}\phi\, x' \;=\; \nabla_x\phi_t \;-\; \nabla^2_{xx}\phi\,\nabla_\xi\lambda\,, \tag{5.5.24}$$

Differentiating $\phi_t = \lambda(y, \nabla_x\phi)$ with respect to x yields

$$\nabla_x\phi_t \;=\; \nabla_\xi\lambda\,\nabla^2_{xx}\phi + \nabla_x\lambda\,. \tag{5.5.25}$$

Equations (5.5.24)-(5.5.25) imply that $\xi' = -\nabla_x\lambda$ completing the proof of the lemma. ∎

It follows that

$$WF(u_\mu(t)) \;\subset\; C_\mu(t) WF(u_\mu(0))\,.$$

Using this same conclusion for the Lax parametrix for the Cauchy problem with initial time at t shows that that

$$WF(u_\mu(0)) \;\subset\; C_\mu(-t) WF(u_\mu(t))\,, \quad \text{equivalently} \quad C_\mu(t)\, WF(u_\mu(0)) \;\subset\; WF(u_\mu(t))\,.$$

Combining implies that

$$WF(u_\mu(t)) \;=\; C_\mu(t) WF(u_\mu(0))\,,$$

completing the proof of **ii**.

The proof of **iii** is similar. The result on $WF(Au)$ shows that the wavefront set of Au is a set of points $\big(t, x, \phi_t(t,x), \nabla_x\phi(t,x)\big)$. Since $\phi_t = \lambda_\mu(t, x, \nabla_x\phi)$ it follows that the wavefront set is a subset of $\tau = \lambda_\mu(t, x, \xi)$.

The formula for $WF(Au)$ together with the formula from **ii** shows that

$$WF_{\mathbb{R}^{1+d}}\, u_\mu(t,x) \;\subset\; \Big\{ \big(t, x, \lambda_\mu(t,x,\xi), \xi\big) \;:\; (x,\xi) \in WF_{\mathbb{R}^d}\, u_\mu(t) \Big\}\,. \tag{5.5.26}$$

We next prove that there is equality in (5.5.26). If (t, x, λ_μ, ξ) on the right were not in the wavefront set then for every real τ, (t, x, τ, ξ) would not be in the wavefront set. The limit point $(t, x, 1, 0)$ is also not in the wavefront set by the microlocal elliptic regularity theorem.

Using a finite covering by cones outsie the wavefront set, it follows that there is a $\zeta \in C_0^\infty$ supported near (t, x) and nonvanishing at t, x and a conic neighborhood Γ of ξ so that

$$\forall n\,, \quad \exists C_n\,, \quad \forall \gamma \in \Gamma\,, \quad \big|\mathcal{F}_{\mathbb{R}^{1+d}}(\zeta u)\big| \;\le\; C_n \langle \tau, \gamma\rangle^{-n}\,.$$

The spatial Fourier Transform is equal to,

$$\mathcal{F}_{\mathbb{R}^d}\big(\zeta(t,\cdot) u(t)\big)(\gamma) \;=\; c \int \mathcal{F}_{\mathbb{R}^{1+d}}(\zeta u)(\tau, \gamma)\; e^{it\tau}\; d\tau,$$

so is rapidly decreasing. Therefore, $(x, \xi) \notin WF(u_\mu(t))$.

Hence,

$$WF_{\mathbb{R}^{1+d}}\, u(t,x) \;=\; \Big\{ \big(t, x, \lambda_\mu(t,x,\xi), \xi\big) \;:\; (x,\xi) \in WF_{\mathbb{R}^d}\, u_\mu(t) \Big\}\,. \tag{5.5.27}$$

To see that this is equivalent to **iii.** reason as follows. For an integral curve $(x(s), \xi(s)) \in T^*(\mathbb{R}^d)\setminus 0$ of the hamilton field with time dependent hamiltonian $-\lambda_\mu(t, x, \xi)$, there is a unique lift to an integral curve of the hamilton field on $T^*(\mathbb{R}^{1+d}) \setminus 0$ with hamiltonian $\tau - \lambda_\mu(t,x,\xi)$ along which $\tau = \lambda_\mu$. The lift is given by

$$(t,x,\tau,\xi)(s) \;=\; \big(s, x(s), \lambda_\mu(s, x(s), \xi(s)), \xi(s)\big). \qquad \blacksquare$$

§5.5.4. Global propagation of singularities

This subsection shows that the analysis restricted to $0 \le t \le T_1$ presented in §5.5.4 implies the analogous global in time result.

First show that the approximate solution $\sum_\mu u_\mu$ has a natural extension to all times, that is beyond the domain where the Lax parametrix construction applies. For each μ, choose a function $g_\mu \in C^\infty(\mathbb{R}^{1+d})$ so that

$$g_\mu \;=\; L\,u_\mu \quad \text{for} \quad 0 \le t \le T_1\,. \tag{5.5.28}$$

Extend u_μ beyond $0 \le t \le T_1$ by solving

$$L\,u_\mu \;=\; g_\mu\,, \qquad u_\mu\big|_{0<t<T_1} \;=\; \text{given } u_\mu\,. \tag{5.5.29}$$

If one makes a different choice, $\tilde g_\mu$, the resulting function $\tilde u_\mu$ satisfies $u_\mu - \tilde u_\mu \in C^\infty(\mathbb{R}^{1+d})$ since

$$L(u_\mu - \tilde u_\mu) \;\in\; C^\infty(\mathbb{R}^{1+d})\,, \qquad \text{and} \qquad u_\mu - \tilde u_\mu \;=\; 0 \;\text{ on }\;]0,T[\times\mathbb{R}^d\,.$$

Thus u_μ and $u_{\text{approx}} = \sum_\mu u_\mu$ are well defined on $\mathbb{R}^{1+d}$ modulo $C^\infty(\mathbb{R}^{1+d})$. One has,

$$u - \sum u_\mu \;\in\; C^\infty(\mathbb{R}^{1+d})\,, \tag{5.5.30}$$

since $L(u - \sum u_\mu) \in C^\infty$ and $u - \sum u_\mu \in C^\infty(]0,T[\times\mathbb{R}^d)$. *To understand the singularities of u it suffices to understand the singularities of u_μ.*

Global Propagation of Singularities Theorem 5.5.6 (Ludwig-Hörmander). i. *The wave front set of u_μ is invariant under the hamilton flow of $\tau - \lambda_\mu(t,x,\xi)$. Precisely, a μ-bichararteristic belongs to $WF\,u_\mu$ if and only if it does so at $\{t=0\}$.*

ii. *If $0 \le t_1 < t_2$, then $WF\,u_\mu(t_2)$ is the image of $WF\,u_\mu(t_1)$ by the symplectic map which is the flow of the hamilton field with time dependent hamiltonian $-\lambda_\mu(t,x,\xi)$.*

Proof. Denote by $\Gamma(t) = (t, \underline{x}(t), \underline{\xi}(t), \lambda_\mu(t, \underline{x}(t), \underline{\xi}(t)), \underline{\xi}(t))$ a μ bicharateristic. It suffices to show that $\Gamma \in WF\,u_\mu$ if and only if $\Gamma(0) \in WF\,u_\mu$. It suffices to prove this for times $t \le T$ for arbitrary $T \in]0,\infty[$.

Consider the set

$$\Big\{\underline{t} \in [0,T] \;:\; \Gamma(t) \in WF\,u_\mu \quad \text{for} \quad 0 \le t \le \underline{t}\Big\}\,.$$

By definition of the wavefront set, this is a closed set. It suffices to prove that this set is open, since once that is known we know that the set is either empty or the entire interval. It is empty when $\Gamma(0) \notin WF\,u_\mu(0)$ and it is the entire interval when $\Gamma(0) \in WF\,u_\mu(0)$.

To prove Γ is open, it suffices to show that if $\Gamma([0,\underline{t}]) \subset WF\,u_\mu$ and $\underline{t} < T$, then $\Gamma([0,\underline{t}+\delta) \subset WF\,u_\mu$ for small positive δ. If $\underline{t} = 0$ this follows from the local in time result.

If $0 < \underline{t} < T$, Choose $\delta := T_1/2$ with T_1 from the local in time result. Define $t' := \underline{t} - \delta$. The Lax parametrix construction yields a solution v_μ defined on $\{|t - t'| \le 2\delta\}$ to

$$Lv_\mu \in C^\infty([\underline{t} - 2\delta, \underline{t} + 2\delta] \times \mathbb{R}^d)\,, \qquad v_\mu(t') - u_\mu(t') \in C^\infty(\mathbb{R}^d)\,.$$

Then, $v_\mu - u_\mu \in C^\infty([\underline{t} - 2\delta, \underline{t} + 2\delta] \times \mathbb{R}^d)$.

The local in time result implies that the wavefront set of v_μ for $\underline{t} - 2\delta \le t \le \underline{t} + 2\delta$ is a union of bicharacteristics. We know that for $0 < t < \underline{t}$ it is a union of only μ bicharacteristics so it follows that the wavefront set of v_μ is invariant under the $\tau - \lambda_\mu$ hamilton flow.

Since in $t \in]t', \underline{t}[$ the wavefront set agrees with that of u_μ it follows that the wavefront set of v_μ consists exactly of the continuations to $\underline{t} < t < t + 2\delta$ of the μ bicharacteristics in $WF(u_\mu)$ for $t < \underline{t}$.

Since $u_\mu - v_\mu \in C^\infty$ this implies that $\Gamma([0, \underline{t} + \delta] \subset WF\, u_\mu$ which completes the proof of **i.**

Denote by $C(t_2, t_1)$ from $T^*(\mathbb{R}^d \setminus 0)$ to itself the flow from time t_1 to t_2 by the flow the hamilton field with time dependent hamiltonian $-\lambda_\mu(t, x, \xi)$. The Lax parametrix construction shows that if $0 < t_1 < t_2 < t_1 + T_1$ then

$$WF\, u_\mu(t_2) \;=\; C(t_2, t_1)\, WF\, u_\mu(t_1)\,.$$

A finite number of applications of this result proves **ii**. ∎

Exercise 5.5.6. Denote by Γ_μ the bicharacteristics passing over $(0, \underline{x}, \underline{\xi})$. Prove that when $Lu \in C^\infty$,

$$\big(\cup_\mu \Gamma_\mu\big) \;\cap\; WF(u) \;=\; \cup_\mu \big(\Gamma_\mu \;\cap\; WF(u_\mu)\big)\,.$$

Exercise 5.5.7 *Under the same hypotheses prove that*

$$(\underline{x}, \underline{\xi}) \;\notin\; WF(u(0)) \quad\iff\quad \big(\cup_\mu \Gamma_\mu\big) \;\cap\; WF(u) \;=\; \phi\,.$$

The first part of the next result restates the conclusion without reference to the decomposition as $\sum u_\mu$. The second part gives an H^s version.

Theorem 5.5.7. i. *If u $Lu \in C^\infty(\mathbb{R}^{1+d})$, then $WF(u)$ is contained in the characteristic variety of L and is invariant under the hamilton flow with hamiltonian equal to $\Pi_\mu\big(\tau - \lambda_\mu(t, x, \xi)\big)$.* **ii.** *The same conclusion is valid with $WF(u)$ replaced by $WF_s(u)$.*

Remarks. 1. The hamilton field restricted to $\tau = \lambda_\mu(t, x, \xi)$ is parallel to the field with hamiltonian $\tau - \lambda_\mu(t, x, \xi)$.

2. It is the reduced hamiltonian and not $\det L(t, x, \tau, \xi)$ which appears. On sheets in the characteristic variety with multiplicity greater than one, the hamiltonian vector field associated to $\det L$ vanishes.

3. Theorem 5.5.6 and its Corollary are global in time. They are not restricted to domains where the eikonal equation is solvable. They go beyond caustics and focussing.

Example. Suppose that $\mathcal{O} \subset \mathbb{R}^d$ is a smoothly bounded open subset and $u(0, x)$ is a function smooth on $\overline{\mathcal{O}}$ and vanishing on $\mathbb{R}^d \setminus \overline{\mathcal{O}}$. The function and its derivatives may jump at the boundary. Exercise 4.6.6, shows that $WF(u(0))$ is contained in the conormal variety $N^*(\partial\mathcal{O})$. Therefore the $\mathbb{R}^{1+d}$ wavefront set of u_ν at time $t = 0$ is contained in the set of points $(0, x, \xi, \lambda_\nu(t0, x, \xi)$ with (x, ξ) conormal to $\partial\mathcal{O}$. \$$WF(u_\nu)$ is then a subset of the flowout by the hamilton flow of $\tau - \lambda_\nu$ of the points $(0, x, \lambda_\nu(0, x, \xi), \xi)$ with $(x, \xi) \in N^*(\partial\mathcal{O})$.

This same set appears in Hamilton-Jacobi theory as follows. Choose a function g smooth on a neighborhood of $\partial\mathcal{O}$ with $g = 0$ and $dg \ne 0$ at all points of $\partial\mathcal{O}$. Solve the initial value problem

$$\psi_t = \lambda(t, x, \partial_x \psi), \qquad \psi|_{t=0} = g\,.$$

The set containing $WF(u)$ is exactly equal to the set of points

$$\big(t, x, \sigma(\psi_t(t,x), \partial_x\psi(t,x))\big) \quad \text{such that} \quad \psi(t,x) = 0\,.$$

Thus as long as the eikonal equation is smootly solvable, $WF(u_\nu) \subset N^*(\{\psi = 0\})$.

The propagation of singularities theorem is global in time so is not limited by the local solvability of the eikonal equation. The result applies after caustics and focussing. On the other hand, locally in time one can show (see the discussion of progressing waves in [Lax 2007]) that the solution remains piecewise smooth with singularities along $\{\psi = 0\}$. For such small times, this example is a generalisation of the piecewise smooth solutions in §1.1.

Proof of Theorem. Part **i** is just a restatement.

For **ii** compute for $\mu \neq \nu$ that on $\tau = \lambda_\mu$,

$$\tau - \lambda_\nu \;=\; \tau - \lambda_\mu + (\lambda_\mu - \lambda_\nu) \;=\; \lambda_\mu - \lambda_\nu \;\neq\; 0.$$

Therefore if Γ_μ is a $\tau - \lambda_\mu$ bicharacteristic along which $\tau = \lambda_\mu$, one has for $\mu \neq \nu$, $\Gamma_\mu \cap WF(u_\nu) = \phi$. Therefore, **ii** is equivalent to the assertion that $WF_s(u_\mu)$ is invariant under the hamilton flow of $\tau - \lambda_\mu(t,x,\xi)$. Must show that if $\Gamma_\mu(\underline{t}) \notin WF_s(u_\mu)$, then $\Gamma_\mu \cap WF_s(u_\mu) = \phi$.

Since the curve γ_μ and WF_s are both closed it follows that

$$\mathcal{B} \;:=\; \big\{t \;:\; \Gamma_\mu(t) \notin WF_s(u_\mu)\big\} \quad \text{is open and nonempty.}$$

It suffices to show that $\mathcal{B}$ is closed.

Suppose that t_2 belongs to the closure, $\overline{\mathcal{B}}$. If $t_2 \in \mathcal{B}$, there is noting to show. Otherwise choose $\mathcal{B} \ni t_1 \neq t_2$ with $|t_1 - t_2| < T_1$ with T_1 from Theorem 5.5.4. It suffices to show that $\Gamma_\mu(t_2) \notin WF_s(u_\mu))$. We treat the case $t_1 < t_2$ the other being similar.

Choose $\beta(t) \in C^\infty(\mathbb{R})$ with β identically equal to one neighborhood of t_2 and vanishing for $t < t_1$. Choose the support of $d\beta/dt$ so close to t_1 so that $\Gamma(t) \notin WF_s(u(t))$ when $t \in \operatorname{supp} d\beta/dt$. Then

$$L(\beta u_\mu) \;=\; \beta\, L\, u_\mu \;+\; [L, \beta] u_\mu \;=\; \beta\, L\, u_\mu \;+\; \;+\; A_0\, \frac{d\beta}{dt}\, u_\mu\,.$$

The first term is smooth and $WF_s([L,\beta]u_\mu) \subset WF_s(u_\mu) \cap \operatorname{supp} d\beta/dt$ by Exercise 4.6.1. The choice of β guarantees that $\Gamma_\mu \cap WF_s(L(\beta u_\mu)) = \phi$

Write $[L,\beta]u_\mu = f_1 + f_2$ with supports in $t \geq t_1$ but very close to t_1, $f_1 \in H^s_{loc}$ and $\Gamma_\mu \cap WF(f_2) = \phi$. Denote by v_j the solutions of $Lv_j = f_j$ which vanish for $t \leq t_1$. Then $v_1 \in H^s_{loc}$ so $\Gamma_\mu(t_2) \notin WF_s(v_1)$.

We are only interested in v_2 near the space time projection of $\Gamma_\mu(t_2)$ so using finite speed we may cutoff f_2 to have compact support without modifying the solution v_2 on a neighborhood of the projection of $\Gamma_\mu(t_2)$.

Use the Lax parametrix and Duhamel's representation to write for t near t_2, $v_2 = \sum_\nu v_{2,\nu} + C^\infty$

$$v_{2,\nu}(t,x) \;:=\; \int a_\nu(t,\sigma,x,\xi)\; e^{i(\phi_\nu(t,\sigma,x,\xi) - x.\xi)}\; \chi(\xi)\; f_1(\sigma,x)\; dx\, d\sigma\; d\xi\,.$$

As in the proof of Theorem 5.5.4, these oscillatory integrals are analysed using Proposition 5.5.3. First, $WF(v_{2,\nu}) \subset \{\tau = \lambda_\nu)$ so $\Gamma_\mu(t_2) \notin WF(v_{2,\nu})$ for $\nu \neq \mu$. The same proposition shows that the behavior of $v_{2,\mu}$ microlocally on Γ_μ is determined by f_μ microlocally on Γ_μ. By construction we have $\Gamma_\mu \cap WF(f_2) = \phi$, and it follows that $\Gamma_\mu \cap WF(v_{2,\mu}) = \phi$.

Exercise 5.5.8. *Give the details of these two appllications of Proposition 5.5.3.*

This completes the proof that $\Gamma_\mu(t_2) \notin WF_s(v_1)$ and thereby proves the Theorem. ∎

§5.6. An application to stabilization

This section requires familiarity with Riemannian geometry, specifically with the geodesic flow. Suppose that M is a compact connected Riemannian manifold without boundary. The metric is

$$g_{ij}(x)\ dx^i\, dx^j\,. \tag{5.6.1}$$

The induced metric on one forms is

$$g^{ij}(x)\ d\xi_i\, d\xi_j\,. \tag{5.6.2}$$

The volume form is

$$dv(x) \;=\; (\det g_{ij}(x))^{1/2}\ dx\,.$$

The Dirichlet integral is

$$D(u,u) \;:=\; \int_M \big\langle du(x), du(x)\big\rangle\ dv(x) \;:=\; \int_M |du(x)|^2\ dv(x)\,. \tag{5.6.3}$$

For functions u supported in a single coordinate patch this is equal to

$$\int \sum_{i,j} g^{ij}(x)\,\partial_i u(x)\,\partial_j u(x)\,(\det g_{ij}(x))^{1/2}\ dx\,.$$

The Laplace Beltrami operator, $\Delta = \Delta_g$, is defined by

$$\big\langle \Delta u, \psi\big\rangle \;:=\; \frac{-1}{2}\,\frac{dD(u+s\psi, u+s\psi)}{ds}\bigg|_{s=0} \;=\; -\int_M \langle du(x), d\psi(x)\rangle\ dv\,. \tag{5.6.4}$$

In local coordinates,

$$\begin{aligned}
-\int_M \langle du(x), d\psi(x)\rangle\ dv(x) \;&=\; -\int g^{ij}(x)\,\partial_i u(x)\,\partial_j\psi(x)\,(\det g_{ij}(x))^{1/2}\ dx\\
&=\; \int (\det g_{ij}(x))^{-1/2}\,\partial_j\Big(g^{ij}(x)\,(\det g_{ij}(x))^{1/2}\,\partial_i u(x)\Big)\ \psi(x)\ (\det g_{ij}(x))^{1/2}\,dx\,,
\end{aligned}$$

so,

$$\Delta u \;=\; (\det g_{ij}(x))^{-1/2}\,\partial_j\Big(g^{ij}(x)\,(\det g_{ij}(x))^{1/2}\,\partial_i u(x)\Big)\,. \tag{5.6.5}$$

Consider the damped Klein Gordon equation,

$$L\,u \;:=\; u_{tt} \;-\; \Delta\,u \;+\; u \;+\; a(x)\,u_t \;=\; 0\,, \qquad C^\infty(M) \ni a \geq 0\,. \tag{5.6.6}$$

One could treat the damped wave equation in the same way with the inconvenience of working modulo the constants.

The principal symbol of the operator L is,

$$h(t,x,\tau,\xi) \ := \ \tau^2 - g^{ij}(x)\,\xi_i\,\xi_j \ = \ \tau^2 - \langle \xi_i dx^i, \xi_j dx^j\rangle \ := \ \tau^2 - |\xi|^2\,.$$

The characteristic variety lies in $\tau \neq 0$ and consists of two smooth sheets,

$$\tau \ = \ \pm|\xi| \ = \ \pm\Big(g^{ij}(x)\,\xi_i\,\xi_j\Big)^{1/2}\,.$$

The roots are simple and the smooth variety hypothesis is everywhere satisfied.

The crucial energy identity is

$$\frac{d}{dt}\int_M |u_t|^2 \ + \ |du(x)^2| \ + \ |u|^2\ dv \ = \ 2\mathrm{Re}\int_M \overline{u}_t\ \big(u_{tt} \ - \ \Delta\, u \ + \ u\big)\ dv\,.$$

Therefore, for solutions of $Lu = 0$ energy is dissipated according to,

$$\frac{d}{dt}\int_M |u_t|^2 \ + \ |du(x)^2| \ + \ |u|^2\ dv \ = \ -\int_M 2a(x)\,|u_t|^2(x)\ dv \ \le \ 0\,. \tag{5.6.7}$$

This yields the uniform *a priori* estimate,

$$\exists C, \quad \forall t \geq 0, \quad \|u(t)\|_{H^1} \ + \ \|u(t)\|_{L^2} \ \le \ C\,\big(\|u(0)\|_{H^1} \ + \ \|u(0)\|_{L^2}\big)\,.$$

If u satisfies $Lu = 0$ then $v = (1-\Delta + a(x))^{s/2}u$ satisfies $Lv = 0$.† The elliptic regularity theorem with elliptic operator $(1-\Delta+a(x))^{s/2}$ implies that $\|v_t\|_{L^2}$ and $\|v\|_{H^1}$ are norms equivalent to $\|u_t\|_{H^s}$ and $\|u\|_{H^{s+1}}$ respectively. Applying the *a priori* estimate proves that,

$$\forall s, \quad \exists C_s, \quad \forall t \geq 0, \quad \|u(t)\|_{H^{s+1}} \ + \ \|u_t(t)\|_{H^s} \ \le \ C_s\,\big(\|u(0)\|_{H^{s+1}} \ + \ \|u_t(0)\|_{H^s}\big)\,.$$

The strategy for proving existence by replacing ∂_x by symmetric difference quotients suitably globalized using partitions of unity, yields the following.

Theorem 5.6.1. *For any $s \in \mathbb{R}$, $f \in H^s(M)$, $g \in H^{s-1}(M)$ there is a unique*

$$u \ \in \ \cap_j C^j\Big([0,\infty[\ ;\ H^{s-j}(M)\Big)$$

satisfying

$$L\,u = 0 \quad \text{on} \quad \mathbb{R}\times M\,, \qquad u|_{t=0} = f\,, \quad u_t|_{t=0} = g\,.$$

Since the coefficients of L do not depend on t, it follows that if $u(t,x)$ is a solution, then so is $u(t+s,x)$ for any s. If $j \in C_0^\infty(\mathbb{R})$ with $\int j(t)\,dt = 1$, then

$$u^\delta(t,x) \ := \ \int u(t+\delta s, x)\ j(s)\ ds \ = \ \frac{1}{\delta}\int u(t+s)\ j(s/\delta)\ ds\,,$$

satisfies $L\,u^\delta = 0$ since the Riemann sums which converge to the integral are solutions. In addition, $u^\delta \to u$ in $C^j([-T,T]\,;\,H^{s-j})$ as $\delta \to 0$.

† Readers can assume that s is an even integer if they do not want to work with fractional powers.

Proposition 5.6.2. *For each $\delta > 0$, $u^\delta \in C^\infty(\mathbb{R} \times M)$.*

Proof. From the definition of u^δ and the continuity of u in time it follows that $u^\delta \in C^\infty(\mathbb{R}\,;\, H^s)$ with

$$\partial_t^k u^\delta(t) \;=\; \int u^\delta(t+\delta s)\, \frac{d^k j(s)}{dt^k}\; ds \;\in\; C(\mathbb{R}\,;\, H^s)\,.$$

This gives a Sobolev regularity for $\partial_t^k u^\delta$ independent of k since

$$C(\mathbb{R}\,;\, H^s) \;\subset\; C_{\rm comp}(\mathbb{R}\,;\, H^{-s})' \;\subset\; H^1_{\rm comp}(\mathbb{R}\,;\, H^{-s})' \;\subset\; H^{-s+1}_{\rm comp}(\mathbb{R}\times M)' \;=\; H^{s-1}_{\rm loc}(\mathbb{R}\times M)\,.$$

The operator ∂_t^k is microlocally elliptic on the characteristic variety of L, since $\mathrm{Char}\, L \subset \tau \neq 0$. The microlocal elliptic regularity theorem applied to the equation $\partial_t^k u \in H^{s-1}_{\rm loc}(\mathbb{R}\times M)\,.$ implies that u is microlocally H^{k+s-1} on the characterictic variety of L.

The microlocal elliptic regularity theorem applied to $Lu = 0$ shows that u also has this regularity away from the characteristic variety of L. Therefore u is everywhere microlocally H^{k+s-1}. Therefore $u \in H^{k+s-1}_{\rm loc}(\mathbb{R}\times M)$.

Since k is arbitrary, Sobolev's lemma completes the proof. ∎

Denote by $S(t)$ the operator

$$S(t)(f,g) \;:=\; \big(u(t), u_t(t)\big)\,,$$

where u is the solution with Cauchy data equal to (f,g). For any s, $S(t)$ is a C^0 group of operators on $H^s \times H^{s-1}$. It is a contraction semigroup on $H^1\times H^0$ for $t \geq 0$. It is uniformly bounded on $H^s\times H^{s-1}$ for $t\geq 0$. †

It a is not identically equal to zero then all solutions tend to zero as $t\to\infty$. The argument leading to the conservative solution v is called *Lasalle's Invariance Principal* in the dynamical systems community.

Iwasaki's Theorem 5.6.3. *If a is not identically equal to zero and the Cauchy data belong to $H^1\times L^2$, then the solution of Theorem 5.6.1 satisfies,*

$$\lim_{t\to\infty}\; \int_M |u_t|^2 \;+\; |d_x u|^2 \;+\; |u|^2\; dx \;=\; 0\,. \tag{5.6.8}$$

Proof. Since the integral on the left in (5.6.8) is a decreasing function of time one has

$$\int_M |u_t|^2 \;+\; |du(x)^2| \;+\; |u|^2\; dv \quad\to\quad E_\infty \;\geq\; 0\,, \tag{5.6.9}$$

monotonically as $t\to\infty$.

In the regularisation operator choose $j\in C_0^\infty(]0,1[)$ and define u^δ as above. The uniform in time *a priori* estimates imply that for $j=0,1$,

$$\lim_{\delta\to 0}\; \sup_{t\geq 0}\; \big\|\partial_t^j u^\delta(t) \;-\; \partial_t^j u(t)\big\|_{H^{1-j}(M)} \;\;=\;\; 0\,,$$

† It is contractive on a norm $\big(\|(1-\Delta+a)^{s/2}u_t\|^2_{L^2} + \|(1-\Delta+a)^{s/2}u\|^2_{H^1}\big)^2$ equivalent to the standard norm.

Thus, it suffices to prove the result for u^δ. Therefore it suffices to consider solutions satisfying,

$$\forall\, j,s \quad \partial_t^j u \;\in\; L^\infty\big([0,\infty[\;;\; H^s(M)\big)\,.$$

Let $v^n(t) := u(t+n)$. Ascoli's theorem implies that there is a subsequence so that, $v^{n_k} \;\to\; v$ uniformly with all derivatives on the compact subsets of $[0,\infty[\times M$. Then, (5.6.9) implies that for all $t \ge 0$,

$$\int_M |v_t|^2 \;+\; |dv(x)^2| \;+\; |v|^2 \; dv(x) \;=\; E_\infty \;\ge\; 0\,. \tag{5.6.11}$$

Therefore (5.6.7) implies that for all $t \ge 0$,

$$\int a(x)\,|v_t|^2\; dx \;=\; 0\,. \tag{5.6.12}$$

It follows that $a(x)\, v_t(t,x) = 0$. Therefore for $t \ge 0$, v satisfies the wave equation $v_{tt} - \Delta\, v = 0$. The idea of the next argument dates at least to Carleman.

Expand v in terms of

Introduce orthogonal eigenfunctions, ϕ_m, of Δ,

$$(1-\Delta)\,\phi_m = \lambda_m^2\,\phi_m\,, \qquad 1<\lambda_m<\lambda_{m+1}, \qquad \|\phi_m\|_{L^2(M)} \;=\; 1\,. \tag{5.6.13}$$

Then,

$$v \;=\; \sum_{m,\pm} a_{m,\pm}\; e^{\pm it\lambda_m}\; \phi_m(x)\,,$$

with rapidly decreasing fourier coefficients $a_{m,\pm} \ne 0$.

Since $v_t = 0$ on Ω, one has for $x \in \Omega$,

$$0 \;=\; \lim_{T\to\infty}\; \frac{1}{T}\int_0^T e^{\mp it\lambda_m}\; v(t,x)\; dt \;=\; a_{m,\pm}\phi_m(x)\,.$$

If one had $E_\infty > 0$, there would be a nonvanishing Fourier coefficient $a_{m,\pm}$. For that coefficient it would follow that $\phi_m = 0$ on Ω.

Apply the unique continuation principal that asserts that solutions of smooth homogeneous linear scalar second order order elliptic equations which vanish on an open set must vanish identically. The equation is $(1-\lambda_m^2-\Delta)\phi_m = 0$ and the conclusion is $\phi_m = 0$. This contradicts $\|\phi_m\|_{L^2} = 1$.

Thus one cannot have $E_\infty > 0$ and (5.6.8) is proved. ∎

Our application of the propagation of singularities theorem is to find necessary and sufficient conditions on $a(x)$ so that solutions decay exponentially in time. The next result shows the equivalence of several notions of decay.

Proposition 5.6.4. *The following are equivalent.*

1. *For each $(f,g) \in H^1 \times H^0$ there are constants $m(f,g) > 0$ and $\gamma(f,g) > 0$ so that*

$$\forall\, t \ge 0, \quad \big\|S(t)(f,g)\big\|_{H^1\times H^0} \;\le\; m\, e^{-\gamma t}\,.$$

2. *There is are constants $M > 0$ and $\Gamma > 0$ so that*

$$\forall\, t\ge 0,\;\; \forall\,(f,g)\in H^1\times H^0, \quad \big\|S(t)(f,g)\big\|_{H^1\times H^0} \;\le\; M\, e^{-\Gamma t}\,\big\|(f,g)\big\|_{H^1\times H^0}\,.$$

3. *There is a* $T > 0$ *so that* $\|S(T)\| < 1$.

4. *For each* $t > 0$ *the spectral radius of* $S(t)$ *is strictly smaller than 1.*

Proof. **1** $\Rightarrow$ **2**. Define

$$\Omega_n \ := \ \Big\{(f,g) \in H^1 \times H^0 \ : \ \forall\, t \geq 0, \ \ \big\|S(t)(f,g)\big\|_{H^1\times H^0} \ \leq \ n\, e^{-t/n}\, \big\|(f,g)\big\|_{H^1\times H^0}\Big\}.$$

Then Ω_n is closed. By hypothesis $\cup_n \Omega_n = H^1 \times H^0$. The Baire Category Theorem implies that there is an n so that Ω_n contains a ball $B_r(U)$ with positive radius.

Therefore $S(t)$ has uniform exponential decay on $U + B_r(0)$. By linearity there is uniform exponential decay on $B_r(0)$ which is the desired conclustion.

2 $\Rightarrow$ **1**. Immediate

2 $\Leftrightarrow$ **3**. Immediate

3 $\Leftrightarrow$ **4**. Follows from the formula for the spectral radius, ρ,

$$\rho \ = \ \lim_{n\to\infty} \big\|S(t)^n\big\|^{1/n} \ = \ \lim_{n\to\infty} \big\|S(nt)\big\|^{1/n}. \qquad ∎$$

Recall that the prinicipal symbol of L is denoted h. The bicharacteristics are integral curves of of the hamilton field of h along which $h = 0$. Vanishing h is the condition that the curve lies in the characteristic variety. The projections of these curves on $\mathbb{R} \times M$ are exactly the geodesics of M traversed at constant speed. Over a point (t, x, ξ) there pass two bichararcteristics, one for each of the two roots $\tau = \pm|\xi|$. The projections of the bicharacteristics on space time are the same geodesic traversed in opposite directions.

The next result gives a necessary and sufficient condition for the equivalent conditions of Proposition 5.6.3 to hold.

Theorem 5.6.5 [Rauch-Taylor]. *The following are equivalent*

1. $\|S(T)\|^2_{H^1\times H^0} < 1$.

2. *Each geodesic of length* T *passes through the set* $\{a > 0\}$.

In particular, solutions decay exponentially in the sense of Proposition 5.6.3 if and only if there is a $T > 0$ *satisfying* **2**.

Proof. **1** $\Rightarrow$ **2**. We show that if **2** is violated, then, $\|S(T)\|_{H^1\times H^0} = 1$.

There is a unit speed geodesic so that $\gamma([0,T])$ does not intersect $\{a > 0\}$. We follow the strategy of Ralston which is to construct smooth solutions on $[0,T] \times M$ which are concentrated on a small neighborhood of γ and therefore have little energy decay.

For each t, define $\xi(t) \in T^*_{\gamma(t)}(M)$ to be the dual vector satisfying

$$\xi(t)(w) \ = \ \langle \gamma'(t)\,,\, w\rangle\,, \qquad \forall w \in T_{\gamma(t)}(M)\,.$$

Then for one of the choices of sign $\Gamma_\pm := (t, \gamma(t), \pm|\xi|, \xi(t))$ is a bicharacteristic of our differential operator which lies over γ. Call that bicharacteristic, Γ.

Lemma 5.6.6. *There is a solution of* $Lu = 0$ *whose wavefront set coincides with the single bicharacteristic* Γ.

Proof of Lemma 5.6.6. Choose $f \in \mathcal{D}'(M)$ such that $WF(f) = (\gamma(0), \mathbb{R}_+\xi(0))$. Consider the solution of $Lv = 0$ with $v(0) = f$, $v_t(0) = 0$.

Over the point $(0, \gamma(0), \xi(0))$ there are two points in the characteritic variety, $(0, \gamma(0), \pm|\xi|, \xi(0))$. Denote by $\Gamma_\pm(t)$ the corresponding bicharacteristics . They pass over the geodesic γ traversed in opposite directions.

The Global Propagation of Singularities Theorem expresses

$$v \;=\; v_+ \;+\; v_- \;+\; C^\infty\,, \qquad L\,v_\pm \in C^\infty \qquad WF\,v_\pm \subset \Gamma_\pm\,.$$

Suppose that Γ_+ is the bicharacteristic passing over the geodesic γ. Define $w \in C^\infty$ to be the solution of

$$L\,w \;=\; -L\,v_+\,, \qquad w|_{t=0} = 0\,, \quad w_t|_{t=0} = 0\,.$$

Then $u := v_+ + w$ satisifies the conditions of the Lemma. If Γ_- passes over γ, simply change the plus to a minus in this argument. ∎

To complete the proof that **1** $\Rightarrow$ **2**, it is sufficient to prove the following lemma.

Lemma 5.6.7. *If* **2** *is violated, then for any $\epsilon > 0$ there is a smooth solution with initial energy equal to one and with*

$$\int_0^T \int_M a(x)\,|u_t|^2(t,x)\;dx\,dt \;<\; \epsilon\,. \tag{5.6.14}$$

Proof of Lemma 5.6.7. Since the solution of Lemma 5.6.6 is not smooth, using the fact that the points of $\mathrm{Char}\,L$ are noncharacteristic for ∂_t^j as in the proof of Proposition 2.6.2, it follows that there is a $j \geq 0$ so that,

$$\partial_t^j u \;\notin\; H^1(]0,T[\times M)\,.$$

Thus replacing u by $\partial_t^j u$ we may suppose that

$$u \;\notin\; H^1(]0,T[\times M)\,. \tag{5.6.15}$$

Define

$$v^\delta \;:=\; \frac{u^\delta}{\int |u_t^\delta(0,x)|^2 + |d_x u^\delta(0,x)|^2 + |u^\delta(0,x)|^2\;dv(x)}\,.$$

Then v^δ is a solution with initial energy equal to one so integrating over points where $a \leq \epsilon/2T$ yields,

$$\int_{[0,T]\times\{a\leq\epsilon/2T\}} a(x)\;|v_t^\delta|^2\;dv(x)\,dt \;\leq\; \epsilon/2\,. \tag{5.6.16}$$

Thanks to (5.6.15), it follows that

$$\lim_{\delta\to 0}\;\|u^\delta\|_{H^1(]0,T[\times M)} \;=\; \infty\,.$$

The basic energy estimate shows that

$$\|u^\delta\|_{H^1(]0,T[\times M)} \;\leq\; C\int |u_t^\delta(0,x)|^2 + |d_x u^\delta(0,x)|^2 + |u^\delta|^2\;dv(x)\,,$$

so

$$\lim_{\delta\to 0} \int |u_t^\delta(0,x)|^2 + |d_x u^\delta(0,x)|^2 + |u^\delta|^2 \, dv(x) \ = \ \infty \,. \tag{5.5.17}$$

The solution u is smooth on the complement of the geodesic which is the projection, $\pi(\Gamma)$, of Γ on space time. It follows from (5.6.17) that for all α, $\partial^\alpha_{t,x} v^\delta$ converges uniformly to zero on compact subsets of $(\mathbb{R}\times M)\setminus \pi(\Gamma)$. Therefore, for δ small,

$$\int_{[0,T]\times\{a\geq\epsilon/2T\}} a(x)\ |v_t^\delta|^2 \ dv(x)\, dt \ \leq\ \epsilon/2\,. \tag{5.6.18}$$

Combining (5.6.16) and (5.6.18) shows that v^δ satisfies the conditions of the Lemma for δ sufficiently small. ∎

This completes the proof that **1** $\Rightarrow$ **2**. To prove that **2** $\Rightarrow$ **1**, the key step is the following Lemma.

Lemma 5.6.8. *Suppose that every geodesic of length T passes through the set $\{a>0\}$. Then if $u\in L^2_{\rm loc}(\mathbb{R}\times M)$ satisfies $Lu=0$ and*

$$u_t \ \in\ L^2_{\rm loc}\big(]0,T[\times\{a>0\}\big), \tag{5.6.19}$$

then $u\in H^1$ on a neighborhood of $\{t=T/2\}\times M$.

Proof of Lemma 5.6.8. It suffices to show that for all $(T/2,x,\tau,\xi)$ that u belongs to H^1 microlocally at $(T/2,x,\tau,\xi)$.

Since $Lu=0$, the microlocal elliptic regularity Theorem 4.6.1 implies that for all s, u is microlocally H^s on the complement of $\operatorname{Char} L$. Thus it suffices to consider $(T/2,x,\pm|\xi|,\xi)\in \operatorname{Char} L$.

The Global Propagation of Singularities Theorem expresses

$$u \ =\ u_+ \ +\ u_- \ +\ C^\infty\,, \qquad WF\, u_\pm \ \subset\ \{\tau=\pm|\xi|\}\,.$$

It suffices to show that for all $x,\xi,\pm$,

$$u_\pm \ \in\ H^1(T/2,x,\pm|\xi|,\xi)\,. \tag{5.6.20}$$

We treat the case $+$, the other being analogous.

Fix x,ξ and denote by $t\mapsto\Gamma(t)$ the bicharacteristic with $\Gamma(T/2)=(T/2,x,|\xi|,\xi)$. By **2**, there is a $\underline{t}\in]0,T[$ so that $\Gamma(\underline{t})=(\underline{t},\underline{x},|\underline{\xi}|,\underline{\xi})$ and $a(\underline{x})>0$.

Since $\int_{[0,T]\times M} a(x)\,|u_t|^2\,dt\,dv(x)<\infty$, we know that $\partial_t u$ is square integrable on a neighborhood of $(\underline{t},\underline{x})$. Since ∂_t is elliptic at $(\underline{t},\underline{x},|\underline{\xi}|,\underline{\xi})$, the Microlocal Elliptic Regularity Theorem implies that $u\in H^1(\underline{t},\underline{x},|\underline{\xi}|,\underline{\xi})$. Since $(\underline{t},\underline{x},|\underline{\xi}|,\underline{\xi})\notin WF(u_-)$ it follows that $u_+\in H^1(\underline{t},\underline{x},|\underline{\xi}|,\underline{\xi})$.

Theorem 5.5.7**ii.** implies that $WF_1(u_+)$ is invariant under the hamilton flow with hamiltonian $\tau-|\xi|$. The image of the point $(\underline{t},\underline{x},|\underline{\xi}|,\underline{\xi})$ at time $T/2$ is the point $(T/2,x,|\xi|,\xi)$. Therefore, $u_+\in H^1(T/2,x,|\xi|,\xi)$. This completes the proof of Lemma 5.6.8. ∎

Lemma 5.6.9. *If $Lu=0$ and there is a time $\underline{t}$ so that u is H^1 on a neighborhood of $\{t=\underline{t}\}\times M$ then for $j=0,1$, $u\in C^j\big(\mathbb{R}\,;\,H^{1-j}\big)$.*

Proof of Lemma 5.6.9. It is given that there is a $\delta>0$ so that $u\in H^1([\underline{t}-\delta,\underline{t}+\delta]\times M)$. The energy decay law show that for all $t>\underline{t}+\delta$, $s\in]\underline{t}-\delta,\underline{t}+\delta[$ and smooth solutions v to $Lv=0$,

$$\int_M |v_t(t)|^2+|d_xv(t)|^2+|v(t)|^2\ dv(x)\ \leq\ \int_M |v_t(s)|^2+|d_xv(s)|^2+|v(s)|^2\ dv(x)\,.$$

Therefore

$$
\begin{aligned}
&\sup_{t\geq \underline{t}+\delta}\ \int_M |v_t(t)|^2+|d_xv(t)|^2+|v(t)|^2\ dv(x) \\
&\quad\leq\ \frac{1}{4\delta}\int_{\underline{t}-\delta/2}^{\underline{t}+\delta/2}\int_M |v_t(s)|^2+|d_xv(s)|^2+|v(s)|^2\ dv(x)\,ds\ \leq\ C\,\big\|v\big\|_{H^1([\underline{t}-\delta/2,\underline{t}+\delta/2]\times M)}
\end{aligned},
$$

with C independent of v.

This inequality applied to the time regularizations u^δ shows that u^δ is a Cauchy sequence in $C^j([\underline{t}+\delta,\infty[\,;\,H^{1-j}(M))$ for $j=0,1$. Since the limit of the u^δ in the sense of distributions is u, it follows that $u\in C^j([\underline{t}+\delta,\infty[\,;\,H^{1-j}(M))$ for $j=0,1$. This suffices to prove the Lemma. ∎

This lemma completes the proof of Lemma 5.6.8. At the same time it can be applied to strengthen the conclusion of Lemma 5.6.8 to $u\in C^j(\mathbb{R}\,;\,H^{1-j}(M))$ for $j=0,1$.

Denote by $\mathcal{B}$ the set of $L^2([0,T]\times M)$ solutions to $Lu=0$ satisfying (5.6.19). It is a Hilbert space with norm squared equal to

$$
\int_{[0,t]\times M}|u|^2\ +\ a(x)\,|u_t|^2\ dt\,dv(x)\,.
$$

The strengthening of Lemma 5.6.8 shows that if T satisfies **2**, then

$$
\mathcal{B}\quad\subset\quad \cap_{j=0,1}\ C^j([0,T]\,;\,H^{1-j}(M))\,.
$$

The graph of the inclusion map is closed. Therefore, the Closed Graph Theorem implies that the inclusion is continuous. Thus, there is a constant C so that

$$
\forall\,u\in\mathcal{B},\quad \int_M |u_t(0)|^2\ +\ |d_xu(0)|^2\ +\ |u(0)|^2\ dv(x)\ \leq\ C\int_{[0,T]\times M}|u|^2\ +\ a(x)\,|u_t|^2\ dt\,dv(x)\,. \tag{5.6.21}
$$

Lemma 5.6.10. *Suppose that* **2** *is satisfied. Define*

$$
K(T)\ =\ \Big\{u\in L^2([0,T]\times M)\ :\ Lu=0\,,\quad\text{and}\quad a(x)\,u_t=0\Big\}\,.
$$

Then, K is finite dimensional subspace of finite energy solutions. $K(T)=\{0\}$ if and only if there is a constant C so that

$$
\forall\,u\in\mathcal{B},\quad \int_M |u_t(0)|^2\ +\ |d_xu(0)|^2\ +\ |u(0)|^2\ dv(x)\ \leq\ C\int_{[0,T]\times M}a(x)\,|u_t|^2\ dt\,dv(x)\,. \tag{5.6.22}
$$

Proof. On the space K, (5.6.21) implies the inequality

$$
\int_M |u_t(0)|^2\ +\ |d_xu(0)|^2\ +\ |u(0)|^2\ dv(x)\ \leq\ C\int_{[0,T]\times M}|u|^2\ dt\,dv(x)\,.
$$

Thus the unit ball in K is precompact proving that K is finite dimensional. At the same time one sees that the solutions have finite energy.

If K is not trivial then (5.6.22) can not hold for the nonzero elements of K, no matter how large C is chosen.

On the other hand, if (5.6.22) does not hold for any C then there is a sequence u^n with

$$\int_{[0,T]\times M} |u^n|^2 \ + \ a(x)\,|u^n_t|^2 \ dt\,dv(x) \ = \ 1\,, \quad \text{and} \quad \int_{[0,T]\times M} a(x)\,|u^n_t|^2 \ dt\,dv(x) \ \to \ 0\,.$$

By (5.6.21) the sequence is compact in $L^2([0,T]\times M)$. Passing to a convergent subsequence we may assume that $u^n \to u$ in $L^2([0,T]\times M)$. The limit satisfies

$$\|u\|_{L^2([0,T]\times M)} \ = \ 1\,, \qquad Lu \ = \ 0\,, \qquad a(x)u_t \ = \ 0\,.$$

That is, u is a nontrivial element of K. ∎

The last, step in the proof of sufficiency is to show that $K(T) = \{0\}$ as soon as T satisfies the geometric condition. The Lemma shows that the elements of K have finite energy. It then follows from the definition of K that $v \in K \Rightarrow \partial_t v \in K$. Since $\partial_t \in \mathrm{Hom}\,(K(T))$ and $K(T)$ is finite dimensional space, it follows that if $K(T) \neq \{0\}$ then there is a $v \in K \setminus \{0\}$ and $\lambda \in \mathbb{C}$ so that $\partial_t v = \lambda v$.

Thus, $v = e^{\lambda t}\phi(x)$ with $\phi \neq 0$, satisfies $Lv = 0$ and $a(x)v_t = 0$ for $t \in]0,T[$ and therefore for all $t \in \mathbb{R}$. Thus v has constant energy violating Theorem 5.6.3. This contradiction shows that $K(T) = \{0\}$.

Therefore inequality (5.6.22) holds. That inequality is equivalent to

$$\|S(T)\|^2_{H^1\times H^0} \ \leq \ \frac{1}{1+C} \ < \ 1\,.$$

This proves **1**. ∎

Exercise 5.6.1. *Show that when* **2** *of Theorem 5.6.5 is violated, then for any N and ϵ there is a linear space V of finite energy Cauchy data so that $\dim V \geq N$ and for all $(f,g) \in V$,*

$$\|S(T)(f,g)\|_{H^1\times H^0} \ \geq \ (1-\epsilon)\|f,g\|\,.$$

Hint. Find N nearby but distinct geodesics which nearly miss $\{a > 0\}$. **Discussion.** This exercise shows that $\{|z| = 1\}$ meets the essential spectrum of $S(T)$.

Appendix 5.I. Hamilton-Jacobi Theory for the Eikonal Equation

§5.I.1. Introduction

In this section the scalar real nonlinear first order partial differential equation

$$F(y, d\phi) = 0 \tag{5.I.1}$$

is solved by reducing to the solution of ordinary differential equations. The function $F(y, \eta)$ is assumed to be a smooth real valued function of its arguments on an open subset of $\mathbb{R}^n_y \times \mathbb{R}^n_\eta$. Written out the equation takes the form

$$F\Big(y_1, \ldots, y_n, \frac{\partial \phi(y)}{\partial y_1}, \ldots, \frac{\partial \phi(y)}{\partial y_n}\Big) = 0\,. \tag{5.I.2}$$

In applications, the function ϕ is usually either a phase function or a function one of whose level sets represents a wave front.

Examples. Three classical examples from optics are the equations

$$|\nabla_x \psi|^2 = 1\,, \qquad F(x, \xi) = |\xi|^2 - 1\,, \tag{5.I.3}$$

$$\phi_t^2 - |\nabla_x \phi|^2 = 0\,, \qquad F(t, x, \tau, \xi) = \tau^2 - |\xi|^2 \tag{5.I.4}$$

and

$$\phi_t^2 - c(x)^2\, |\nabla_x \phi|^2 = 0\,, \qquad F(t, x, \tau, \xi) = \tau^2 - c(t, x)^2\, |\xi|^2\,, \quad c > 0\,. \tag{5.I.5}$$

The first describes solutions of the second which have the special form $\phi(t, x) = t \pm \psi(x)$. For equation (5.I.5) the rays

bend or refract in a medium of variable speed of propagation $c(t, x)$. When $c(t, x)$ is independent of t, Solutions of (5.I.5) which are of the form $t \pm \psi(x)$ lead to a generalization of (5.I.3),

$$|\nabla_x \psi|^2 = 1/c(x)^2 := n^2(x)\,, \qquad F(x, \xi) = |\xi|^2 - n^2(x)\,. \tag{5.I.6}$$

The next three examples exhibit explicit solutions.

Example. Seek solutions of (5.I.4) which are linear functions of the coordinates, $\phi(t, x) = \tau t + \xi.x$. It is a solution if and only if $\tau = \pm|\xi|$. Thus, for any linear initial function $g(x) = \xi.x$ with $\xi \neq 0$, this yields two solutions of (5.I.4) with $\phi(0, x) = g$. The solutions come from two determinations of ϕ_t from ϕ_x. There is one solution for each of

$$\phi_t = \pm\, |\nabla_x \phi|\,. \tag{5.I.7}$$

Example. The linear solutions of (5.I.3) are precisely the functions $\xi.x$ with $|\xi| = 1$. If one imposes initial data $\psi(0, x_2, \ldots, x_n) = \xi_2 x_2 + \ldots + \xi_n x_n$ then if $|\xi_2, \ldots, \xi_n| < 1$ there are two linear solutions given by $\xi_1 = \pm(1 - |\xi_2, \ldots, \xi_n|^2)^{1/2}$. If $|\xi_2, \ldots, \xi_n| > 1$ there are no solutions, since the equation (5.I.1) cannot be solved even at a single point of the initial surface.

Example. The solutions of (5.I.3) which depend only on $r := |x|$ are of the form $c \pm r$ with constant c. These functions measure signed distance to the sphere $|x| = c$. More generally, if M is a piece of hypersurface in $\mathbb{R}^n$ and $\psi(x)$ is the signed distance to M then ψ is well defined locally and solves (5.I.3). The case where M is a sphere shows that the solution need not exist globally as there is a singularity at $x = 0$.

Example. The spherically symmetric solutions $\phi(t, r)$ of (5.I.4) are functions $f(t \pm |x|)$ defined for $x \neq 0$. The level surfaces of ϕ are either outgoing or incoming spheres. The incoming (resp. ouotgoing) solutions degenerate to a point in finite positive (resp. negative) time.

Example. If $c = c(r)$ the spherically symmetric solutions $\phi(t, r)$ of (5.I.5) are the solutions of the equations

$$\big(\partial_t \pm c(r)\,\partial_r\big)\phi = 0\,.$$

The level surfaces of ϕ are spheres which move outward for the plus sign and inward with the minus sign. The speed $c(r)$ depends on the position. The integral curves of the vector fields $\partial_t \pm c(r)\partial_r$ describe refractive effects.

§5.I.2. Determining the germ of ϕ at the initial manifold

Consider the initial value problem defined by equation (1) with initial data

$$\phi|_M = g\,. \tag{5.I.8}$$

Here M is a hypersurface in $\mathbb{R}^n$ and g is smooth on M. Differentiating (8) tangent to M determines $n-1$ components of $d\phi$. Equivalently, knowing $\phi|_M$ determines the restriction of $d\phi$ to the tangent space of M,

$$d\phi\big|_{TM} = dg\,. \tag{5.I.9}$$

The differential equation (5.I.1) must be used to determine the remaining component of $d\phi$. The next example is a generalisation of (5.I.7).

Example. Consider equation (5.I.4) with the initial condition

$$\phi\big|_{t=0} = g(x)\,, \qquad dg \neq 0\,. \tag{5.I.10}$$

In this case $M = \{t = 0\}$. At $t = 0$, $\partial\phi/\partial x_j = \partial g/\partial x_j$ are known functions of x. The time derivative must be found by solving (5.I.4) for ϕ_t yielding

$$\frac{\partial\phi}{\partial t} = \pm\,|\nabla_x\phi| = \pm\,|\nabla_x g|\,. \tag{5.I.11}$$

If $dg = 0$ then the equation $F(t, x, \phi_t, \phi_x) = 0$ need not be smoothly solvable for ϕ_t as a function of the other variables. This is the case for example near $x = 0 \in \mathbb{R}^n$ with $g(x) = |x|^2$.

More generally, consider equation (5.I.1) and $M = \{y_1 = 0\} \subset \mathbb{R}^n$. The initial data (5.I.8) determine $\partial\phi/\partial y_2, \ldots, \partial\phi/\partial y_n$ along M. The missing derivative $\partial\phi/\partial y_1$ must be determined by solving the equation (5.I.1). The preceding examples show that there may be multiple solutions or no solution at all. In favorable cases, picking a solution $\partial\phi/\partial y_1$ at one point $\underline{x} \in M$ uniquely determines $d\phi$ locally on M.

Infinitesimal Determination Lemma 5.I.1. *Suppose that $g \in C^\infty(M)$, $\underline{y} \in M$, and $\underline{\eta} \in T^*_{\underline{x}}(\mathbb{R}^n)$ satisfy*

$$F(\underline{y}, \underline{\eta}) = 0\,, \qquad \text{and} \qquad \underline{\eta}|_{T_{\underline{y}}M} = dg(\underline{y})\,. \tag{5.I.12}$$

Suppose in addition that $\nabla_\xi F(\underline{y},\underline{\eta})$ is not tangent to M at $\underline{y}$. Then on a neighborhood $\omega \subset M$ of $\underline{y}$ in M, there is one and only one smooth function $\omega \ni y \mapsto \eta(y)$ which satisfies

$$F(y,\eta(y)) = 0 \quad \text{on} \quad M\,, \qquad \eta|_{TM} = dg\,, \qquad \text{and} \qquad \eta(\underline{y}) = \underline{\eta}\,. \tag{5.I.13}$$

In particular, a C^1 solution of the initial value problem (5.I.1),(5.I.8) which satisfies $d\phi(\underline{y}) = \underline{\eta}$ must satisfy $d\phi(y) = \eta(y)$ for all $y \in \omega$.

Proof. Introduce coordinates near $\underline{y}$ so that M is locally given by $y_1 = 0$. Then

$$\underline{\eta} = (\underline{\eta}_1, \partial g(\underline{y})/\partial y_2, \ldots, \partial g(\underline{y})/\partial y_n)\,.$$

Let with $y' := (y_2, \;\ldots\;, y_n)$. Then,

$$\eta(y') \;=\; \Big(\eta_1(y'), \frac{\partial g(y')}{\partial y_2}, \ldots, \frac{\partial g(y')}{\partial y_n}\Big)$$

must satisfy,

$$F\Big(0, y', \eta_1, \frac{\partial g(y')}{\partial y_2}, \ldots, \frac{\partial g(y')}{\partial y_n}\Big) = 0\,, \qquad \eta_1(\underline{x}) = \underline{\eta}_1\,. \tag{5.I.14}$$

The implicit function theorem shows that this uniquely determines $\eta_1(y')$ on a neighborhood of $\underline{y}$ provided that $0 \neq \partial F/\partial \eta_1$. This is equivalent to $\nabla_\eta F$ not being tangent to M. ∎

Remark. If M is connected and $\nabla_\eta F$ is nowhere tangent to M , then connecting $\underline{y}$ to an arbitrary point of M by an arc of a continuous curve, then covering the arc by overlapping neighborhoods ω extends the determination lemma to all of M.

Remark. The linearization of the equation (5.I.1) at a solution ϕ is the partial differential operator

$$\sum_\mu \frac{\partial F}{\partial \eta_\mu} \frac{\partial}{\partial y_\mu}\,. \tag{5.I.15}$$

When $\nabla_\eta F$ is not tangent to M, the surface M is noncharacteristic for the linearized operator. That is equivalent to the surface M being noncharacteristic along the solution ϕ of (5.I.1) (see [Rauch,*Partial Differential Equations*, §1.5]). In that case, not only is $d\phi$ determined on M, but so are all the partial derivatives of ϕ.

§**5.I.3. Propagation laws for $\phi, d\phi$**

To analyse equation (5.I.1), differentiate with respect to y_ν. This is an example of a general strategy whereby differentiating a fully nonlinear equation yields a quasilinear equation for its derivatives. In the case of a first order real scalar equation, this simple idea solves the problem. Differentiating (5.I.1) with respect to y_ν for $1 \le \nu \le n$ yields the n equations,

$$\frac{\partial F}{\partial y_\nu}(y, d\phi(y)) + \sum_\mu \frac{\partial F}{\partial \eta_\mu}(y, d\phi(y))\, \frac{\partial^2 \phi(y)}{\partial y_\nu \partial y_\mu} = 0\,, \qquad 1 \le \nu \le n\,. \tag{5.I.16}$$

Identity (5.I.16) shows that the derivative of $d\phi$ in the direction $\nabla_\eta F$ is equal to $-\nabla_y F$. One recognizes the y and η parts of the Hamilton vector field

$$V_F := \sum_\mu \; \frac{\partial F(y,\eta)}{\partial \eta_\mu} \frac{\partial}{\partial y_\mu} - \frac{\partial F(y,\eta)}{\partial y_\mu} \frac{\partial}{\partial \eta_\mu}\,. \tag{5.I.17}$$

associated to the hamiltonian F.

Propagation Lemma 5.I.2. *Suppose that ϕ is a solution of (1) and that $(y(s),\eta(s))$ is an integral curve of the vector field V_F, that is*

$$\frac{dy(s)}{ds} = \frac{\partial F}{\partial \eta}(y(s),\eta(s))\,, \qquad \frac{d\eta(s)}{ds} = -\frac{\partial F}{\partial y}(y(s),\eta(s))\,. \tag{5.I.18}$$

If $\eta(0) = d\phi(y(0))$, then $\eta(s) = d\phi(y(s))$ so long as $y([0,s])$ belongs to the domain on which ϕ satisfies (1). If the system (18) is expanded to include

$$\frac{d\rho(s)}{ds} = \sum_\mu \eta_\mu(s)\,\frac{\partial F}{\partial \eta_\mu}(y(s),\eta(s))\,, \qquad \rho(0) = \phi(y(0))\,, \tag{5.I.19}$$

then $\rho(s) = \phi(y(s))$ on the same interval of s.

Proof. Define $Y(s)$ as the solution of $dY/ds = -\nabla_\eta F(Y(s), d\phi(Y(s))$ with $Y(0) = y(0)$. Define $\Xi(s) := d\phi(Y(s))$. Then equation (5.I.16) is equivalent to $d\Xi/ds = -\nabla_y F(Y(s),\Xi(s))$. Thus $(Y(s),\Xi(s))$ solves the same initial value problem as $(y(s),\eta(s))$. By uniqueness, $(y(s),\eta(s)) = (Y(s),\Xi(s))$.

Similarly,

$$\frac{d\phi(y(s))}{ds} = \sum_\mu \frac{\partial\phi(y(s))}{\partial y_\mu}\,\frac{\partial y_\mu(s)}{ds} = \sum_\mu \eta_\mu(s)\,\frac{\partial F}{\partial \eta_\mu}(y(s),\eta(s))\,,$$

so $(y(s),\eta(s),\phi(y(s))$ solves the y,η,ρ system of ordinary differential equations and the last assertion of the proposition follows from uniqueness. ∎

Example. The integral curves of V in the case of equation (5.I.5) are solutions of the system of ordinary differential equations

$$\frac{dt}{ds} = 2\tau\,, \qquad \frac{dx}{ds} = -2c^2\xi\,, \qquad \frac{d\tau}{ds} = 0\,, \qquad \frac{d\xi}{ds} = 2\xi^2 c\,\frac{\partial c(x)}{\partial x}\,. \tag{5.I.20}$$

The velocity

$$\frac{dx}{dt} \;=\; \frac{dx/ds}{dt/ds} \;=\; \frac{-2c^2\xi}{2\,\tau} \;=\; \frac{-2c^2\xi}{\pm c|\xi|} \;=\; \mp c\,|\xi|.$$

This is equal to the group velocity associated to the root $\tau = \pm c|\xi|$. This gives three very distinct ways to arrive at the group velocity, stationary phase as in §1.3, the purely geometric construction in §2.4, and Hamilton-Jacobi Theory.

If you know the values of $d\phi(0,x)$, then the Proposition tells you that $d\phi(t(s),x(s)) = (\tau(s),\xi(s))$ where $(t(s),x(s),\tau(s),\xi(s))$ is a solution of (5.I.20). This allows you to compute $d\phi$ in $t>0$ from its values at $t=0$. The curves $(t(s),x(s),\tau(s),\xi(s))$ are called **bicharachteristic strips** or simply **bicharachteristics**, and their projections on (t,x) space are called **rays**. The bicharacteristics describe how the values of $d\phi$ are propagated along rays. The speed of the rays are equal to the local group velocity.

There are two intuitive ways to think of the integral curves $(y(s),\eta(s))$ The first is to note that $\eta(s) = d\phi(y(s))$ determines the tangent plane to the graph of ϕ at $y(s)$. One has a curve of tangent planes to the solution surface. I like to think of the level surfaces to ϕ as defining surfaces of constant phase and therefore $d\phi(y(s))$ describes how the surfaces of constant phase are positioned at $y(s)$ It can be viewed as giving an infinitesimal element of an oscillatory solution. Then the Proposition can viewed as a propagation law for infinitesimal oscillations. The example $F = \tau^2 - \xi_1^2 - 4\xi_2^2$ shows that the direction of propagation dx/dt need not be parallel to ξ (see Exercise 2.4.9).

Covering Lemma 5.I.3. *Suppose that the restriction of $d\phi$ to M is known and that $\nabla_\eta F(\underline{y}, d\phi(\underline{y}))$ is not tangent to M at $\underline{y}$. For $q \in M$ denote by $(y(s,q), \eta(s,q))$ the solution of (5.I.18) with $y(0) = q$, $\eta(0) = d\phi(q)$. Then the map $s, q \mapsto y(s,q)$ is a diffeomorphism of a neighborhood of $(0, \underline{y})$ in $\mathbb{R} \times M$ to a neighborhood of $\underline{x}$ in $\mathbb{R}^n$. Equivalently the family of rays $y(\cdot, q)$ parametrized by $q \in M$ simply covers a neighborhood of $\underline{y}$.*

Proof. Introduce local coordinates on M and consider the Jacobian matrix of the mapping y at $(s,q) = (0, \underline{y})$. The last $n-1$ columns of J span the tangent space of M. The first column is parallel to $dy/ds|_{(0,q)} = \nabla_\eta F(\underline{y}, \underline{\eta})$ which is not tangent to M. Thus the columns span $\mathbb{R}^n$ and the result follows from the Inverse Function Theorem. ∎

Main Theorem 5.I.4. *Suppose that data are given satisfying the following conditions.*
(i.) M is a hypersurface in $\mathbb{R}^n$.
(ii.) $g \in C^\infty(M)$.
(iii.) $(\underline{y}, \underline{\eta})$ satisfies $F(\underline{y}, \underline{\eta}) = 0$ and $(\sum_\mu \underline{\eta}_\mu \, dy_\mu)|_{T_{\underline{y}}(M)} = dg(\underline{y})$.
(iv.) $\nabla_\eta F(\underline{y}, \underline{\eta})$ is not tangent to M at $\underline{y}$.
Then, there is a smooth solution ϕ of (5.I.1) satisfying (5.I.17) on a neighborhood of $\underline{y}$ in M and $d\phi(\underline{y}) = \underline{\eta}$. Any two such solutions must coincide on a neighborhood of $\underline{y}$.

Proof. The previous results combine to prove uniqueness as follows. The Infinitesimal Determination Lemma determines $d\phi$ along M. Let $y(s,q), \eta(s,q), \rho(s,q)$ be the solutions of the system (5.I.18),(5.I.19). They are parametrized by $q \in M$. The Propagation Lemma implies that

$$\phi(y(s,q)) = \rho(s,q)\,. \tag{5.I.21}$$

The Covering Lemma shows that (5.I.21) uniquely determines ϕ on a neighborhood of $\underline{y}$.

To prove existence we show that the function defined by (5.I.21) on a neighborhood of $\underline{y}$ in $\mathbb{R}^n$ furnishes a solution.

The initial conditions (5.I.19) for ρ, show that ϕ defined by (5.I.21) satisfies $\phi|_M = g$.

The initial values $y(0,q), \eta(0,q)$ determined in the Infinitesimal Determination Lemma, satisfy $F(y,\eta) = 0$. The classic computation of conservation of energy shows that F is constant along solution curves of (5.I.18), which proves that $F(y(s,q), \eta(s,q)) = 0$. Thus, to prove that ϕ solves (1) it suffices to prove that

$$\frac{\partial \phi}{\partial y_\mu}(y(s,q)) = \eta_\mu(s,q)\,. \tag{5.I.22}$$

Define $\zeta(y) = (\zeta_1(y) \ldots, \zeta_n(y))$ on a neighborhood of $\underline{y}$ by

$$\zeta_\mu(x(s,q)) = \eta_\mu(s,q)\,. \tag{5.I.23}$$

The desired relation (5.I.22) reads

$$\frac{\partial \phi}{\partial y_\mu} = \zeta_\mu\,, \qquad 1 \le \mu \le n\,. \tag{5.I.24}$$

That is, the predicted value of the differential is the same as what one obtains by differentiating the predicted value of the function. Define a vector field on a neighborhood of $\underline{x}$ by

$$W := \sum_\mu \frac{\partial F}{\partial \eta_\mu}(y,\zeta(y))\,\frac{\partial}{\partial y_\mu}\,.$$

The defining relations for ϕ and ζ yield

$$W\,\phi = \sum_\mu \zeta_\mu(y)\,\frac{\partial F}{\partial \eta_\mu}(y,\zeta(y))\,, \tag{5.I.25}$$

$$W\,\zeta_\mu = -\frac{\partial F}{\partial y_\mu}(y,\zeta(y))\,, \qquad 1\le \mu\le n\,. \tag{5.I.26}$$

Differentiating (5.I.25) yields

$$\begin{aligned} W\,\frac{\partial\phi}{\partial y_\mu} &+ \Big(\sum_{\mu,\nu}\frac{\partial^2 F}{\partial\eta_\mu\partial y_\nu} + \frac{\partial^2 F}{\partial\eta_\mu\partial\eta_\nu}\,\frac{\partial\zeta_\nu}{\partial y_\mu}\Big)\,\frac{\partial\phi}{\partial y_\mu} \\ &= \sum_\mu\left(\frac{\partial\zeta_\mu}{\partial y_\nu}\,\frac{\partial F}{\partial\eta_\mu} + \zeta_\mu\sum_\nu\Big(\frac{\partial^2 F}{\partial\eta_\mu\partial y_\nu} + \frac{\partial^2 F}{\partial\eta_\mu\partial\eta_\nu}\,\frac{\partial\zeta_\nu}{\partial y_\mu}\Big)\right). \end{aligned}$$

The first sum on the right is equal to $W\eta_\mu$ so,

$$W\Big(\frac{\partial\phi}{\partial y_\mu} - \eta_\mu\Big) + \Big(\sum_{\mu,\nu}\frac{\partial^2 F}{\partial\eta_\mu\partial y_\nu} + \frac{\partial^2 F}{\partial\eta_\mu\partial\eta_\nu}\,\frac{\partial\zeta_\nu}{\partial y_\mu}\Big)\Big(\frac{\partial\phi}{\partial y_\mu} - \zeta_\mu\Big) = 0\,. \tag{5.I.27}$$

This homogeneous linear ordinary differential equation shows that $\partial\phi/\partial y_\mu - \zeta_\mu$ vanishes on a ray as soon as it vanishes at the foot of that ray on M. Thus it suffices to show that (5.I.24) is satisfied on M.

To verify (5.I.24) along M it suffices to find n linearly independent vectors v so that $v.d\phi = v.\zeta$. The Infinitesimal Determination Lemma gives $\zeta(x) = \eta(x)$ on M so in particular for v tangent to M, $v.\zeta = v.\eta = v.dg = v.d\phi$. For v equal to the field W, equation (5.I.25) yields $v.d\phi = \zeta.\nabla_\eta F(y,\zeta) = \zeta.v$. Hypothesis (iv.) shows that W is not tangent to M near $\underline{y}$ so the fields just checked span and the proof is complete. ∎

Remark. The reality of F and the hypothesis that $\nabla_\eta F$ is not tangent to M, shows that (5.I.1) is a strictly hyperbolic partial differential equation at $\underline{y}$ with timelike direction given by the conormal to M. One generalization of the Main Theorem is a local solvability result for strictly hyperbolic nonlinear initial value problems. From that perspective, the proof just presented is the **method of characteristics** from §1.1 for fully nonlinear scalar equations.

§5.I.4. The symplectic approach

The appearance of the hamilton field V_F shows that the construction has a link with symplectic geometry. The connection offers an alternative way to prove the Main Theorem. Begin by reinterpreting the Propagation Lemma. Introduce the graph of the differential $d\phi$

$$\Lambda := \big\{\,(y,\eta)\;:\;\eta = d\phi(y)\,\big\}\,.$$

This is a smooth n dimensional surface in the $2n$ dimensional space of (y,η). It is defined by the n equations

$$\eta_\mu = \frac{\partial \phi(y)}{\partial y_\mu}\,, \qquad 1\le \mu\le n\,. \tag{5.I.28}$$

Instead of looking for ϕ we look for Λ. Identify the space of (x,η) as the cotangent bundle of $\mathbb{R}^n$ with its symplectic form

$$\sigma := \sum_{\mu=1}^{n} d\eta_\mu \wedge dy_\mu\,.$$

The equality of mixed partials shows that Λ is **Lagrangian** in the sense that it is an n-manifold such that $\sigma(v,w)=0$ whenever v and w are tangent to Λ. Equation (5.I.1) is thus turned into the search for a Lagrangian manifold which lies in the set $\{(y,\eta)\,:\,F(y,\eta)=0\,\}$.

Tangency Lemma 5.I.5. *If ϕ is a solution of (5.I.1), then the vector field V_F is tangent to the surface Λ. Equivalently, an integral curve of V_F which touches Λ lies in Λ.*

Proof. Since the surface Λ lies in the level set $\{F=0\}$, one has $\langle dF, v\rangle=0$ for all tangent vectors v to Λ. The definition of hamilton field then shows that $\sigma(V_F,v)=\langle dF,v\rangle=0$ for all such v. Thus V_F belongs to the σ annihilator of the tangent space of Λ. Since Λ is Lagrangrian, this annihilator is the tangent space to Λ so V_F is tangent to Λ. ∎

An $n-1$ dimensional piece Λ_0 of Λ is given in the Infinitesimal Determination Lemma. Precisely

$$\Lambda_0 := \Big\{\,(y, d\phi(y))\,:\, y\in M\,\Big\} \tag{5.I.29}$$

is known from the initial data. Then one takes the union of the bicharacteristics through Λ_0 to define Λ. The next result follows from the Inverse Function Theorem.

Flow Out Lemma 5.I.6. *If Λ_0 is a smooth embedded $n-1$ dimensional surface in $2n$ dimensional (y,η) space which is nowhere tangent to the the vector field V_F, then locally the union of integral curves of V_F starting in Λ_0 defines a smooth n dimensional manifold.*

The next question to resolve is whether Λ so defined is a graph, that is whether there are smooth functions ζ_μ so that $\Lambda=\{(y,\zeta_1(y),\dots,\zeta_n(y))\}$. Denote by $\pi(x,\xi):=x$ the natural projection from (x,ξ) space to x space.

Clean Projection Lemma 5.I.7. *Suppose that Λ_0 is defined as in (5.I.29), $\underline{y}\in M$ and that Λ is constructed by the Flow Out Lemma. Then Λ is a graph on a neighborhood of $(\underline{y}, d\phi(\underline{y}))$ if and only if $\nabla_\eta F(\underline{y}, d\phi(\underline{y}))$ is not tangent to M at $\underline{y}$.*

Proof. The variety Λ is a smooth graph if and only if π is a diffeomorphism from a neighborhood of $(\underline{y}, d\phi(\underline{y}))$ in Λ to a neighborhood of $\underline{y}$ in $\mathbb{R}^n$. The Inverse Function Theorem implies that the necessary and sufficient condition is that the differential of π is an invertible map of tangent spaces. At a point $(y,\eta)\in\Lambda_0$ the tangent space is equal to $T_{y,\eta}(\Lambda_0)\oplus V_F(y,\eta)$. This implies that

$$d\pi\big(T_{(\underline{y},d\phi(\underline{y}))}\Lambda\big) = T_{\underline{y}}M \oplus \mathbb{R}\frac{\partial F}{\partial \eta}(\underline{y}, d\phi(\underline{y}))\,. \tag{5.I.30}$$

The right hand side of (30) is all of $\mathbb{R}^n$ if and only if $\partial F/\partial\eta$ is not tangent to M at $\underline{y}$, proving the Lemma. ∎

Alternate proof of the Main Theorem. To prove existence, it suffices to find a function ϕ defined on an $\mathbb{R}^n$ neighborhood of $\underline{y}$ so that

$$\frac{\partial \phi}{\partial y_\mu}(y) = \zeta_\mu(y)\,, \quad 1 \le \mu \le n\,, \qquad \phi(\underline{y}) = g(\underline{y})\,. \tag{5.I.31}$$

That such a function satisfies (5.I.8) follows from $d\phi|_{TM} = dg$. A necessary and sufficient condition for the existence of such a ϕ is the equality of mixed partials,

$$\frac{\partial \zeta_\mu}{\partial y_\nu} = \frac{\partial \zeta_\nu}{\partial y_\mu} \qquad \forall\, \mu \neq \nu\,. \tag{5.I.32}$$

It is not hard to show that (5.I.32) is equivalent to $\sigma|_{T\Lambda} = 0$.

Turn next to the proof that $\sigma|_{T\Lambda} = 0$. Since Λ is the flowout of Λ_0 and the flow by a hamiltonian vector field preserves the two form σ, it suffices to verify that $\sigma(v, w) = 0$ whenever v and w are tangent to Λ over a point of M.

The tangent space at such a point is the direct sum of the tangent space to Λ_0 and $\mathbb{R}\, V_F$. Thus it suffices to consider

$$v = v_0 + a\, V_F\,, \qquad w = w_0 + b\, V_F\,,$$

with v_0 and w_0 tangent to Λ_0 and real a, b. Use bilinearity to express $\sigma(v, w)$ as a sum of four terms. One has $\sigma(v_0, , w_0) = 0$ since this is the symplectic form of $T^*(M)$ evaluated at a pair of tangent vectors to the Lagrangian submanifold $\{\text{graph}\, dg\}$. The term $ab\, \sigma(V_F, V_F)$ vanishes since σ is antisymmetric. Finally, the cross terms are evaluated using the definition of hamiltonian vector fields,

$$\sigma(v_0, V_F) = \langle dF, v_0 \rangle\,, \qquad \sigma(V_F, w_0) = -\langle dF, w_0 \rangle\,. \tag{5.I.33}$$

Since Λ_0 lies in the set $\{F = 0\}$, the tangent vectors v_0 and w_0 are annihilated by dF, so the terms (5.I.33) vanish. This completes the proof. ∎

Chapter 6. The Nonlinear Cauchy Problem

§6.1. Introduction.

Nonlinear equations are classified according to the strength of the nonlinearity. The key criterion is what order terms in the equation are nonlinear.

A secondary condition is the growth of the nonlinear terms at infinity. When the functions which enter the nonlinear terms are uniformly bounded in absolute value, the behavior at infinity is not important.

Among the nonlinear equations in applications two sorts are most common. *Semilinear equations* are linear in their principal part. First order semilinear symmetric hyperbolic systems take the form

$$L(y, \partial_y)\, u + F(y,u) = f(y)\,, \qquad F(y,0) \;=\; 0 \tag{6.1.1}$$

where L is a symmetric hyperbolic operator, and the nonlinear function is a smooth map from $\mathbb{R}^{1+d} \times \mathbb{C}^N \to \mathbb{C}^N$ whose partial derivatives of all orders are uniformly bounded on sets of the form $\mathbb{R}^{1+d} \times K$, with compact $K \subset \mathbb{C}^N$. The derivatives are standard partial derivatives and not derivatives in the sense of complex analysis. A translation invariant semilinear equation with principal part equal to the d'alembertian is of the form

$$\Box u + F(u, u_t, \nabla_x u) \;=\; 0 \qquad F(0,0,0) \;=\; 0\,.$$

More strongly nonlinear, and typical of compressible inviscid fluid dynamics, are the quasilinear systems,

$$L(y, u, \partial_y)\, u \;=\; f(y)\,, \tag{6.1.2}$$

where

$$L(y,u,\partial_y) = \sum_{j=0}^{d} A_j(y,u)\, \partial_j \tag{6.1.3}$$

has coefficients which are smooth hermitian symmetric matrix valued functions with derivatives bounded on $\mathbb{R}^{1+d} \times K$ as above. A_0 is assumed uniformly positive on such sets.

For semilinear equations there is a natural local existence theorem requiring data in $H^s(\mathbb{R}^d)$ for some $s > d/2$. The theorem gives solutions which are continuous functions of time with values in $H^s(\mathbb{R}^d)$. This shows that the spaces $H^s(\mathbb{R}^d)$ with $s > d/2$, are natural configuration spaces for the dynamics. Once a solution belongs to such a space, it is bounded and continuous so that $F(y,u)$ is well defined, bounded, and continuous. Nonlinear ordinary differential equations are a special case, so for general problems one expects at most a local existence theorem.

For quasilinear equations, the local existence theorem requires an extra derivative, that is initial data in $H^s(\mathbb{R}^d)$ with $s > 1 + d/2$. Again the solution is a continuous functions of time with values in $H^s(\mathbb{R}^d)$. The classic example is Burgers' equation

$$u_t + u\, u_x \;=\; 0\,.$$

We treat first the semilinear case. The quasilinear case is treated in §6.6. The key step in the proof uses Schauder's Lemma, which shows that $u \mapsto F(y,u)$ takes $H^s(\mathbb{R}^d)$ to itself.

§6.2. Schauder's Lemma and Sobolev embedding.

The fact that H^s is invariant under nonlinear maps is closely connected to the Sobolev Embedding Theorems which assert that elements of the spaces H^s are in L^p for approriate $p(s,d)$. The simplest such L^p estimate for Sobolev spaces is the following.

Theorem 6.2.1 Sobolev. *If $s > d/2$, $H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ and*

$$\|w\|_{L^\infty(\mathbb{R}^d)} \ \leq \ C(s,d)\,\|w\|_{H^s(\mathbb{R}^d)}\,. \tag{6.2.1}$$

Proof. Inequality (6.2.1) for elements of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is an immediate consequence of the Fourier Inversion Formula,

$$w(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix.\xi}\ \hat{w}(\xi)\ d\xi = (2\pi)^{-d/2}\int_{\mathbb{R}^d} \frac{e^{-ix.\xi}}{<\xi>^s}\ <\xi>^s\ \hat{w}(\xi)\ d\xi\,.$$

The Schwarz inequality yields

$$|w(x)| \ \leq \ \Big|\Big|\frac{1}{<\xi>^s}\Big|\Big|_{L^2(\mathbb{R}^d)}\,\|w\|_{H^s(\mathbb{R}^d)}.$$

The first factor on the right is finite if and only if $s > d/2$.
For $w \in H^s$, choose $w^n \in \mathcal{S}$ with

$$w^n \to w \quad \text{in} \quad H^s\,, \qquad \|w_n\|_{H^s(\mathbb{R}^d)} \ \leq \ \|w\|_{H^s(\mathbb{R}^d)}\,.$$

Inequality (6.2.1), yields $\|w^n - w^m\|_{L^\infty(\mathbb{R}^d)} \leq C\,\|w^n - w^m\|_{H^s(\mathbb{R}^d)}$. Therefore the w^n converge uniformly on $\mathbb{R}^d$ to a continuous limit γ. Therefore $w^n \to \gamma$ in $\mathcal{D}'(\mathbb{R}^d)$ with $\|\gamma\|_{L^\infty} \leq C\,\|w\|_{H^s}$ However, $w^n \to w$ in H^s and therefore in $\mathcal{D}'$, so $w = \gamma$. This proves the continuity of w and the estimate (6.2.1). ∎

Consider the proof that $u \in H^2(\mathbb{R}^2)$ implies that $u^2 \in H^2(\mathbb{R}^2)$. One must show that u^2, $\partial(u^2)$, and, $\partial^2(u^2)$ are square integrable.
The function u^2 and its first derivative $\partial(u^2) = 2u\partial u$, are both the product of a bounded function and a square integrable function and so are in L^2. Compute the second derivative,

$$\partial^2(u^2) \ = \ u\partial^2 u + 2(\partial u)^2\,.$$

The first is a product $L^\infty \times L^2$ so is L^2. For the second, one needs to know that $\partial u \in L^4$. The fact that H^2 is invariant is equivalent to that Sobolev embedding.

Theorem 6.2.2 Schauder's Lemma. *Suppose that $G(x,u) \in C^\infty(\mathbb{R}^d \times \mathbb{C}^N\,;\mathbb{C}^N)$ such that $G(x,0) = 0$, and for all $|\alpha| \leq s+1$ and compact $K \subset \mathbb{C}^N$, $\partial^\alpha_{x,u}G \in L^\infty(\mathbb{R}^d \times K)$. Then the map $w \mapsto G(x,w)$ sends $H^s(\mathbb{R}^d)$ to itself provided $s > d/2$. The map is uniformly lipschitzian on bounded subsets of $H^s(\mathbb{R}^d)$.*

Proof of Schauder's Lemma for integer s. Consider $G = G(w)$. The case of G depending on x is uglier but requires no additional ideas. The key step is to estimate the H^s norm of $G(w)$ assuming that $w \in \mathcal{S}$, We prove that

$$\forall\,R,\ \ \exists\,C = C(R),\ \ \forall\,w \in \mathcal{S}(\mathbb{R}^d),\ \ \|w\|_{H^s(\mathbb{R}^d)} \leq R \quad \Longrightarrow \quad \|G(w)\|_{H^s(\mathbb{R}^d)} \leq C(R)\,.$$

This suffices to prove the first assertion of the theorem since for $w \in H^s$, choose $w^n \in \mathcal{S}$ with

$$w^n \to w \quad \text{in} \quad H^s\,, \qquad \|w_n\|_{H^s(\mathbb{R}^d)} \ \le\ \|w\|_{H^s(\mathbb{R}^d)}\,.$$

Then Sobolev's Theorem implies that w^n converges uniformly on $\mathbb{R}^d$ to w, so $G(w^n)$ converges uniformly to $G(w)$. In particular, $G(w^n) \to G(w)$ in $\mathcal{D}'(\mathbb{R}^d)$.

However, $G(w^n)$ is bounded in H^s so passing to a subsequence we may suppose that $G(w^n) \to v$ weakly in H^s. Therefore $G(w^n) \to v$ in $\mathcal{D}'(\mathbb{R}^d)$. Equating the $\mathcal{D}'$ limits proves that $G(w) \in H^s$.

For $\|w\|_{H^s} \le R$ there is a constant so that $\|w\|_{L^\infty} \le C$. Choose $\Gamma > 0$ so that,

$$\|w_j| \le C \quad \Longrightarrow \quad |G(w_1) - G(x_2)| \ \le\ \Gamma\,|w_1 - w_2|\,.$$

Apply with $w_2 = G(w_2) = 0$ to show that $\|G(w_1)\|_{L^2} \le C$.

It remains to estimate the derivatives of $G(w)$. For $w \in \mathcal{S}$, Leibniz' rule implies that $\partial_x^\beta G(w(x))$ with $|\beta| \le s$ is a finite sum of terms of the form

$$G^{(\gamma)}(w)\ \Pi_{j=1}^J\, \partial_x^{\alpha_j} w\,, \qquad |\gamma| = J\ \le\ |\beta|\,, \qquad \alpha_1 + \cdots + \alpha_J = \beta\,. \tag{6.2.2}$$

This is proved by induction on $|\beta|$. Increasing the order by one, the additional derivative either falls on the G term yielding an expression of the desired form with $|\gamma| = J$ increased by one, or on one of the factors in $\Pi\,\partial_x^{\alpha_j} w$ yielding an expression of the desired form with the same value of γ.

Sobolev's Theorem implies that

$$\|G^{(\gamma)}(w)\ \Pi_{j=1}^J\, \partial_x^{\alpha_j} w\|_{L^2}\ \le\ \|G^{(\gamma)}(w)\|_{L^\infty} \|\Pi_{j=1}^J\, \partial_x^{\alpha_j} w\|_{L^2}\ \le\ C(R)\, \|\Pi_{j=1}^J\, \partial_x^{\alpha_j} w\|_{L^2}\,. \tag{6.2.3}$$

Following [Rauch 1983] we use the Fourier transform to prove the key estimate.

Lemma 6.2.3. *If $s > d/2$ there is a constant $C = C(s,d)$ so that for all $w_j \in \mathcal{S}(\mathbb{R}^d)$ and all multiindices α_j with $s' := \sum |\alpha_j| \le s$,*

$$\|\,\Pi_{j=1}^J \partial_x^{\alpha_j} w_j\,\|_{L^2(\mathbb{R}^d)}\ \le\ C\,\Pi_{j=1}^J\, \|w_j\|_{H^s(\mathbb{R}^d)}\,.$$

Example. For $u \in H^2(\mathbb{R}^2)$, $(\partial u\, \partial u) \in L^2$ so $\partial u \in L^4$.

Proof. By Plancherel's theorem, it suffices to estimate the L^2 norm of the Fourier transform of the product. Set

$$g_i\ := <\xi>^{s-|\alpha_i|}\, \mathcal{F}\big(\partial_x^{\alpha_i} w_i\big)\,, \quad <\xi> :=\ (1+|\xi|^2)^{1/2}\,, \qquad \text{so,} \qquad \|g_i\|_{L^2} \le c\|w_i\|_{H^s}\,.$$

Compute,

$$\begin{aligned} \mathcal{F}\big(\Pi_{j=1}^J \partial_x^{\alpha_j} w_{l_j}\big)(\xi_1)\ &=\ \frac{g_1}{<\xi>^{s-|\alpha_1|}} * \frac{g_2}{<\xi>^{s-|\alpha_2|}} * \cdots * \frac{g_J}{<\xi>^{s-|\alpha_J|}}(\xi_1) \\ &= \int_{\mathbb{R}^{d(J-1)}} \frac{g_1(\xi_1-\xi_2)}{<\xi_1-\xi_2>^{s-|\alpha_1|}}\ \frac{g_2(\xi_2-\xi_3)}{<\xi_2-\xi_3>^{s-|\alpha_2|}} \cdots \frac{g_J(\xi_J)}{<\xi_j>^{s-|\alpha_J|}}\ d\xi_2 \ldots d\xi_J\,. \end{aligned} \tag{6.2.4}$$

For each ξ, at least one of the J numbers $<\xi_1 - \xi_2>, \ldots, <\xi_{J-1} - \xi_J>, <\xi_J>$ is maximal. Suppose it's the b^{th} number $<\xi_b - \xi_{b+1}>$ with the convention that $\xi_{J+1} \equiv 0$. Then since $\sum |\alpha_i| \le s$,

$$\langle \xi_b - \xi_{b+1}\rangle^{s-|\alpha_b|}\ \ge\ \langle \xi_b - \xi_{b+1}\rangle^{\sum_{j\ne b} |\alpha_j|}\ \ge\ \Pi_{j\ne b}\langle\, \xi_j - \xi_{j+1}\rangle^{|\alpha_j|}\,,$$

which implies that,

$$\Pi_{j=1}^J \langle \xi_j - \xi_{j+1}\rangle^{s-|\alpha_j|} \;=\; \langle \xi_b - \xi_{b+1}\rangle^{s-|\alpha_b|}\; \Pi_{j\neq b}\, \langle \xi_j - \xi_{j+1}\rangle^{s-|\alpha_j|} \;\geq\; \Pi_{j\neq b} \langle \xi_j - \xi_{j+1}\rangle^{s}\,.$$

Thus the integrand on the right side of (6.2.4) is dominated by

$$\Big|\, g_b(\xi_b - \xi_{b+1})\; \Pi_{j\neq b}\; \frac{g_j(\xi_j - \xi_{j+1})}{\langle \xi_j - \xi_{j+1}\rangle^s}\, \Big|\,. \tag{6.2.5}$$

Thus for any $\xi_1 \in \mathbb{R}^d$, the integrand in (6.2.4) is dominated by the sum over b of the terms (6.2.5). Hence

$$\begin{aligned} \|\mathcal{F}\big(\Pi_{j=1}^J \partial_x^{\alpha_j} w_{l_j}\big)\|_{L^2(\mathbb{R}^d)} \;&\leq\; \Big\| \sum_{b=1}^J \frac{|g_1|}{<\xi>^s} * \cdots * |g_b| * \frac{|g_{b+1}|}{<\xi>^s} * \cdots * \frac{|g_J|}{<\xi>^s} \Big\|_{L^2} \\ &\leq\; \sum_{b=1}^J \Big\| \frac{g_1}{<\xi>^s}\Big\|_{L^1} \cdots \Big\| \frac{g_{b-1}}{<\xi>^s}\Big\|_{L^1} \|g_b\|_{L^2} \Big\| \frac{g_{b+1}}{<\xi>^s}\Big\|_{L^1} \cdots \Big\| \frac{g_J}{<\xi>^s}\Big\|_{L^1} \end{aligned}$$

where the last step uses Young's inequality.
As in Sobolev's Theorem, $s > d/2 \;\Rightarrow <\xi>^{-s} \in L^2(\mathbb{R}^d)$ and the Schwarz inequality yields

$$\Big\| \frac{g_j}{<\xi>^s} \Big\|_{L^1} \;\leq\; C_1\, \|g_j\|_{L^2} \;\leq\; C_2\, \|w_j\|_{H^s}.$$

Using this in the previous estimate proves the lemma. ∎

To prove the Lipschitz continuity asserted in Schauder's Lemma it suffices to show that for all R there is a constant $C(R)$ so that

$$w_j \in \mathcal{S}(\mathbb{R}^d) \quad \text{for } j = 1,2 \qquad \text{and} \qquad \|w_j\|_{H^s(\mathbb{R}^d)} \leq R$$

imply

$$\|G(w_1) - G(w_2)\|_{H^s(\mathbb{R}^d)} \;\leq\; C\, \|w_1 - w_2\|_{H^s(\mathbb{R}^d)}\,.$$

Taylor's Theorem expresses

$$G(w_1) - G(w_2) = \int_0^1 G'(w_2 + \theta(w_1 - w_2))\, d\theta\; (w_1 - w_2)\,.$$

The estimates of the first part show that the family of functions $G'(w_2 + \theta(w_1 - w_2))$ parametrized by θ is bounded in $H^s(\mathbb{R}^d)$. Thus

$$\Big\| \int_0^1 G'(w_2 + \theta(w_1 - w_2))\; d\theta \Big\|_{H^s(\mathbb{R}^d)} \;\leq\; C(R)\,.$$

Applying the Lemma to the expression for $G(w_1) - G(w_2)$ as a product of two terms completes the proof. ∎

The standard proof of Schauder's Lemma for integer s uses the L^p version of the Sobolev Embedding Theorem.

Sobolev Embedding Theorem 6.2.4. *If $1 \leq s \in \mathbb{R}$ and $\alpha \in \mathbb{N}^d$ is a multiindex with $0 < s - |\alpha| < d/2$, there is a constant $C = C(\alpha, s, d)$ independent of $u \in H^s(\mathbb{R}^d)$ so that*

$$\| \partial_y^\alpha u \|_{L^{p(\alpha)}} \ \leq \ C \| \ |\xi|^s \, \hat{u}(\xi) \ \|_{L^2(\mathbb{R}^d)} \,, \tag{6.2.6}$$

where

$$p(\alpha) \ := \ \frac{2d}{d - 2s + 2|\alpha|} \,. \tag{6.2.7}$$

For $s - |\alpha| \ > \ d/2$, $\partial_y^\alpha u$ is bounded and continuous and

$$\|\partial_y^\alpha u\|_{L^\infty} \ \leq \ C \|u\|_{H^s(\mathbb{R}^d)} \,.$$

For $s - |\alpha| = d/2$, one has

$$\|\partial_y^\alpha u\|_{L^p(\mathbb{R}^d)} \ \leq \ C(p, s, \alpha) \, \|u\|_{H^s(\mathbb{R}^d)}$$

for all $2 \leq p < \infty$.

Proofs can be found in [Hörmander I.4.5, Taylor III.13.6.4]. When $p(\alpha)$ is an integer the estimate can be proved using Lemma 6.2.3. The formula for $p(\alpha)$ is forced by dimensional analysis. For a fixed nonzero $\psi \in C_0^\infty$, consider $u_\lambda(x) := \psi(\lambda x)$. The left hand side of (6.2.6) then is of the form $c\lambda^a$ for some a. Similarly the right hand side is of the form $c'\lambda^b$ for some b. In order for the inequality to hold, one must have $\lambda^a \leq c''\lambda^b$ for all positive λ so it is necessary that $a = b$.

Exercise 6.2.1. *Show that $a = b$ if and only if p is given by (6.2.7).*

Another way to look at the scaling argument is that for dimensionless u the left hand side of (6.2.6) has dimensions $length^{(d-p|\alpha|)/p}$ while the right hand side has dimensions $length^{(d-2s)/2}$. The formula for p results from equating these two expressions.

Standard proof of Schauder's Lemma. The usual proof for integer s uses the Sobolev estimates. together with Hölder's inequality. Hölder's inequality yields

$$\sum_{k=1}^J \frac{1}{p_k} \ = \ \frac{1}{2} \qquad \Longrightarrow \qquad \big\| \, \partial_x^{\alpha_1} w_{l_1} \cdots \partial_x^{\alpha_J} w_{l_J} \, \big\|_{L^2} \ \leq \ \Pi_{k=1}^J \big\| \partial_x^{\alpha_k} w_{l_k} \big\|_{L^{p_k}} \,.$$

Since each factor $\partial_x^{\alpha_k} w_{l_k}$ belongs to L^2 it suffices to find q_k so that

$$\partial_x^{\alpha_k} w_{l_k} \in L^{q_k} \quad \text{and} \quad \sum \frac{1}{q_k} \leq \frac{1}{2} \,.$$

Let $\mathcal{B}$ denote the set of $k \in \{1, \dots, J\}$ so that $s - |\alpha_k| > d/2$. For these indices the factor in our product is bounded, and so for $k \in \mathcal{B}$ set $q_k := \infty$.

Let $\mathcal{A} \subset \{1, \dots, J\}$ denote those indices i for which $s - |\alpha_i| < \frac{d}{2}$. For $k \in \mathcal{A}$, q_k is chosen as in Sobolev's Theorem,

$$q_k \ := \ \frac{2d}{d - 2s + 2|\alpha_k|} \,.$$

If $s - |\alpha_k| = \frac{d}{2}$, the factor in the product belongs to L^p for all $2 \leq p < \infty$ and the choice of q_k in this range is postponed.

With these choices, the Sobolev embedding theorem estimates

$$\|\partial_x^{\alpha_k} w_{l_k}\|_{L^{q_k}} \;\leq\; C\|w\|_{H^s(\mathbb{R}^d)}\,.$$

Then since $\sum|\alpha_i| \leq s$, and $s > d/2$,

$$\sum_{i\in\mathcal{A}\cup\mathcal{B}} \frac{1}{q_i} \;=\; \sum_{i\in\mathcal{A}} \frac{1}{q_i} \;=\; \sum_{i\in\mathcal{A}} \frac{d-2s+2|\alpha_i|}{2d} \;\leq\; \frac{Jd-2Js+2s}{2d}$$
$$=\; \frac{Jd-2(J-1)s}{2d} \;<\; \frac{Jd-(J-1)d}{2d} \;=\; \frac{1}{2}\,.$$

This shows there is room to pick large q_k corresponding to the case $s - |\alpha_k| = d/2$ so that $\sum 1/q_k < 1/2$, and the proof is complete. ∎

Another nice proof of Schauder's Lemma can be found in [Beals, pp 11-12]. Other arguments can be built on Littlewood-Paley decomposition of $G(w)$ as in [Bony, Meyer] and presented in [Alinhac-Gerard, Taylor III.13.10], or, on the representation

$$G(u) \;=\; \int \hat{G}(\xi)\,(e^{iu\xi}-1)\;d\xi\,.$$

The latter requires that one prove a bound on the norm of $e^{iu\xi}-1$ (see [Rauch-Reed 1982]) which grows at most polynomially in ξ. The last two arguments have the advantage of working when s is not an integer.

§6.3. Basic existence theorem.

The basic local existence theorem follows from Schauder's Lemma and the linear existence theorem. Schauder proved a quasilinear second order scalar version, but his argument, which is recalled in [Courant, §VI.10], works without essential modification once you add the linear energy inequalities of Friedrichs. The following existence proof is inspired by Picard's argument for ordinary differential equations. As in §1.1, Picard's bounds (6.3.8) replace the standard and less precise contraction argument.

Theorem 6.3.1. *If $s > d/2$ and $f \in L^1_{\rm loc}\big([0,\infty[\,;\,H^s(\mathbb{R}^d)\big)$, then there is a $T\,\in]0,1]$ and a unique solution $u \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ to the semilinear initial value problem defined by the partial differential equation (6.2) together with the initial condition*

$$u(0,x) \;=\; g(x) \in H^s(\mathbb{R}^d). \tag{6.3.1}$$

The time T can be chosen uniformly for f and g from bounded subsets of $L^1([0,1]\,;\,H^s(\mathbb{R}^d))$ and $H^s(\mathbb{R}^d)$ respectively. Consequently, there is a $T^*\,\in]0,\infty]$ and a maximal solution $u \in C\big([0,T^*[\,;\,H^s(\mathbb{R})^d)\big)$. If $T^*<\infty$ then

$$\lim_{t\to T^*}\;\|u(t)\|_{H^s(\mathbb{R}^d)} \;=\; \infty\,. \tag{6.3.2}$$

Proof. The solution is constructed as the limit of Picard iterates. The first approximation is not really important. Set

$$\forall\, t,x\,, \qquad u^1(t,x) \;:=\; g(x)\,.$$

For $\nu > 1$, the basic linear existence theorem implies that the Picard iterates defined as solutions of the linear initial value problems

$$L(y, \partial_y)\, u^{\nu+1} + F(y, u^\nu) \;=\; f(y)\,, \qquad u^{\nu+1}(0) = g$$

are well defined elements of $C([0,\infty[\,;\, H^s(\mathbb{R}^d)\,)$.

Let C denote the constant in the linear energy estimate (2.2.2). Choose a real number

$$R \;>\; 2C\, \|g\|_{H^s(\mathbb{R}^d)}\,. \tag{6.3.3}$$

Schauder's lemma implies that there is a constant $B(R) > 0$ so that

$$\|w(t,\cdot)\|_{H^s(\mathbb{R}^d)} \le R \qquad \Longrightarrow \qquad \|F(t,\cdot,w(\cdot))\|_{H^s(\mathbb{R}^d)} \;\le\; B\,.$$

Thanks to (6.3.3) one can choose $T > 0$ so that

$$C\left(e^{CT}\|g\|_{H^s(\mathbb{R}^d)} + \int_0^T e^{C(T-\sigma)}\big(B + \|f(\sigma)\|_{H^s(\mathbb{R}^d)}\big)d\sigma \right) \le R\,. \tag{6.3.4}$$

Using (2.2.2) shows that for all $\nu \ge 1$ and all $0 \le t \le T$

$$\|u^\nu(t)\|_{H^s(\mathbb{R}^d)} \;\le\; R\,. \tag{6.3.5}$$

Schauder's Lemma implies that there is a constant Λ so that for all t,

$$\|w_j\|_{H^s(\mathbb{R}^d)} \le R \Rightarrow \|F(t,x,w_1(x)) - F(t,x,w_2(x))\|_{H^s(\mathbb{R}^d_x)} \le \Lambda \|w_1 - w_2\|_{H^s(\mathbb{R}^d_x)}\,. \tag{6.3.6}$$

Then for $\nu \ge 2$, (2.2.2) applied to the difference $u^{\nu+1} - u^\nu$ implies that

$$\|u^{\nu+1}(t) - u^\nu(t)\|_{H^s(\mathbb{R}^d)} \;\le\; C\,\Lambda \int_0^t e^{C(t-\sigma)}\, \|u^\nu(\sigma) - u^{\nu-1}(\sigma)\|_{H^s(\mathbb{R}^d)}\; d\sigma\,. \tag{6.3.7}$$

Define

$$M_1 \;:=\; \sup_{0\le t\le T} \|u^1(t) - u^2(t)\|_{H^s(\mathbb{R}^d)} \qquad \text{and} \quad M_2 \;:=\; C\,\Lambda\, e^{CT}.$$

An induction on ν using (6.3.7) shows that for all $\nu \ge 2$

$$\|u^{\nu+1}(t) - u^\nu(t)\|_{H^s(\mathbb{R}^d)} \;\le\; M_1 \frac{(M_2 t)^{\nu-1}}{(\nu-1)!}. \tag{6.3.8}$$

Exercise 6.3.1. *Prove (6.3.8).*

Estimate (6.3.8) shows that the sequence $\{u^\nu\}$ is Cauchy in $C([0,T]\,;\, H^s(\mathbb{R}^d))$. Denote by u the limit. Passing to the limit in the initial value problem defining $u^{\nu+1}$ shows that u satisfies the initial value problem (6.1.1), (6.3.1). This completes the proof of existence.

Uniqueness is a consequence of the inequality

$$\|u_1(t) - u_2(t)\|_{H^s(\mathbb{R}^d)} \;\le\; C_1 \int_0^t e^{C(t-\sigma)}\, \|u_1(\sigma) - u_2(\sigma)\|_{H^s(\mathbb{R}^d)}\; d\sigma\,, \tag{6.3.9}$$

which is proved exactly as (6.3.7). Gronwall's inequality implies that $\|u_1 - u_2\| \equiv 0$. ∎

Remarks. Similar estimates show that there is continuous dependence of the solutions when the data f and g converge in $L^1_{loc}(\mathbb{R}\,;\,H^s(\mathbb{R}^d))$ and $H^s(\mathbb{R}^d)$ respectively.

Exercise 6.3.2. *Prove this.* **Discussion.** Concerning the first, more precise results are presented in §6.6.

Exercise 6.3.3. *Show that if the source term f satisfies $\partial_t^k f \in L^1_{\rm loc}([0,T^*[\,;\,H^{s-k}(\mathbb{R}^d))$ for $k = 1, 2, \dots, m$ as in Theorem 2.2.2, then $u \in \cap_k C^k\big([0,T^*[\,;\,H^{s-k}(\mathbb{R}^d)\big)$ for the same k.*

Finite speed of propagation for nonlinear equations is usually proved by writing a linear equation for the difference of two solutions. When $Lu_j + G(y, u_j) = 0$, denote by

$$w \;:=\; u_1 - u_2, \qquad B(y) \;:=\; \int_0^1 G'(y, u_2 + \theta(u_2(y) - u_1(y)))\; d\theta\,.$$

Taylor's theorem implies that

$$G(y,u_2) - G(y,u_1 = B(y)(u_2 - u_1), \qquad \text{so,} \qquad L\,w + B(y)\,w \;=\; 0\,.$$

This is a linear equation with coefficient $B \in C(H^s)$ which need not be smooth. For the L^2 estimates that are used to prove finite speed it is sufficient to know that $B \in L^\infty$.

The finite speed of propagation is determined entirely by the linear operator $L(y,\partial)$. Sharp estimates were proved in §2.5.

§6.4. Moser's inequality and the nature of the breakdown.

The breakdown (6.3.2) could in principal occur in a variety of ways. For example, the function might stay bounded and become more and more rapidly oscillatory. In fact this does not occur. Where the domain of existence ends the maximal amplitude of the solution must diverge to infinity. To prove this requires more refined inequalities than those of Sobolev and Schauder.

The proofs of Schauder's Lemma show that

$$\|G(y,w)\|_{H^s(\mathbb{R}^d_x)} \;\le\; h(\|w\|_{H^s(\mathbb{R}^d_x)})\,,$$

with a nonlinear function h which depends on G. There is a sharper estimate which grows only linearly in $\|w\|_{H^s}$ when one has L^∞ bounds.

Theorem 6.4.1 Moser's Inequality. *With the same hypotheses as Schauder's Lemma, there is a smooth function $h : [0,\infty[\to [0,\infty[$ so that for all $w \in H^s(\mathbb{R}^d)$ and t,*

$$\|G(x,w)\|_{H^s(\mathbb{R}^d_x)} \le h(\|w\|_{L^\infty(\mathbb{R}^d_x)})\,\|w\|_{H^s(\mathbb{R}^d_x)}\,. \tag{6.4.1}$$

This is proved by using Leibniz' rule and Hölder's inequality as in the standard proof of Schauder's Lemma. However in place of the Sobolev inequalities one uses the Gagliardo-Nirenberg interpolation inequalities.

Theorem 6.4.2 Gagliardo-Nirenberg Inequalities. *If $w \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $0 < |\alpha| < s$ then*

$$\partial_x^\alpha\, w \in L^{2s/|\alpha|}(\mathbb{R}^d)$$

In addition, there is a constant $C = C(|\alpha|, s, d)$ so that

$$\|\partial^\alpha w\|_{L^{2s/|\alpha|}(\mathbb{R}^d)} \ \leq\ C\, \|w\|_{L^\infty(\mathbb{R}^d)}^{1-|\alpha|/s} \Big(\sum_{|\beta|=s} \|\partial^\beta w\|_{L^2(\mathbb{R}^d)} \Big)^{|\alpha|/s}. \tag{6.4.2}$$

Remarks. 1. The second factor on the right in (6.4.2) is equivalent to the L^2 norm of the operator $|\partial_x|^s$ applied to u where $|\partial_x|^s$ is defined to be the Fourier multiplier by $|\xi|^s$. This gives the correct extension to non integer s.

2. The indices in (6.4.2) are nearly forced. Consider which inequalities

$$\|\partial^\alpha w\|_{L^p(\mathbb{R}^d)} \ \leq\ C\, \|w\|_{L^\infty(\mathbb{R}^d)}^{1-\theta} \Big(\sum_{|\beta|=s} \|\partial^\beta w\|_{L^2(\mathbb{R}^d)} \Big)^{\theta}$$

homogeneous of degree one in w might be true. The test functions $w = e^{ix.\xi/\epsilon}\psi(x)$ with $\epsilon \to 0$ show that a necessary condition is $|\alpha| \leq s\theta$. The idea is to use the L^∞ norm as much as possible and the s-norm as little as possible, which yields $|\alpha| = s\theta$. Considering $w = \psi(\epsilon x)$, or equivalently comparing the dimensions of the two sides forces $p = 2s/\alpha$.

3. Evans, Hörmander (nonlinear hyperbolic), Chemin *et.al.* and [Taylor III.10.3] are convenient references.

Proof of Moser's Inequality. For $w \in \mathcal{S}(\mathbb{R}^d)$, G independent of x, and $\sigma := |\alpha| \leq s$, the quantity $\partial_x^\alpha (G(w))$ is a sum of terms of the form

$$G^{(\gamma)}(w)\ \Pi_{j=1}^J\, \partial_x^{\alpha_j} w \tag{6.4.3}$$

where $|\gamma| = J$, and $\alpha_1 + \cdots + \alpha_J = \alpha$. The first factor in (6.4.3) is bounded with L^∞ norm bounded by a nonlinear function of the L^∞ norm of w.

For the second factor, Hölder's inequality yields

$$\|\, \partial_x^{\alpha_1} w \cdots \partial_x^{\alpha_J} w \,\|_{L^2} \ \leq\ \Pi_{k=1}^J\ \|\partial_x^{\alpha_k} w\|_{L^{2/\lambda_k}}$$

provided the nonnegative λ_k satisfy $\sum \lambda_k = 1$.

The Gagliardo-Nirenberg inequalities yield

$$\|\partial_x^{\alpha_k} w\|_{L^{2\sigma/|\alpha_k|}} \ \leq\ C\, \|w\|_{L^\infty}^{(\sigma-|\alpha_k|)/\sigma}\ \||\partial|^\sigma w\|_{L^2}^{|\alpha_k|/\sigma}\,.$$

With these choices

$$\sum \lambda_k = \sum \frac{|\alpha_k|}{\sigma} = 1\,,$$

and one has

$$\|\, \partial_x^{\alpha_1} w \cdots \partial_x^{\alpha_J} w \,\|_{L^2} \ \leq\ C\, \|w\|_{H^s}\,.$$

∎

Exercise 6.4.1. *Carry out the proof for G which depend on x.*

Theorem 6.4.3. *If $T^* < \infty$ in Theorem 6.3.1, then*

$$\limsup_{t\to T^*} \|u(t)\|_{L^\infty} \ =\ \infty\,. \tag{6.4.4}$$

Proof. It suffices to show that it is impossible to have $T^* < \infty$ and $|u| \le R < \infty$ on $[0,T^*[\times\mathbb{R}^d$. The strategy is to show that if $|u(t,x)| \le R < \infty$ on $[0,T^*[\times\mathbb{R}^d$, then (6.3.2) is violated.

Use the linear inequality for $0 \le t < T$,

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \ \le\ C\Big(\, \|u(0)\|_{H^s(\mathbb{R}^d)} + \int_0^t \ \|(Lu)(\sigma)\|_{H^s(\mathbb{R}^d)}\ d\sigma\Big)\,. \tag{6.4.5}$$

Then use Moser's inequality to give

$$\|(Lu)(\sigma)\|_{H^s(\mathbb{R}^d)} = \|F(\sigma,x,u(\sigma,x)) - f(\sigma,x)\|_{H^s(\mathbb{R}^d)} \ \le\ C(R)\Big(\,\|u(\sigma)\|_{H^s(\mathbb{R}^d)} + 1\Big)\,. \tag{6.4.6}$$

Insert (6.4.5) in (6.4.6) to find

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \le C\bigg(\,\|u(0)\|_{H^s(\mathbb{R}^d)} + \int_0^t \ \big(\|u(\sigma)\|_{H^s(\mathbb{R}^d)} + 1\big)\ \ d\sigma\bigg)\,. \tag{6.4.7}$$

Gronwall's inequality shows that there is a constant $C'' < \infty$ so that for $t \in [0,T^*[$

$$\|u(t)\|_{H^s(\mathbb{R}^d)} \le C''\,. \tag{6.4.8}$$

This violates (6.3.2), and the proof is complete. ∎

A mild sharpening of this argument (following [Yudovich]) shows that weaker norms than L^∞, for example the BMO norm, must also blow up at T^*.

Corollary 6.4.4. *If the data f and g in Theorem 6.4 belong to $L^1_{loc}\big([0,\infty[\,;\,H^s(\mathbb{R}^d)\big)$ and $H^s(\mathbb{R}^d)$ respectively, then they belong for all $d/2 < \tilde s \le s$. The blowup time $T^*(\tilde s)$ is independent of $\tilde s$. In particular, if the data belong for all s then the solution belongs to $C\big([0,T^*[\,;\,H^s(\mathbb{R}^d)\big)$ for all s.*

Proof. For $s \ge \tilde s > d/2$ denote by $u_{\tilde s}(t,x)$ the corresponding maximal solution. Since u_s is a $C(H^{\tilde s})$ solution it follows that if u_s is defined on $[0,T]$ then $u_s = u_{\tilde s}$ on this this interval so $T^*(\tilde s) \ge T$. Therefore, $T^*(\tilde s) \ge T^*(s)$.

On the other hand if $T^*(\tilde s) > T^*(s)$, it follows that $u_{\tilde s} \in L^\infty([0,T^*(s)] \times \mathbb{R}^d)$. By uniqueness of $H^{\tilde s}$ valued solutions one has

$$u_s = u_{\tilde s} \qquad \text{for} \qquad 0 \le t < T^*(s)\,.$$

So

$$\|u_{\tilde s}\|_{L^\infty([0,T^*(s)[\times\mathbb{R}^d)} \ =\ \|u_{\tilde s}\|_{L^\infty([0,T^*(s)[\times\mathbb{R}^d)} \ <\ \infty\,,$$

violating the blowup criterion of Theorem 6.4.1 ∎

§6.5. Perturbation theory and smooth dependence.

In this section the dependence of solutions on data is investigated. The first result yields two versions of lipschitz dependence.

Theorem 6.5.1. i. *If u and v are two solution in $C([0,T]\,;\,H^s(\mathbb{R}^d))$, then there is a constant C depending only on $\sup_{[0,T]}\max\{\|u(t)\|_s, \|v(t)\|_s\}$ so that*

$$\forall\ 0 \le t \le T, \quad \|u(t) - v(t)\|_s \ \le\ C\,\|u(0) - v(0)\|_s\,. \tag{6.5.1}$$

ii. *If $u \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ is a solution then there are constants $C, \delta > 0$ so that if $\|u(0)-h\|_s < \delta$ then the solution v with $v(0) = h$ belongs to $C([0,T]\,;\,H^s(\mathbb{R}^d))$ and $\sup_{0\le t]\le T} \|v(t)-u(t)\|_s < C\,\delta$.*

Proof. i. Choose Λ so that for w_1 and w_2 in H^s with

$$\|w_j\|_s \ \le\ \sup_{[0,T]} \max\{\|u(t)\|_s, \|v(t)\|_s\}\,,$$

and $0 \le t \le T$

$$\|F(t,x,w_1(x)) - F(t,x,w_2(x))\|_{H^s(\mathbb{R}^d_x)} \le \Lambda \|w_1 - w_2\|_{H^s(\mathbb{R}^d_x)}\,.$$

Then subtracting the equations for u and v yields

$$\|u(t) - v(t)\|_{H^s(\mathbb{R}^d)} \ \le\ \|u(0) - v(0)\|_s \ +\ \Lambda \int_0^t e^{C(t-\sigma)}\,\|u(\sigma) - v(\sigma)\|_{H^s(\mathbb{R}^d)}\,d\sigma\,.$$

Gronwall's inequality completes the proof of **i.**.

To prove **ii** it suffices to consder $\delta < 1$. Write $v = u + w$ so the initial value problem is equivalent to

$$L\,w + F(u+w) - F(u) = 0\,, \qquad w(0) = h\,.$$

So long as

$$\sup_{[0,t]} \ \|w(s)\| \ \le\ 2\,,$$

one estimates $\|F(w+u) - F(u)\|_s \le K\|w\|_s$ to find

$$\|w(t)\|_s \ \le\ \|h\|_s \ +\ \int_0^t K\,\|w(\sigma)\|_s\,d\sigma\,.$$

Gronwall implies that

$$\|w(t)\|_s \ \le\ \|h\|_s\,e^{Kt}\,.$$

Choose $C := e^{KT}$ and consider only δ so small that $\delta C < 2$. It follows that a local solution $w \in C([0,\underline{t}]\,,\,H^s)$ with $\underline{t} < T$ satisfies

$$\sup_{[0,\underline{t}]} \|w(t)\|_s \ <\ \min\ \big\{C\delta\,,\,2\big\}\,.$$

Therefore the maximal solution is defined at least on $[0,T]$, and on that interval satisfies $\|w(t)\|_s \le 2\,\|h\|_s$ which completes the proof of **ii.** ∎

Given a solution u we compute a perturbation expansion for the solution with initial data $u(0)+g$ with small g. To simplify the notation, consider the semilinear equation

$$L(y,\partial)\,u \ +\ F(u) \ =\ 0\,, \qquad F(0) = 0, \quad F'(0) = 0\,.$$

Consider the map, $\mathcal{N} : u(0) \mapsto u$ from H^s to $C([0,T]\,;\,H^s(\mathbb{R}^d))$. At the end we will show that this map is smooth. For the moment we simply compute the Taylor expansion, assuming that it exists. Assuming smoothness, the solution with data $u(0)+g$ has expansion

$$\mathcal{N}(u(0)+g) \ \sim\ u + M_1(g) + M_2(g) + \ldots \ \sim\ \sum_{j=1}^{\infty} M_j(g)\,, \tag{6.5.1}$$

where the M_j are continuous symmetric j-linear operators from H^s to $C([0,T]\,;\,H^s(\mathbb{R}^d))$.

To compute them, fix g and consider the initial data equal to $u(0)+\delta g$. The resulting solution has an expansion in δ

$$\mathcal{N}(u+\delta g) \ \sim\ u + \delta u_1 + \delta^2 u_2 + \cdots\,. \tag{6.5.2}$$

However,

$$\mathcal{N}(u+\delta g) \ \sim\ u + M_1(\delta g) + M_2(\delta g) + \ldots \ \sim\ u + \sum_{j=1}^{\infty} \delta^j\, M_j(g)\,.$$

Comparing with (6.5.2) one sees that

$$u_j \ = \ M_j(g, g, \cdots, g)\,, \qquad j \ \text{ copies of } \ g\,.$$

To compute u_j plug the expansion (6.5.2) into the equation

$$L(y,\partial)\Big(u + \sum_{j\geq 1} \delta^j u_j\Big) \ + \ F\Big(u + \sum_{j\geq 1} \delta^j u_j\Big) \ \sim \ 0\,.$$

The initial condition yields

$$u_1(0) \ = \ g\,, \qquad u_j(0) \ = \ 0\,, \quad j \geq 2\,. \tag{6.5.3}$$

Expanding the left hand side in powers of δ, the terms u_j are determined by setting the coefficients of the successive powers of δ equal to zero. Introduce the compact notation for the Taylor expansion

$$F(v+h) \ \sim\ F(v) + F_1(v;h) + F_2(v;h,h) + \ldots\,,$$

where for $v \in \mathbb{C}^N$, $F_j(v\,;\,\cdot)$ is a symmetric j linear map from $(\mathbb{C}^N)^j \to \mathbb{C}^N$.

Setting the coefficients of δ^j equal to zero for $j = 1,2,3$ yields the initial value problems

$$L\,u_1 \ = \ 0\,, \qquad u_1(0,x) \ = \ g\,, \tag{6.5.4}$$

$$L\,u_2 \ + \ F_2(u;u_1,u_1) \ = \ 0\,, \qquad u_2(0,x) \ = \ 0\,, \tag{6.5.5}$$

$$L\,u_3 + F_2(u;u_2,u_1) + F_2(u;u_1,u_2) + F_3(u;u_1,u_1,u_1) \ = \ 0\,, \quad u_3(0,x) = 0\,, \tag{6.5.6}$$

which determine u_j for $j = 1,2,3$. The pattern is clear. The initial value problem determining u_j is linear in u_j with source terms which are nonlinear functions of $u_1, \ldots, u_{j-1}$.

Exercise 6.5.1. *Suppose that the u_j are determined by solving these initial value problems. Then define $u_{\rm approx}(\delta)$ using Borel's theorem so that*

$$u_{\rm approx}(\delta) \ \sim\ \sum_{j\geq 1} \delta^j\, u_j\,, \quad \text{in} \quad C([0,T]\,;\,H^s(\mathbb{R}^d))\,.$$

Prove that for δ sufficiently small the exact solution of the initial value problem exists on $[0,T]$ and $u_{\rm exact)}(\delta) - u_{\rm approx}(\delta) \sim 0$ in $C([0,T]\,;\,H^s(\mathbb{R}^d))$. **Hint.** Compute a nonlinear equation for the error which has source terms $O(\delta^\infty)$. Use the method of Theorem 6.5.1.**ii. Discussion.** The key element is the stability argument at the end which shows that a nonlinear problem with infinitely small sources has a solution which is infinitely small. In science texts it is routine to overlook the need for such stability arguments.

The next result is stronger than that of the exercise.

Theorem 6.5.2. *If $u \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ is a solution then the map $\mathcal{N}$ from initial data to solution is smooth from a neighborhood of $u(0)$ to $C([0,T]\,;\,H^s(\mathbb{R}^d))$. The derivative is given by $\mathcal{N}_1(u(0),g) = u_1$ from (6.5.4). Derivatives of each order are uniformly bounded on the neighborhood.*

Proof. The preceding computations show that if $\mathcal{N}$ is differentiable then $\mathcal{N}_1(u(0),g) = u_1$ from (6.5.4). It suffices to show that this is the derivative of $\mathcal{N}$ and that the map from $u(0)$ to $\mathcal{N}_1(u(0),\cdot)$ is locally bounded and smooth with values in the linear maps from $H^s(\mathbb{R}^d)$ to $C([0,T]\,;\,H^s(\mathbb{R}^d))$.

To prove differentiability let $u := \mathcal{N}(u(0))$ be the base solution and $v := u + u_1$ be the first approximation. Then

$$L\,u + F(u) \;=\; 0\,, \qquad L\,v + F(v) \;=\; F(u+u_1) - F_1(u,u_1)\,.$$

Schauder's lemma together with Taylor's theorem shows that

$$\big\|F(u+u_1) - F_1(u,u_1)\big\|_{C([0,T]\,;\,H^s)} \;\le\; C\,\big\|u_1\big\|^2_{C([0,T]\,;\,H^s)}$$

Since the initial values of u and v are equal, the basic linear energy estimate proves that

$$\big\|u - v\big\|_{C([0,T]\,;\,H^s)} \;\le\; C\,\big\|u_1\big\|^2_{C([0,T]\,;\,H^s)}\,.$$

This proves that $\mathcal{N}$ is differentiable and the formula for the derivative. The formula implies that the derivative is locally bounded.

The derivative is computed by solving (6.5.4). Since u is a differentiable function of $u(0)$ with locally bounded derivative. Then $F_1(u,\cdot)$ is differentiable with locally bounded derivative. As in the proof of differentiability, it follows that $\mathcal{N}_1(u(0),\cdot)$ is a differentiable function of $u(0)$ with locally bounded derivative. The higher differentiability follows by an inductive argument. ∎

§6.6 The Cauchy problem for quasilinear symmetric hyperbolic systems.

For ease of reading we present only the case of real solutions of real equations. This includes most quasilinear examples from applications. The equations have the form

$$L(u,\partial)u \;:=\; \sum_{\mu=0}^{d} A_\mu(u)\,\partial_\mu u \;=\; f\,, \tag{6.6.1}$$

where the coefficient matrices A_μ are smooth symmetric matrix valued functions of u defined on an open subset of $\mathbb{R}^d$. The leading coefficient, $A_0(u)$, is assumed to be strictly positive. The leading coefficient, $A_0(u)$, is assumed to be strictly positive. One can almost as easily treat coefficients which are function of y and u.

The existence theorem is local in time, and for small times the values of u are close to values of the initial data. Thus for convenience we can modify the coefficients outside a neighborhood of the values taken by the initial data to arrive at a system with everywhere defined smooth matrix valued coefficients. Even more we may suppose that the coefficients take constant values outside a compact subset of u space.

In contrast to the linear case, one cannot reduce to the case $A_0 = I$. However, if one is interested only in solutions which take values near a constant value $\underline{u}$, changing variable to $v := A_0(\underline{u})^{1/2}u$ one can reduce to the case $A_0(\underline{u}) = I$. This is useful for quasilinear geometric optics.

§6.6.1. Existence of solutions.

Local existence is analogous to Theorem 6.4, except that it is important that the coefficients $A_\mu(u(x))$ be lipschitz continuous functions of y. For this reason we work in Sobolev spaces $H^s(\mathbb{R}^d)$ with $s > 1 + d/2$. The importance of the Lipschitz condition is seen from the basic $L^2(\mathbb{R}^d)$ energy law when $f = 0$,

$$\frac{d}{dt}\Big(A_0(u)\,u(t)\,,\,u(t)\Big) = \Big(\big(\sum_\mu \partial_\mu(A_\mu(u))\big)\ u(t)\,,\,u(t)\Big), \quad \mathrm{div}\,A := \sum_\mu \partial_\mu(A\mu(u)). \tag{6.6.2}$$

To control the growth of the L^2 norm uses the lipschitz bound. It is not obvious but is true, that the same bound suffices to control the growth of higher derivatives. The existence part of the following Theorem is essentially due to Schauder [Sch].

Theorem 6.6.1. *If $\mathbb{N} \ni s > 1 + d/2$, $f \in C^j\big([0,\infty[\,;\,H^{s-j}(\mathbb{R}^d)\big)$, and $g \in H^s(\mathbb{R}^d)$, then there is a $T > 0$ and a unique solution*

$$u \ \in\ \cap_{j=0}^s\ C^j([0,T]\,;\,H^{s-j}(\mathbb{R}^d))$$

to the initial value problem

$$L(u,\partial)u \ =\ f\,, \qquad u(0,x) \ =\ g(x)\,. \tag{6.6.3}$$

The time T can be chosen uniformly for f, g belonging to bounded subsets of $L^1_{loc}\big([0,\infty[\,;\,H^s(\mathbb{R}^d)\big)$ and $H^s(\mathbb{R}^d)$ respectively. Therefore, there is a $T_* \in]0,\infty]$ and a maximal solution in $\cap_j C^j([0,T_*[\,;\,H^{s-j}(\mathbb{R}^d))$. If $T_* < \infty$ then $\lim_{t\nearrow T_*}\|u(t)\|_{H^s(\mathbb{R}^d)} = \infty$. A more precise result is

$$\limsup_{t\nearrow T_*}\ \|u(t), \nabla_y u(t)\|_{L^\infty(\mathbb{R}^d)} \ =\ \infty\,. \tag{6.6.4}$$

Remark. If we had not modified the coefficients to be everwhere defined and smooth, the blow up criterion would be that either (6.6.4) occurs or, the values of u approach the boundary of the domain where the coefficients are defined.

This is so since if one has a solution of the original system whose values are taken in a compact subset K of the domain of definition of the coefficients, one can modify the coefficients outside a compact neighborhood of K. Theorem 6.6.2 implies that there is a solution on a larger time interval.

The standard proof of Theorem 6.6.1 proceeds by considering the sequence of approximate solutions satisfying

$$L(u^\nu,\partial)u^{\nu+1} \ =\ f\,, \qquad u^{\nu+1}\big|_{t=0} \ =\ g\,.$$

The linear equation satisfied by $u^{\nu+1}$ has coefficients $A_\mu(u^\nu)$ depending on u^ν for which one has only H^s control. The key to the proof is to derive *a priori* estimates for solutions of linear symmetric hyperbolic intitial value problems with coefficient matrices having only H^s regularity (see [Metivier, Lax]).

Schauder's approach was to approximate the functions A_μ by polynomials in u and the data f, g by real analytic functions and to use the Cauchy-Kowalsekaya Theorem (see [Courant-Hilbert]). *A priori* estimates are used to control the approximate solutions on an fixed, possibly small, time interval. We solve the equation by the method of finite differences. A disadvantage of this method is that it reproves the linear existence theorem. An advantage is that the basic *a priori* estimate for the difference scheme allows one to prove existence and the sharp blowup criterion at the same time.

Proof. For ease of reading, we present the case $f = 0$. The approximate solution u^h is the unique local solution of the ordinary differential equation in $H^s(\mathbb{R}^d)$

$$A_0(u)\partial_t u^h \;+\; \sum_j A_j(u^h)\,\delta^h_j u^h \;=\; 0\,, \qquad u^h(0,x) \;=\; g(x)\,. \tag{6.6.5}$$

For each fixed $h > 0$, the map

$$w \;\mapsto\; A_0(u)^{-1}\sum_j A_j(u^h)\,\delta^h_j w^h$$

from $H^s(\mathbb{R}^d)$ to itself is uniformly lipschitzean on bounded subsets. It follows that there is a unique maximal solution

$$u^h \;\in\; C^1([0,T^h_*[\,;\,H^s(\mathbb{R}^d))\,, \qquad T^h_* \in]0,\infty]\,.$$

If $T^h_* < \infty$, then $\lim_{t\nearrow T^h_*}\|u^h(t)\|_{H^s(\mathbb{R}^d)} = \infty$.

The heart of the existence proof are uniform estimates for u^h on an h independent interval. The starting point is an $L^2(\mathbb{R}^d)$ estimate,

$$\frac{d}{dt}\Big(A_0(u^h)\,u^h(t)\,,\,u^h(t)\Big) \;=\; \Big(\big(\partial_t(A_0(u^h)\big)\,u^h\,,\,u^h\Big) \;+\; \sum_j\Big(\big(A_j\delta^h_j + (A_j\delta^h_j)^*\big)u^h\,,\,u^h\Big)\,.$$

Thanks to the symmetry of A_j and the antisymmetry of δ^h_j,

$$A_j\delta^h_j + (A_j\delta^h_j)^* \;=\; [A_j(u^h)\,,\,\delta^h_j]\,.$$

There is a constant, $C = C(A_\mu)$, so that

$$\|\partial_t A_0(u^h)\|_{\mathrm{Hom}(L^2(\mathbb{R}^d))} \;+\; \|A_j\delta^h_j + (A_j\delta^h_j)^*\|_{\mathrm{Hom}(L^2(\mathbb{R}^d))} \;\le\; C\,\|\nabla_y u^h(t)\|_{L^\infty(\mathbb{R}^d)}\,. \tag{6.6.6}$$

For $|\alpha| \le s$ and $\partial = \partial_x$, compute

$$\begin{aligned}
\frac{d}{dt}\Big(A_0(u^h)\,\partial^\alpha u^h(t)\,,\,\partial^\alpha u^h(t)\Big) \;&=\; \Big(A_0(u^h)\partial^\alpha\partial_t u^h\,,\,\partial^\alpha u^h\Big) + \\
&\Big(A_0(u^h)\partial^\alpha u^h\,,\,\partial^\alpha\partial_t u^h\Big) + \Big(\big(\partial_t A_0(u^h)\big)\partial^\alpha u^h\,,\,\partial^\alpha u^h\Big) \\
&:=\; \Big(A_0(u^h)\partial^\alpha\partial_t u^h\,,\,\partial^\alpha u^h\Big) + \Big(A_0(u^h)\partial^\alpha u^h\,,\,\partial^\alpha\partial_t u^h\Big) + E_1\,,
\end{aligned} \tag{6.6.7}$$

beginning the collection of terms which we will prove are acceptably large.

The first term on the right of (6.6.7) is equal to

$$\Big(\partial^\alpha A_0(u^h)\partial_t u^h\,,\,\partial^\alpha u^h\Big) + \Big([A_0(u^h),\partial^\alpha]\partial_t u^h, \partial^\alpha u^h\Big) \;:=\; \Big(\partial^\alpha A_0(u^h)\partial_t u^h\,,\,\partial^\alpha u^h\Big) + E_2\,. \tag{6.6.8}$$

Analogously, the symmetry of A_0 shows that the second term in (6.6.7) is equal to

$$\begin{aligned}\Big(\partial^\alpha u^h\,,\,A_0(u^h)\partial^\alpha\partial_t u^h\Big) \;&=\; \Big(\partial^\alpha u^h\,,\,\partial^\alpha A_0(u^h)\partial_t u^h\Big) \;+\; \Big(\partial^\alpha u^h, [A_0(u^h),\partial^\alpha]\partial_t u^h\Big)\\ &:=\; \Big(\partial^\alpha u^h\,,\,\partial^\alpha A_0(u^h)\partial_t u^h\Big) \;+\; E_3\,.\end{aligned} \tag{6.6.9}$$

Using the differential equation, the sum of the nonerror terms in (6.6.8-9) is equal to the sum on j of

$$\begin{aligned}&\Big(\partial^\alpha u^h\,,\,\partial^\alpha A_j(u^h)\,\delta_j^h u^h\Big) \;+\; \Big(\partial^\alpha A_j(u^h)\,\delta_j^h u^h\,,\,\partial^\alpha u^h\Big)\\ &\quad=\; \Big(\partial^\alpha u^h\,,\,A_j(u^h)\,\delta_j^h\partial^\alpha u^h\Big) \;+\; \Big(A_j(u^h)\,\delta_j^h\partial^\alpha u^h\,,\,\partial^\alpha u^h\Big) \;+\; E_4\\ &\quad=\; \Big(\big(A_j(u^h)\,\delta_j^h + (A_j(u^h)\delta_j^h)^*\big)\partial^\alpha u^h\,,\,\partial^\alpha u^h\Big) \;+\; E_4\\ &\quad:=\; E_5 \;+\; E_4\,,\end{aligned} \tag{6.6.10}$$

where

$$E_4 \;:=\; \Big(\partial^\alpha u^h\,,\,[\partial^\alpha, A_j(u^h)]\,\delta_j^h u^h\Big) \;+\; \Big([\partial^\alpha, A_j(u^h)]\,\delta_j^h u^h\,,\,\partial^\alpha u^h\Big)\,.$$

Denote by

$$\mathcal{E}(w) \;:=\; \sum_{|\alpha|\le s}\Big(A_0(w)\partial_x^\alpha w\,,\,\partial_x^\alpha w\Big)\,.$$

Since A_0 is strictly positive, there is a constant C independent of w so that

$$\frac{1}{C}\sum_{|\alpha|\le s}\|\partial_x^\alpha w(t)\|^2_{L^2(\mathbb{R}^d)} \;\le\; \mathcal{E}(w) \;\le\; C\sum_{|\alpha|\le s}\|\partial_x^\alpha w\|^2_{L^2(\mathbb{R}^d)}\,. \tag{6.6.11}$$

Summing over all $|\alpha|\le s$ yields

$$\frac{d\mathcal{E}(u^h(t))}{dt} \;=\; \sum_{j=1}^{5} E_j\,. \tag{6.6.12}$$

Lemma 6.6.2. *For all $R>0$, $1\le j\le 5$, and $0<h<1$, there is a constant $C(R)$ depending only on L and R so that*

$$\big\|u^h(t),\nabla_y u^h(t)\big\|_{L^\infty(\mathbb{R}^d)} \;\le\; R \quad\Longrightarrow\quad |E_j| \;\le\; C(R)\sum_{|\alpha|\le s}\|\partial_x^\alpha u^h(t)\|^2_{H^s(\mathbb{R}^d)}\,.$$

Proof of Lemma. The cases $j=1$ and $j=5$ follow from (6.6.6). The remaining three cases are similar and we present only $j=3$ which is the worst. It suffices to show that

$$\|[\partial^\alpha, A_j(u^h)]\,\partial_t u^h\|^2_{L^2(\mathbb{R}^d)} \;\le\; C(R)\sum_{|\alpha|\le s}\|\partial_x^\alpha u^h(t)\|^2_{H^s(\mathbb{R}^d)}\,. \tag{6.6.13}$$

The quantity on the left of (6.6.13) is a linear combination of terms

$$\partial^\beta(A_j(u^h))\,\partial^\gamma\big((A_0^{-1}A_j)(u^h)\delta_j^h\,u^h\big)\,,\qquad \beta+\gamma=\alpha,\quad \beta\ne 0\,.$$

Since $\beta \neq 0$ this is equal to

$$(\partial^\beta A_j(u^h) - A_j(0))\, \partial^\gamma\big((A_0^{-1}A_j(u^h) - A_0^{-1}A_j(0))\delta_j^h\, u^h\big) \\ +\ \partial^\beta(A_j(u^h) - A_j(0))\,(A_0^{-1}A_j)(0)\partial^\gamma\delta_j^h\, u^h\,.$$

Estimate

$$\|A_j(u^h) - A_j(0)\|_{L^\infty}\ +\ \|(A_0^{-1}A_j)(u^h)\ -\ (A_0^{-1}A_j)(0)\|_{L^\infty}\ \le\ C(R)\,,$$

and from Moser's inequality,

$$\|A_j(u^h) - A_j(0)\|_{H^s}\ +\ \|(A_0^{-1}A_j)(u^h)\ -\ (A_0^{-1}A_j)(0)\|_{H^s}\ \le\ C(R)\,\|u^h\|_{H^s}^{1/2}\,.$$

The Gagliardo-Nirenberg estimates then imply (6.6.13). ∎

The local solution is constructed so as to take values in the set

$$\mathcal{W}\ :=\ \Big\{w \in H^s(\mathbb{R}^d)\ ;\ \mathcal{E}(w) \le \mathcal{E}(g) + 1\Big\}\,.$$

Choose $R > 0$ so that

$$w \in \mathcal{W}\quad\Longrightarrow\quad \|w\|_{L^\infty} + \|\nabla_x w\|_{L^\infty} + \|\sum_j A_j(w)\delta_j^h w\|_{L^\infty}\ <\ R\,.$$

So long as $u^h(t)$ stays in $\mathcal{W}$, one has

$$\frac{d\mathcal{E}(u^h(t))}{dt}\ \le\ C(R)\,C\,\mathcal{E}(u^h(t))\ \le\ C(R)\,C\,\Big(\mathcal{E}(g) + 1\Big)\,.$$

with C from (6.6.11) Therefore,

$$\mathcal{E}(u^h(t)) - \mathcal{E}(g)\ \le\ T\,C(R)\,C\,\Big(\mathcal{E}(g) + 1\Big)\,.$$

Define T by

$$T\,C(R)\,C\,\Big(\mathcal{E}(g) + 1\Big)\ =\ \frac{1}{2}\,.$$

If follows that for all h, u^h takes values in $\mathcal{W}$ for $0 \le t \le T$.
This uniform bound implies a subsequence which converges weak star in $L^\infty([0,T]\,;\,H^s(\mathbb{R}^d))$ and stongly in $C^j([0,T]\,;\,H^{s-j}(\mathbb{R}^d))$ for $1 \le j \le s$.
The limit satisfies the initial value problem and also

$$\frac{d\mathcal{E}(u(t))}{dt}\ \le\ C(R)\,\mathcal{E}(u(t))\,. \tag{6.6.14}$$

This together with the uniform continuity of u implies that $\|u(t)\|_{H(\mathbb{R}^d)}$ is continuous. It follows that $u \in C([0,T]\,;\,H^s(\mathbb{R}^d))$. That $\partial_t^j u \in C([0,T]\,;\,H^{s-j}(\mathbb{R}^d))$ follows by using the differential equation to express these derivatives in terms of spatial derivatives as in the semilinear case.

Uniqueness is proved by deriving a linear equation for the difference $w := u - v$ of two solutions u and v. Toward that end compute

$$A_\mu(u)\partial_\mu u - A_\mu(v)\partial_\mu v\ =\ A_\mu(u)\partial_\mu(u - v)\ +\ (A_\mu(u) - A_\mu(v))\partial_\mu v\,.$$

Write $A_\mu(u) - A_\mu(v) \ = \ \mathcal{G}_\mu(u,v)\,(u-v)\,$, to find

$$A_\mu(u)\partial_\mu u - A_\mu(v)\partial_\mu v = \mathcal{A}_\mu \partial w + \mathcal{B}_\mu w,$$

$$\mathcal{A}_\mu(y) := A_\mu(u(y)), \quad \mathcal{B}_\mu(y) := \mathcal{G}_\mu(u(y), v(y))\,\partial_\mu v(y).$$

Therefore

$$\mathcal{L}(y,\partial)\,w \ = \ 0\,, \qquad \mathcal{L}(y,\partial_y) := \sum \Big(\mathcal{A}_\mu \partial_\mu + \mathcal{B}_\mu\Big)\,. \tag{6.6.15}$$

The energy method yields

$$\frac{d}{dt}\big(\mathcal{A}_0\, w(t), w(t)\big) \ \le \ C\,\big(\mathcal{A}_0\, w(t)\,,\, w(t)\big)\,, \tag{6.6.16}$$

Since $w|_{t=0} = 0$, it follows that $w = 0$ which is the desired uniqueness.

All that remains is the proof of the precise blow up criterion (6.6.4). This is immediate since if the lipschitz norm does not blow up, then (6.6.14) implies that the $H^s(\mathbb{R}^d)$ norm does not blow up. This completes the proof of Theorem 6.6.1. ∎

Example. In addition to the numerous examples from mathematical physics we point out the elegant proof of Garabedian reducing the Cauchy-Kowaleskaya Theorem to the solution of quasilinear symmetric hyperbolic initial value problems, [Garabedian, Taylor III].

§6.6.2. Examples of breakdown.

In this section we exhibit a simple mechanism, wave breaking, for the breakdown of solutions with u bounded and $\nabla_x u$ diverging to infinity as $t \nearrow T^*$. The method of proof leads to two Liouville type theorems.

The classic example is Burgers' equation

$$u_t + u\,u_x \ = \ 0\,. \tag{6.6.17}$$

For a smooth solution on $[0,T]\times\mathbb{R}^d$ the equation shows that u is constant on the integral curves of $\partial_t + u\,\partial_x$. Therefore those integral curves are straight lines.

For the solution of the initial value problem with

$$u(0,x) \ = \ g(x) \in C_0^\infty(\mathbb{R})\,, \tag{6.6.18}$$

the value of u on the line $(t, x+g(x)t)$ must be equal to $g(x)$. This is an implicit equation,

$$u(t, x+tg(x)) \ = \ g(x)\,, \tag{6.6.19}$$

uniquely determining a smooth solutions for t small.

However, if g is not monotone increasing, consider the lines starting from two points $x_1 < x_2$ where $g(x_1) > g(x_2)$. The lines intersect in $t > 0$ at which point the conditions that u take value $g(x_1)$ and $g(x_2)$ contradict. Thus the solution must break down before this time. While the solution is smooth, $u(t)$ is a rearrangement of $u(0)$ so the sup norm of u does not blow up. The existence theorem shows that the gradient must explode.

That the gradient explodes can also be proved by differentiating the equation to show that $v := \partial_x u$ satisfies

$$v_t + u\,\partial_x v + v^2 \ = \ 0\,.$$

This equation is exactly solvable since

$$\frac{d}{dt}\, v(t, x + g(x)t) \;=\; v_t + u\,\partial_x v \;=\; -\,v^2\,.$$

Therefore,

$$v(t, x + g(x)t) \;=\; \frac{g'(x)}{1 - g'(x)t}\,.$$

Proposition 6.6.2. *The maximal solution of the initial value problem (6.6.4-5) satisfies*

$$T_* \;=\; \frac{1}{-\min g'(x)}\,. \tag{6.6.20}$$

Proof. The preceding computations shows that T_* can be no larger than the right hand side of (6.6.20). On the other hand, the implicit function theorem provides a smooth solution of (6.6.18) so long as the map $x \mapsto x + tg(x)$ is a diffeomorphism from $\mathbb{R}$ to itself. This holds exactly for t smaller than the right hand side of (6.6.20). ∎

The method of proof yields the following results of Liouville type.

Theorem 6.6.3. i. *The only global solutions $u \in C^1(\mathbb{R}^{1+d})$ of Burgers' equation 6.6.5 are the constants.*
ii. *The only global solutions $\psi(x) \in C^3(\mathbb{R}^d)$ of the eikonal equation $|\nabla_x \psi| = 1$ are affine functions.*

Proof. i. Denote $g(x) := u(0, x)$. If there is a point with $g'(\underline{x}) < 0$ the above proof shows that $u_x(t, \underline{x} + g(\underline{x})t)$ diverges as $t \nearrow T^*$. If there is a point with $g'(\underline{x}) > 0$ then an analogous argument shows that $u_x(t, \underline{x} + g(\underline{x})t)$ diverges as $t \searrow -1/g'(\underline{x})$. Therefore g is constant and the result follows.
ii. Denote by

$$V \;:=\; 2 \sum \partial_j\psi\; \partial_j\,,$$

a C^1 vector field. Differentiating $\sum(\partial_j\psi)^2 = 1$, yields for each partial derivative $\partial\psi$,

$$V\,\partial\psi \;=\; 0\,, \qquad 0 \;=\; V\,\partial^2\psi \;+\; 2\sum_j(\partial_j\partial\psi)^2 \;\geq\; V\,\partial^2\psi \;+\; (\partial^2\psi)^2\,. \tag{6.6.21}$$

The first implies that $\nabla_x\psi$ is constant on the integral curves of V. Therefore the integral curves are stationary points or straight lines $\underline{x} + s\nabla_x\psi(\underline{x})$.
If ψ is not linear, there is a point $\underline{x}$ at which the matrix of second derivatives at $\underline{x}$ is not equal to zero. The same holds on a neiborhood of $\underline{x}$ so we can choose $\underline{x}$ so that $\nabla_x\psi(\underline{x}) \neq 0$. A linear change of coordinates yields $\partial_1^2\psi(\underline{x}) \neq 0$.
Then

$$h(s) \;:=\; \partial_1^2\psi(\underline{x} + 2s\nabla_x\psi(\underline{x}))\,, \quad \text{satisfies} \quad \frac{dh}{ds} \;\leq\; -h(s)^2\,.$$

If $h(0) < 0$ then h diverges to $-\infty$ at a finite positive value of s. Similarly if $h(0) > 0$ then h diverges to $+\infty$ at a finite negative value of s. Thus ψ cannot be globally C^2. ∎

§6.6.3. Dependence on initial data.

Theorem 6.6.1 shows that the map from $u(0)$ to $u(t)$ maps $H^s(\mathbb{R}^d)$ to itself and takes bounded sets to bounded sets. In contrast to the case of semilinear equations, this mapping is not smooth. It is not even lipschitzean. It is lipschitzean as a mapping from $H^s(\mathbb{R}^d)$ to $H^{s-1}(\mathbb{R}^d)$.

Suppose that $v \in C([0,T]\,;\,H^s(\mathbb{R}^d))$ with $s > 1 + d/2$ solves (6.6.1). Denote by $\mathcal{N}$ the map $u(0) \mapsto u(\cdot)$ from initial data to solution. It is defined on a neighborhood, $\mathcal{U}$, of $v(0)$ in $H^s(\mathbb{R}^d)$ to $\cap_j C^j([0,T]\,;\,H^{s-j}(\mathbb{R}^d))$.

Theorem 6.6.4. *Decreasing the neighborhood $\mathcal{U} \subset H^s(\mathbb{R}^d)$ if necessary, the map*

$$\mathcal{U} \ni u(0) \;\mapsto\; u(\cdot) \;\in\; \cap_{\{j\,:\,s-j-1>d/2\}}\, C^j([0,T]\,;\,H^{s-1-j}(\mathbb{R}^d))$$

is uniformly lipschitzean.

Proof. The assertion follows from the linear equation (6.6.15) for the difference of two solutions. The coefficients $\mathcal{A}_\mu$ belong to $C^j([0,T]\;:\;H^{s-j}(\mathbb{R}^d))$ for $0 \le j \le s$. On the other hand, the coefficients $\mathcal{B}_\mu \in C^j([0,T]\;:\;H^{s-j-1}(\mathbb{R}^d))$ for $0 \le j \le s-1$ have one less derivative. For this linear equation, the change of variable $\tilde{w} = \mathcal{A}_0^{-1/2} w$ reduces to the case $\mathcal{A}_0 = I$.
The estimate is proved by computing

$$\frac{d}{dt} \sum_{|\alpha| \le s-1} (\partial^\alpha \tilde{w}(t)\,,\,\partial^\alpha \tilde{w})\,.$$

The restriction to $s-1$ comes from the fact that $\mathcal{B}$ is only $s-1$ times differentiable.

Exercise. *Carry out this proof using the proof of Theorem 6.6.1 as model.* ∎

We next prove differentiable dependence by the perturbation theory method of §6.5. Suppose that

$$L(v,\partial)\,v \;=\; 0\,,$$

and consider the perturbed problem

$$L(u,\partial)u \;=\; 0\,, \qquad u|_{t=0} \;=\; v(0) \;+\; g\,, \tag{6.6.22}$$

with g small. To compute the Taylor expansion, introduce the auxiliary problems

$$L(\tilde{u},\partial)\tilde{u} \;=\; 0\,, \quad \tilde{u}|_{t=0} \;=\; v(0) \;+\; \delta g\,, \qquad \tilde{u} \;\sim\; u_0 + \delta u_1 + \delta^2 u_2 + \cdots\,. \tag{6.6.23}$$

Then $L(\tilde{u},\partial)\tilde{u}$ has expansion in powers of δ computed from the expression

$$0 = \sum_\mu \Big(A_\mu(u_0) + \delta A'_\mu(u_0)(u_1) + \delta^2 A''_\mu(u_0)(u_1,u_1) + \cdots\Big)\,\partial_\mu\Big(u_0 + \delta u_1 + \delta^2 u_2 + \cdots\Big).$$

The $O(\delta^0)$ term yields

$$L(u_0,\partial)u_0 \;=\; 0\,, \qquad u_0|_{t=0} \;=\; u(0)\,, \tag{6.6.24}$$

yielding, $u_0 = v$, is the unperturbed solution.
The $O(\delta)$ term yields

$$\sum_\mu A_\mu(v)\partial_\mu u_1 \;+\; \sum_\mu \big[A'_\mu(v)u_1\big]\partial_\mu v \;=\; 0\,, \qquad u_1|_{t=0} \;=\; g\,. \tag{6.6.25}$$

Introduce the linearization of L at the solution v by

$$\mathbf{L}\, w \ :=\ \sum_\mu A_\mu(v)\partial_\mu w \ +\ \sum_\mu \big[A'_\mu(v)(w)\big]\partial_\mu v\,. \tag{6.6.26}$$

The equation of first order perturbation theory becomes

$$\mathbf{L}\, u_1 \ =\ 0\,, \qquad u_1\big|_{t=0} \ =\ g\,. \tag{6.6.27}$$

In the zero order term of $\mathbf{L}$, the coefficient depends on ∂v so in general u_1 will be one derivative less regular than v.

The $O(\delta^2)$ terms yield

$$\mathbf{L}\, u_2 \ +\ \sum_\mu \big[A'_\mu(v)(u_1)\big]\partial_\mu u_1 \ +\ \sum_\mu \big[A''_\mu(v)(u_1,u_1)\big]\partial_\mu v \ =\ 0\,, \qquad u_2\big|_{t=0} \ =\ 0\,. \tag{6.6.28}$$

There is a source term depending on ∂u_1 so typically, u_2 will be one derivative less regular than u_1 and therefore two derivatives less regular than v.

Continuing in this fashion yields initial value problems determining u_j as symmetric j-multilinear functionals of g provided that v is sufficiently smooth.

Theorem 6.6.5. *Suppose that $s > 1 + d/2$, and $v \in C([0,T] \ :\ H^s(\mathbb{R}^d))$ satisfies (6.6.1). Then the map, $\mathcal{N}$, from initial data to solution is a differentiable function from a neighborhood of $v(0)$ in $H^s(\mathbb{R}^d)$ to $C([0,T]\,;\, H^{s-1}(\mathbb{R}^d))$. The derivative is locally bounded. If $s - j > d/2$ then $\mathcal{N}$ is j times differentiable as a map with values in $C([0,T]\,;\, H^{s-j}(\mathbb{R}^d))$. The derivatives are locally bounded.*

Sketch of Proof. The linear equation determining u_1 has coefficient which involve the first derivative of v. As a result u_1 will in general be one derivative less regular than v. That is as bad as it gets. It is not difficult to show using an estimate as in Theorems 6.5.2, 6.6.4 that

$$\Big\|\mathcal{N}(u(0)+g) - \Big(\mathcal{N}(u(0)) + u_1\Big)\Big\|_{C\left([0,T]\,;\,H^{s-1}(\mathbb{R}^d)\right)} \ \le\ C\,\|g\|^2_{H^s(\mathbb{R}^d)}\,.$$

This yields differentiability, the formula for the derivative, and local boundedness.

Similarly, the calculations before the Theorem show that if $\mathcal{N}$ is twice differentiable then one must have

$$\mathcal{N}_2(v(0), g, g) \ =\ u_2\,,$$

where u_2 is the solution of (6.6.28). It is straight forward to show that $\mathcal{N}_2$ so defined is a continuous quadratic map from $H^s \mapsto C\big([0,T]\,;\, H^{s-2}(\mathbb{R}^d)\big)$.

A calculation like that in Theorem 6.5.2 shows that

$$\Big\|\mathcal{N}(u(0)+g) - \Big(\mathcal{N}(u(0)) + u_1 + u_2\Big)\Big\|_{C\left([0,T]\,;\,H^{s-2}(\mathbb{R}^d)\right)} \ \le\ C\,\|g\|^3_{H^s(\mathbb{R}^d)}\,.$$

This is not enough to conclude that $\mathcal{N}$ is twice differentiable. What is needed is a formula for the variation of $\mathcal{N}_1(v(0), g)$ when $v(0)$ is varied. The derivative $\mathcal{N}_1(v(0), g) = u_1$ is determined by solving the linear Cauchy problem (6.6.27) which has the form

$$L(v,\partial)\, u_1 + B(v,\partial v)\, u_1 \ =\ 0\,, \qquad u_1(0) \ =\ g\,.$$

The map from $v(0)$ to the coefficients in (6.6.26) is differentiable and locally bounded from $H^s \to C([0,T]\,;\,H^{s-1})$. Provided that $s-1 > d/2+1$ it follows from a calculation like that used to show that $\mathcal{N}$ is differentiable, that the map from $v(0)$ to u_1 is differentiable from H^s to $C([0.T]\,;\,H^{s-2}(\mathbb{R}^d))$, that $\mathcal{N}$ is twice differentiable, and the second derivative is locally bounded. The straight forward but notationally challenging computations are left to the reader.

The inductive argument for higher derivatives is similarly passed to the reader. ▮

We next show by example that the loss of one derivative expressed in Theorems 6.6.4 and 6.6.5 is sharp. Choose

$$0 \le \chi \in C_0^\infty, \qquad \chi = 1 \ \text{ on } \ \{|x| \le 1/2\}.$$

and denote $x_+ = \max\{x, 0\}$.. Consider Burgers' equation, $v_t + v\,v_x = 0$, with initial data

$$v(0,x) = (x_+)^{3/2+\delta}\chi(x) \qquad 0 < \delta < 1/2,$$

belonging to $H^2(\mathbb{R})$ but not $H^3(\mathbb{R})$. Choose $\underline{t} > 0$ so that v the local solution valued inH^2 exists for $0 \le t \le \underline{t}$. That solution is compared with the solution u with initial value equal to $v(0,x) + \epsilon\chi(x)$. The solution u vanishes for $x \le 1 + \epsilon t$. So, The difference $(u-v)(\underline{t})$ is equal to v on an interval of lenght $\epsilon\underline{t}$ to the right of the origin. Therefore

$$\|(u-v)(\underline{t})\|^2_{H^2(\mathbb{R})} \ \ge \ \int_0^{\epsilon\underline{t}} (v_{xx})^2 dx \ \ge \ C \int_0^{\epsilon\underline{t}} x^{-1+2\delta}\, dx \ \ge \ C\,\epsilon^{2\delta}\,.$$

Since this is not $O(\epsilon)$ the example shows that the map from data to solution is not Lipschitzean on H^2. Moreover it is not Hölderian with any index $\alpha > 0$. If we had taken $\delta > 1/2$ then the data would be H^3 and consistent with the Theorems the map would be Lipschitzean with values in H^2.

§6.7. Global small solutions for maximally dispersive nonlinear systems.

In dimensions greater than one, solutions of linear constant coefficient hyperbolic systems, no lower order terms, and no hyperplanes in their characteristic variety, tend to zero as $t \to \infty$. The maximally dispersive systems decay as fast as is possible, consistent with L^2 conservation. Consider a nonlinear system

$$L(\partial)\,u + G(u) \ = \ 0\,, \qquad G(0) = 0\,, \quad \nabla_u G(0) = 0\,.$$

Solutions with small initial data, say $u\big|_{t=0} = \epsilon\, f$ are approximated by solutions of the linearized equation

$$L(0,\partial)u \ = \ 0\,,$$

with the same initial data. On bounded time intervals, the error is $O(\epsilon^2)$ since the nonlinear term is at least quadratic at the origin. When solutions of $L\,u = 0$ decay in L^∞, $G(u)$ is even smaller. There is a tendency to approach linear behavior for large times. For $G = O(|u|^p)$ at the origin, the higher is p the stronger is the tendency. The higher is the dimension, the more dispersion is possible and the stronger can be the effect.

We prove that for maximally dispersive systems in dimension $d \ge 4$ and $p \ge 3$, the Cauchy problem is globally solvable for small data. This line of investigation has been the subject of much research. The CBMS lectures of Strauss present a nice selection of topics. The important special case of perturbations of the wave equation was the central object of a program of F. John in which the contributions of S. Klainerman were capital. I recommend the books of Sogge, Hörmander, Shatah-Struwe, and Strauss for more information. The analysis we present follows ideas predating the John-Klainerman revolution. A quasilinear version including refined estimates for scattering operators can be found in [Satoh, Kajitani-Satoh] . The sharper result in the spirit of John-Klainerman is that there is global existence of small solutions when $(d-1)(p-1)/2 > 1$. Estimates sufficient for the sharper result are proved in the article of Georgiev, Lucente, and Ziliotti. The sharp condition can be understood as follows. The nonlinear equation is like a linear equation with potential $\sim u^{p-1} \sim t^{-(d-1)(p-1)/2}$. The Cook criterion (see Reed and Simon vol. III) suggests that there is scattering behavior when this is integrable in time, that is $(d-1)(p-1)/2 > 1$.

The global existence result is in sharp contrast to the example

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} - u^2 = 0, \qquad u(0,x) = \epsilon\,\phi(x), \quad 0 \le \phi \in C_0^\infty(\mathbb{R}^d) \setminus 0,$$

for which solutions blow up in time $O(\epsilon^{-1})$ independent of dimension. The associated linear problem is completely nondispersive.

Assumption 1. $L(\partial)$ is a maximally dispersive symmetric hyperbolic system with constant coefficients as in §3.4.

Assumption 2. $G(u)$ is a smooth nonlinear function whose leading Taylor polynomial at the origin is homogeneous of degree $p \ge 3$.

Theorem 6.7.1. *Suppose that $(d-1)/2 > 1$ and $(d-1)(p-2)/2 > 1$, and σ is an integer greater than $(d+1)/2$. For each $\delta_1 > 0$, there is a $\delta_0 > 0$ so that if*

$$\|f\|_{H^\sigma(\mathbb{R}^d)} + \|f\|_{W^{\sigma,1}(\mathbb{R}^d)} \le \delta_0, \qquad \Big(\|f\|_{W^{\sigma,1}(\mathbb{R}^d)} := \sum_{|\alpha|\le\sigma} \|\partial_x^\alpha f\|_{L^1(\mathbb{R}^d)}\Big), \tag{6.7.1}$$

then the solution of the Cauchy problem

$$L\,u + G(u) = 0, \qquad u\big|_{t=0} = f, \tag{6.7.2}$$

exists globally and satisifes for all $t \in \mathbb{R}$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \langle t\rangle^{-(d-1)/2}\,\delta_1, \quad \text{and} \quad \|u(t)\|_{H^\sigma(\mathbb{R}^d)} \le \delta_1. \tag{6.7.3}$$

There is a $c > 0$, so that for δ_1 small one can take $\delta_0 = c\,\delta_1$.

Proof. We treat the case of $t \ge 0$. For simplicity we treat only the case of G equal to a homogeneous polynomial. The modifications for the general case are outlined in an exercise after the proof.

Decreasing δ_1 makes the task more difficult. If $\delta_1 \le 1$ is given, choosing δ_0 sufficiently small, the solution satisfies (6.7.3) on some maximal interval $[0, T[$, $T \in]0, \infty]$. The proof relies on *a priori* estimates for the solution on this maximal interval.

Denote by $S(t) := e^{-it\sum_j A_j \partial_j}$ the unitary operator on $H^s(\mathbb{R}^d)$ giving the time evolution for the linear equation $Lu = 0$,

$$\|S(t)\|_{H^s(\mathbb{R}^d)} = \|f\|_{H^s(\mathbb{R}^d)}. \tag{6.7.4}$$

The Theorem in §3.4.2 yields the estimate

$$\begin{aligned} \|S(t)\,f\|_{L^\infty(\mathbb{R}^d)} &\le C_0\,\langle t\rangle^{-(d-1)/2} \Big(\|f\|_{H^s(\mathbb{R}^d)} + \sum_{j=-\infty}^{\infty} \big\|\,|D|^{(d+1)/2} f_j\big\|_{L^1}\Big) \\ &\le C_1 \langle t\rangle^{-(d-1)/2}\,\delta_0. \end{aligned} \tag{6.7.5}$$

Duhamel's formula reads

$$u(t) \;=\; S(t)\, f \;+\; \int_0^t S(t-s)\; G(u(s))\; ds\,. \tag{6.7.6}$$

For the homogeneous polynomial G we have Moser's inequality,

$$\|G(u)\|_{H^\sigma(\mathbb{R}^d)} \;\le\; C_2\, \|u\|^{p-1}_{L^\infty(\mathbb{R}^d)}\; \|u\|_{H^\sigma(\mathbb{R}^d)}\,. \tag{6.7.7}$$

Exercise 6.7.1. *Prove (6.7.7).*

Use this and (the sharp condition) $(d-1)(p-1)/2 > 1$ to estimate

$$\begin{aligned} \|u(t)\|_{H^\sigma} \;&\le\; \delta_0 \;+\; \int_0^t C_2\, \big(\langle t-s\rangle^{-(d-1)/2}\,\delta_1\big)^{p-1}\; \delta_1\; ds \\ &\le\; \delta_0 + C_3\, \delta_1^p\,, \qquad C_3 := C_2 \int_0^\infty \langle t\rangle^{-(p-1)(d-1)/2}\, dt\,. \end{aligned} \tag{6.7.8}$$

The L^∞ norm satisfies,

$$\|u(t)\|_{L^\infty} \;\le\; C_1\, \langle t\rangle^{-(d-1)/2}\, \delta_0 \;+\; \int_0^t \big\|S(t-s)\; G(u(s))\big\|_{L^\infty}\; ds\,. \tag{6.7.9}$$

Use the dispersive estimate (3.4.8-3.4.9) to find

$$\big\|S(t-s)\, G(u(s))\big\|_{L^\infty} \;\le\; C_6\, \langle t-s\rangle^{-(d-1)/2}\, \|G(u(s))\|_{W^{\sigma,1}}\,. \tag{6.7.10}$$

Lemma 6.7.2. *There is a constant C so that for all u one has*

$$\|G(u)\|_{W^{\sigma,1}} \;\le\; C\, \|u\|^{p-2}_{L^\infty}\; \|u\|^2_{H^\sigma}\,. \tag{6.7.11}$$

Proof of Lemma. Leibniz' rule shows that it suffices to show that if $|\alpha_1 + \ldots + \alpha_p| = s \le \sigma$ then

$$\big\|\partial^{\alpha_1} u\, \partial^{\alpha_2} u \;\;\cdots\;\; \partial^{\alpha_p} u\,\big\|_{L^1} \;\le\; C\, \|u\|^{p-2}_{L^\infty}\; \|u\|_{L^2} \|\, |D|^s u\|_{L^2}\,.$$

Both sides have the dimensions ℓ^{d-s}.

Define $\theta_i := |\alpha_i|/s$ so $\sum \theta_i = 1$. The Gagliardo-Nirenberg estimate interpolating between $u \in L^\infty$ and $|D|^s u \in L^2$ is

$$\|\partial^{\alpha_i} u\|_{L^{p_i}} \;\le\; C\, \|u\|^{1-\theta_i}_{L^\infty}\; \|\, |D|^s u\|^{\theta_i}_{L^2}\,, \qquad \frac{1}{p_i} = \frac{1-\theta_i}{\infty} + \frac{\theta_i}{2} = \frac{\theta_i}{2}\,.$$

Define $\theta := 1/(p-1)$ and interpolate between $\partial^{\alpha_i} u \in L^{p_i}$ and $\partial^{\alpha_i} u \in L^2$ to find

$$\|\partial^{\alpha_i} u\|_{L^{r_i}} \;\le\; \|\partial^{\alpha_i} u\|^{1-\theta}_{L^{p_i}}\; \|\partial^{\alpha_i} u\|^{\theta}_{L^2}\,, \qquad \frac{1}{r_1} \;=\; \frac{1-\theta}{p_i} + \frac{\theta}{2}\,.$$

Therefore,

$$\|\partial^{\alpha_i} u\|_{L^{r_i}} \ \le\ \|u\|_{L^\infty}^{(1-\theta_i)(1-\theta)}\ \|u\|_{L^2}^{(1-\theta_i)\theta}\ \|\,|D|^s u\|_{L^2}^{\theta_i}\,, \qquad 1 = \sum 1/r_i\,.$$

Hölder's inequality implies

$$\big\|\partial^{\alpha_1} u\,\partial^{\alpha_2} u\ \cdots\ \partial^{\alpha_p} u\big\|_{L^1}\ \le\ \Pi_{i=1}^p \|\partial^{\alpha_i} u\|_{L^{r_i}}\ \le\ C\,\|u\|_{L^\infty}^{p-2}\ \|u\|_{L^2}\ \|\,|D|^s u\|_{L^2}\,,$$

which completes the proof. ∎

Estimates (6.7.10-11) yield,

$$\int_0^t \big\|S(t-s)\,G(u(s))\big\|_{L^\infty}\,ds\ \le\ C_7\int_0^t \langle t-s\rangle^{-(d-1)/2}\,\langle s\rangle^{-(p-2)(d-1)/2}\,\delta_1^2\,ds\,.$$

Consider $0\le s\le 1$ to see that this integral cannot decay faster than $\langle t\rangle^{-(d-1)/2}$. On the other hand on $s\ge t/2$ (resp. $s\le t/2$), the first (resp. second) factor in the integral is bounded above by $C\langle t\rangle^{-(d-1)/2}$ and the other factor uniformly integrable since using the hypotheses $(d-1)/2>1$ and $(p-2)(d-1)/2>1$. Therefore,

$$\int_0^t \big\|S(t-s)\,G(u(s))\big\|_{L^\infty}\,ds\ \le\ C_8\,\langle t\rangle^{-(d-1)/2}\,\delta_1^2\,. \tag{6.7.12}$$

Combining yields

$$\|u(t)\|_{L^\infty}\ \le\ \Big(C_1\,\delta_0 + C_8\,\delta_1^2\Big)\langle t\rangle^{-(d-1)/2}$$

Since $p>2$, decreasing δ_1 if necessary we may suppose that,

$$C_3\,\delta_1^2\ <\ \frac{\delta_1}{2}\,,\qquad \text{and}\qquad C_8\,\delta_1^2\ <\ \frac{\delta_1}{2}\,.$$

Then, choose $\delta_0>0$ so that

$$\delta_0\ +\ C_3\,\delta_1^2\ <\ \frac{\delta_1}{2}\,,\qquad \text{and}\qquad C_1\delta_0\ +\ C_8\,\delta_1^2\ <\ \frac{\delta_1}{2}\,.$$

With these choices, the estimates show that on the maximal interval $[0,T[$, one has

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)}\ \le\ \langle t\rangle^{-(d-1)/2}\,\frac{\delta_1}{2}\,,\quad \text{and}\quad \|u(t)\|_{H^s(\mathbb{R}^d)}\ \le\ \frac{\delta_1}{2}\,. \tag{6.7.13}$$

If T were finite, the solution would satisfy (6.7.3) on the interval $[0,T+\epsilon]$ for small positive ϵ violating the maximality of T. Therefore $T=\infty$. The estimate (6.7.13) on $[0,T[$ completes the proof. ∎

Remark. The estimates of the L^∞ norm using $W^{\sigma,1}$ are crude compared to those obtained using the weighted L^2 estimates of John-Klainerman.

Exercise 6.7.2. *For the case of G which are not homogeneous show that there are smooth functions H_α and functions G_α homogeneous of degree p so that*

$$G(u)\ =\ \sum\ G_\alpha(u)\ H_\alpha(u)\,,$$

the sum being finite. Modify the Moser inequality arguments appropriately to prove the general result.

§6.8. The subcritical nonlinear Klein-Gordon equation in the energy space.

§6.8.1. Introductory remarks.

The mass zero nonlinear Klein-Gordon equation is

$$\Box_{1+d} u \ + \ F(u) \ = \ 0\,. \tag{6.8.1}$$

where

$$F \in C^1(\mathbb{R})\,, \qquad F(0) = 0\,, \qquad F'(0) = 0\,. \tag{6.8.2}$$

The classic examples from quantum field theory are the equations with $F(u) = u^p$ with $p \geq 3$ an odd integer. For ease of reading we consider only real solutions.
The equation (6.8.1) is Lorentz invariant and if G denotes the primitive,

$$G'(s) \ = \ F(s)\,, \qquad G(0) \ = \ 0\,, \tag{6.8.3}$$

then the local energy density is defined as

$$e(u) \ := \ \frac{u_t^2 + |\nabla_x u|^2}{2} \ + \ G(u)\,. \tag{6.8.4}$$

Solutions $u \in H^2_{\rm loc}(\mathbb{R}^{1+d})$ satisfy the differential energy law,

$$\partial_t e - {\rm div}\big(u_t\, \nabla_x u\big) \ = \ u_t\Big(\Box u \ + \ F(u)\Big) = 0\,. \tag{6.8.5}$$

The corresponding integral conservation law for solutions suitably small at infinity is,

$$\partial_t \int_{\mathbb{R}^d} \frac{u_t^2 + |\nabla_x u|^2}{2} \ + \ G(u)\ dx \ = \ 0\,, \tag{6.8.6}$$

is one of the fundamental estimates in this section. Solutions are stationary for the Lagrangian,

$$\int_0^T \int_{\mathbb{R}^d} \frac{u_t^2 - |\nabla_x u|^2}{2} - G(u)\ dt\, dx\,.$$

When F is smooth, the methods of §6.3-6.4 yield local smooth existence.

Theorem 6.8.1. *If $F \in C^\infty$, $s > d/2$, $f \in H^s(\mathbb{R}^d)$, and $g \in H^{s-1}(\mathbb{R}^d)$, then there is a unique maximal solution*

$$u \ \in \ C\big([0,T_*[\ ;\ H^s(\mathbb{R}^d)\big) \ \cap \ C^1\big([0,T_*[\ ;\ H^{s-1}(\mathbb{R}^d)\big)\,.$$

satisfying

$$u(0,x) = f\,, \quad u_t(0,x) = g\,.$$

If $T_* < \infty$ then

$$\limsup_{t\to T_*}\ \|u(t)\|_{L^\infty(\mathbb{R}^d)} \ = \ \infty\,.$$

In favorable cases, the energy law (6.8.6) gives good control of the norm of $u, u_t \in H^1 \times L^2$. Controling the norm of the difference of two solutions is, in contrast, a very difficult problem for which many fundamental questions remain unresolved.

An easy first case is nonlinearities F which are uniformly lipschitzean. In this case, there is global existence in the energy space.

Theorem 6.8.2. *If F satisfies $F' \in L^\infty(\mathbb{R})$, then for all Cauchy data $f, g \in H^1 \times L^2$ there is a unique solution*

$$u \in C\big(\mathbb{R}\,;\; H^1(\mathbb{R}^d)\big) \;\cap\; C^1\big(\mathbb{R}\,;\; L^2(\mathbb{R}^d)\big)\,.$$

For any finite T, the map from data to solution is uniformly lipschitzean from $H^1 \times L^2$ to $C([-T,T\,;\,H^1) \cap C^1([-T,T]\,;\,L^2)$. If $f, g \in H^2 \times H^1$ then

$$u \;\in\; L^\infty\big(\mathbb{R};\; H^2(\mathbb{R}^d)\big)\,, \qquad u_t \;\in\; L^\infty\big(\mathbb{R};\; H^1(\mathbb{R}^d)\big)\,.$$

If $f, g \in H^s \times H^{s-1}$ with $1 \le s < 2$, then

$$u \;\in\; C\big(\mathbb{R};\; H^s(\mathbb{R}^d)\big)\,, \qquad u_t \;\in\; C\big(\mathbb{R};\; H^{s-1}(\mathbb{R}^d)\big)\,.$$

Sketch of Proof. The key estimate is the following. If u and v are solutions then

$$\Box(u-v) \;=\; F(v) - F(u), \qquad |F(u) - F(v)| \;\le\; C|u-v|\,.$$

Multiplying by $u_t - v_t$ yields

$$\frac{d}{dt}\int (u_t - v_t)^2 + |\nabla_x(u-v)|^2 dx \;=\; 2\int (u_t - v_t)\,\big(F(v) - F(u)\big)\,dx \;\le\; C\,\|u_t - v_t\|_{L^2}^2\,\|u-v\|_{L^2}^2\,.$$

It follows that for any T there is an *a priori* estimate

$$\sup_{|t|\le T}\;\Big(\|u(t) - v(t)\|_{H^1} \;+\; \|u_t - v_t\|_{L^2}\Big) \;\le\; C(T)\Big(\|u(0) - v(0)\|_{H^1} \;+\; \|u_t(0) - v_t(0)\|_{L^2}\Big)\,.$$

This estimate exactly corresponds to the asserted Lipschitz continuity of the map from data to solutions.

Applying the estimate to $v = u(x+h)$ and taking the supremum over small vectors h, yields an *a priori* estimate

$$\sup_{|t|\le T}\;\Big(\|u(t)\|_{H^2} \;+\; \|u_t\|_{L^2}\Big) \;\le\; C(T)\Big(\|u(0)\|_{H^2} \;+\; \|u_t(0)\|_{H^1}\Big)\,,$$

which is the estimate correponding to the H^2 regularity. ∎

Higher regularity for dimensions $d \ge 10$ is an outstanding open problem. For example, for $d \ge 10$, smooth compactly supported initial data, and $F \in C_0^\infty$ or $F = \sin u$, it is not known if the above global unique solutions are smooth. For $d \le 9$ the result can be found in [Brenner-vonWahl 1982]. Smoothness would follow if one could prove that $u \in L^\infty_{\rm loc}$. What is needed is to show that the solutions do not get large in the pointwise sense. Compared to the analogous regularity problem for

Navier-Stokes this problem has the advantage that solutions are known to be unique and depend continuously on the data.

§6.8.2. The ordinary differential equation and nonlipshitzean F.

Concerning global existence for functions $F(u)$ which may grow more rapidly than linearly as $u \to \infty$, the first considerations concern solutions which are independent of x and therefore satisfy the ordinary differential equation,

$$u_{tt} \;+\; F(u) \;=\; 0\,. \tag{6.8.7}$$

Global solvability of the ordinary differential equation is analysed using the energy conservation law

$$\Big(\frac{u_t^2}{2} \;+\; G(u)\Big)' \;=\; u_t\Big(u_{tt} + F(u)\Big) \;=\; 0\,.$$

Think of the equation as modeling a nonlinear spring. The spring force is attractive, that is pulls the spring toward the origin when

$$F(u) > 0 \quad \text{when} \quad u > 0 \qquad \text{and,} \qquad F(u) < 0 \quad \text{when} \quad u < 0\,.$$

In this case one has $G(u) > 0$ for all $u \neq 0$. Conservation of energy then gives a pointwise bound on u_t uniform in time

$$u_t^2(t) \;\le\; u_t^2(0) + 2G(u(0))\,, \qquad |u_t(t)| \;\le\; \big(u_t^2(0) + 2G(u(0))\big)^{1/2}\,.$$

This gives a pointwise bound

$$|u(t)| \;\le\; |u(0)| + |t|\big(u_t^2(0) + 2G(u(0))\big)^{1/2}\,.$$

In particular the ordinary differential equation has global solutions.

In the extreme opposite case consider the replusive spring force $F(u) = -u^2$ and $G(u) = -u^3/3$. The energy law asserts that $u_t^2/2 - u^3/3 := E$ is independent of time. Consider solutions with

$$u(0) > 0, \quad u_t(0) > 0 \qquad \text{so} \quad E > -\frac{u^3(0)}{3}\,.$$

For all $t > 0$,

$$|u_t| \;=\; \Big|\frac{u^3}{3} + E\Big|^{1/2}\,,$$

At $t = 0$ one has

$$u_t(0) \;=\; \Big(\frac{u^3(0)}{3} + E\Big)^{1/2} > 0\,.$$

Therefore u increases and $u^3/3 + E$ stays positive and one has for $t \ge 0$

$$u_t(t) \;=\; \Big(\frac{u^3(t)}{3} + E\Big)^{1/2} > 0\,.$$

Both u and u_t are strictly increasing.

Since

$$\frac{du}{\big(\frac{u^3}{3} + E\big)^{1/2}} \;=\; dt\,,$$

$u(t)$ approaches ∞ at time

$$T \;:=\; \int_{u(0)}^{\infty}\; \frac{du}{\left(\frac{u^3}{3}+E\right)^{1/2}}\,.$$

Exercise 6.8.1. *Show that if there is an $M>0$ so that $G(s)<0$ for $s\geq M$ and*

$$\int_M^\infty \frac{1}{\sqrt{|G(s)|}}\, ds \;<\; \infty$$

then there are solutions of the ordinary differential equation which blow up in finite time.

Keller's Blowup Theorem 6.8.3. [1957]. *If*

$$a,\delta>0, \qquad d\leq 3, \qquad E:=\delta^2/2-a^3/3, \qquad T \;:=\; \int_a^\infty \Big|\frac{u^3}{3}+E\Big|^{-1/2}\,du\,,$$

and $\phi,\psi\in C^\infty(\mathbb{R}^d)$ satisfy

$$\phi\geq a \quad \text{and} \quad \psi\geq\delta \qquad \text{for} \quad |x|\leq T\,,$$

the the smooth solution of

$$\Box_{1+d}u \;-\; u^2\,, \qquad u(0)=\phi, \quad u_t(0)=\psi$$

blows up on or before time T.

Proof. Denote by $\underline{u}$ the solution of the ordinary differential equation with initial data $\underline{u}(0)=a$, $\underline{u}_t(0)=\delta$.
If $u\in C^\infty\big([0,\underline{t}]\times\mathbb{R}^d\big)$, then finite speed of propagation and positivity of the fundamental solution of $\Box_{1+d}$ imply that

$$u \;\geq\; \underline{u} \qquad \text{on} \quad \big\{|x|\leq T-\underline{t}\big\}\,.$$

Since $\underline{u}$ diverges as $t\to T$ it follows that $\underline{t}\leq T$ ∎

In the case of attractive forces where $G\geq 0$ one can hope that there is global smooth solvability for smooth initial data. This question has received much attention and is very far from being understood. For example even in the uniformly lipschitzean case where solutions H^2 in x exist globally, C^∞ regularity is unknown in high dimensions.

§6.8.3. Subcritical nonlinearities.

In the remainder of this section we will study solvability in the energy space defined by $u,u_t\in H^1\times L^2$. This regularity is suggested by the basic energy law. For uniformly lipschitzean nonlinearities the global solvability is given by Theorem 6.8.2. The interest is in attractive nonlinearities with superlinear growth at infinity.

A crucial role is played by the rate of growth of F at infinity. There is a critical growth rate so that for nonlinearities which are subcritical and critical there is a good theory based on Strichartz estimates. The analysis is valid in all dimensions.

To concentrate on essentials, we present the family of attractive (repulsive) nonlinearities $F=u|u|^{p-1}$ ($F=-u|u|^{p-1}$) with potential energies given by $\pm\int|u|^{p+1}/(p+1)dx$. Start with four natural notions of subcriticality. They are in increasing order of strength. One could expect to call p subcritical when

1. $H^1(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ so the nonlinear term makes sense for elements of H^1.
2. $H^1(\mathbb{R}^d) \subset L^{p+1}(\mathbb{R}^d)$ so the potential energy makes sense for elements of H^1.
3. $H^1(\mathbb{R}^d)$ is compact in $L^{p+1}_{\rm loc}(\mathbb{R}^d)$ so the potential energy is in a sense small compared to the kinetic energy.
4. $H^1(\mathbb{R}^d) \subset L^{2p}(\mathbb{R}^d)$ so the nonlinear term belongs to $L^2(\mathbb{R}^d)$ for elements of H^1.

The Sobolev embedding is

$$H^1(\mathbb{R}^d) \ \subset\ L^q(\mathbb{R}^d)\,, \quad \text{for,} \quad q \ =\ \frac{2d}{d-2}\,. \tag{6.8.8}$$

The above conditions then read (with the values for $d = 3$ given in parentheses),

1. $p \le 2d/(d-2)$, $\qquad (p \le 6)$,
2. $p + 1 \le 2d/(d-2)$, $\qquad \big\{\text{equiv.} \quad p \le (d+2)/(d-2)\big\}$, $\qquad (p \le 5)$,
3. $p < (d+2)/(d-2)$, $\qquad (p < 5)$,
4. $p \le d/(d-2)$, $\qquad (p \le 3)$.

The correct answer is **3**. Much that will follow can be extended to the critical case $p = (d+2)/(d-2)$. The case **1** in contrast is supercritical and comparatively little is known. It is known that in the supercritical case, solutions are very sensitive to initial data. The dependence is not lipschitzean, and it is lipschitzean in the subcritical and critical cases. The books of Sogge, and Shatah-Struwe and the orignal 1985 article of Ginibre and Velo are good references. The sensitive dependence is a recent result of [Lebeau 2001, 2005].

Notation. *Denote by* $L^q_tL^r_x([0,T])$ *the space* $L^q_tL^r_x([0,T]\times\mathbb{R}^d)$, *For an open interval*

$$L^q_tL^r_x([0,T[) \ :=\ \cup_{0<\underline{T}<T}\ L^q_tL^r_x([0,\underline{T}])\,.$$

Theorem 6.8.4. i. *If p is subcritical for H^1, that is $p < (d+2)/(d-2)$, then for any $f \in H^1(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$ there is $T_* > 0$ and a unique solution*

$$u \in C([0,T_*[\ \ H^1(\mathbb{R}^d)) \ \cap\ C^1([0,T_*[\,;\,L^2(\mathbb{R}^d)) \ \cap\ L^p_tL^{2p}_x([0,T_*[) \tag{6.8.9}$$

of the repulsive problem

$$\Box u \ -\ u|u|^{p-1} \ =\ 0\,, \qquad u(0) = f, \quad u_t(0) = g\,. \tag{6.8.10}$$

If $T_* < \infty$ then

$$\liminf_{t\nearrow T_*}\ \|\nabla_{t,x}u\|_{L^2(\mathbb{R}^d)} \ =\ \infty\,. \tag{6.8.11}$$

The energy conservation law (6.8.6) is satisfied.
ii. *For the attractive problem*

$$\Box u \ +\ u|u|^{p-1} \ =\ 0\,, \qquad u(0) = f, \quad u_t(0) = g\,. \tag{6.8.12}$$

one has the same result with $T_* = \infty$ and with $u \in L^p_t L^{2p}_x(\mathbb{R})$. For any $T > 0$, the map from Cauchy data to solution is uniformly lipschitzean

$$H^1 \times L^2 \quad \to \quad C([0,T]\,;\,H^1)\ \cap\ C([0,T]\,;\,L^2)\ \cap\ L^p_t L^{2p}_x([0,T]))\,.$$

In the proof of this result and all that follows a central role is played by the linear wave equation and its solution for which we recall the basic energy estimate

$$\|\nabla_{t,x} u(t)\|_{L^2(\mathbb{R}^d)}\ \le\ \|\nabla_{t,x} u(0)\|_{L^2(\mathbb{R}^d)}\ +\ \int_0^t \|\Box u(t)\|_{L^2(\mathbb{R}^d)}\ dt\,.$$

This is completed by the L^2 estimate

$$\|u(t)\|_{L^2(\mathbb{R}^d)}\ \le\ \int_0^t \|u_t(t)\|_{L^2(\mathbb{R}^d)}\ dt\,.$$

In particular, for $h \in L^1_{\rm loc}\big(\mathbb{R}\,;\,L^2(\mathbb{R}^d)\big)$ there is a unique solution

$$u\ \in\ C\big(\mathbb{R}\,;\,H^1(\mathbb{R}^d)\big)\ \cap\ C^1\big(\mathbb{R}\,;\,L^2(\mathbb{R}^d)\big)\,,$$

to

$$\Box u\ =\ h\,,\qquad u(0)\ =\ 0\,,\quad u_t(0)\ =\ 0\,.$$

This solution is denoted

$$\Box^{-1} h\,.$$

In order to take advantage of this we seek solutions so that

$$F_p(u)\ :=\ \pm\, u|u|^{p-1}\ \in\ L^1_t L^2_x\,.$$

Compute

$$\|F_p(u)\|_{L^1_t L^2_x}\ =\ \int_0^T \Big(\int |u^p|^2\ dx\Big)^{1/2}\ dt\,,$$

where

$$\Big(\int |u|^{2p}\ dx\Big)^{1/2}\ =\ \Big[\Big(\int |u|^{2p}\Big)^{1/2p}\Big]^p\ =\ \|u\|^p_{L^{2p}(\mathbb{R}^d)}\,,$$

so

$$\|F_p(u)\|_{L^1_t L^2_x}\ =\ \int_0^T \|u\|^p_{L^{2p}_t \mathbb{R}^d_x}\ dt\ =\ \|u\|^p_{L^p_t L^{2p}_x}\,. \tag{6.8.13}$$

The above calculation proves the first part of the next lemma.

Lemma 6.8.5. *The mapping $u \mapsto F_p(u)$ takes $L^p_t L^{2p}_x([0,T]$ to $L^1_t L^2_x([0,T])$. It is uniformly Lipshitzean on bounded subsets.*

Proof. Write

$$F_p(v)\ -\ F_p(w)\ =\ G(v,w)\big(v-w\big)\,,\qquad |G(v,w)|\ \le\ C\Big(|v|^{p-1}\ +\ |w|^{p-1}\Big)\,.$$

Write

$$\|G(v,w)(v-w)\|^2_{L^2_x} \;=\; \int |G|^2\;|v-w|^2\;dx\,.$$

Use Hölder's inequality for $L^{p/(p-1)}_x \times L^p_x$ to estimate by

$$\le\;\Big(\int |G(v,w)|^{2p/(p-1)}dx\Big)^{\frac{p-1}{p}}\Big(\int |v-w|^{2p}dx\Big)^{\frac{1}{p}}\,.$$

Then

$$\|F_p(v)\;-\;F_p(w)\|_{L^2}\;\le\;C\,\|v,w\|^{p-1}_{L^{2p}_x}\;\|v-w\|_{L^{2p}_x}\,.$$

Finally estimate the integral in time using Hölder's inequality for $L^{p/(p-1)}_t \times L^p_t$. ∎

It is natural to seek solutions $u\in L^p_tL^{2p}_x([0,T])$. With that as a goal we ask when it is true that

$$\Box^{-1}\big(L^1_tL^2_x\big)\;\subset\;L^p_tL^{2p}_x\,.$$

This is exactly in the family of questions addressed by the Strichartz inequalities. The next Lemma gives the inequalities adapted to the present situation.

Lemma 6.8.6. *If*

$$q>2\,,\qquad\text{and}\qquad \frac{1}{q}\;+\;\frac{d}{r}\;=\;\frac{d}{2}-1\,,\tag{6.8.14}$$

then there is a constant $C>0$ so that for all $T>0$, $h,f,g\in L^1_t(L^2_x)\times H^1\times L^2$ the solution of

$$\Box u\;=\;h\,,\qquad u(0)\;=\;f\,,\qquad u_t(0)\;=\;g\,,$$

satisfies

$$\|u\|_{L^q_tL^r_x([0,T])}\;\le\;C\Big(\|h\|_{L^1_tL^2_x([0,T])}\;+\;\|\nabla_xf\|_{L^2(\mathbb{R}^d)}\;+\;\|g\|_{L^2(\mathbb{R}^d)}\Big)\,.\tag{6.8.15}$$

Proof. 1. Rewrite the wave equation as a symmetric hyperbolic pseudodifferential system motivated by D'Alembert's solution of the $1-d$ wave equation. Factor,

$$\partial_t^2-\Delta\;=\;(\partial_t+i|D|)\,(\partial_t-i|D|)\;=\;(\partial_t+i|D|)\,(\partial_t-i|D|)\,.$$

Introduce

$$v_\pm\;:=\;(\partial_t\mp i|D|)u\,,\qquad V\;:=\;(v_+,v_-)\,,$$

so

$$V_t\;+\;\begin{pmatrix}1&0\\0&-1\end{pmatrix}i|D|\,V\;=\;\begin{pmatrix}h\\h\end{pmatrix}\,.$$

Lemma 3.4.8 implies that for $\sigma=d-1$, $q>2$, (q,r) σ- admissible, and h,f,g with spectrum in $\{R_1\le|\xi|\le R_2\}$ one has

$$\|u\|_{L^q_tL^r_x}\;\le\;C\,\|\nabla_{t,x}u\|_{L^q_tL^r_x}\;\le\;C\,\|V\|_{L^q_tL^r_x}\;\le\;C\Big(\|h\|_{L^1_tL^2_x}\;+\;\||D|f\|_{L^2}\;+\;\|g\|_{L^2}\Big)\,.$$

2. Denote by ℓ the dimensions of t and x. With dimensionless u , the terms on right of this inequality have dimension $\ell^{d/2\,-1}$.

The dimension of the term on the left is equal to

$$\left(\ell^{dq/r}\,\ell\right)^{1/q} \;=\; \ell^{\frac{d}{r}+\frac{1}{q}}\,.$$

The two sides have the same dimensions if and only if

$$\frac{d}{r} \;+\; \frac{1}{q} \;=\; \frac{d}{2} - 1\,. \tag{6.8.16}$$

Under this hypothesis it follows that the same inequality holds, with the same constant C for data with support in $\lambda R_1 \le |\xi| \le \lambda R_2$.

Comparing (6.8.16) with σ-admissibility which is equivalent to

$$\frac{d}{r} \;+\; \frac{1}{q} \;\le\; \frac{d}{2} \;-\; \frac{1}{2} \;-\; \frac{1}{r}\,,$$

shows that (6.8.16) implies admissibility since $r \ge 2$.

3. Lemma 6.8.6 follows using Littlewood-Paley theory as at the end of §3.4.3. ∎

We now answer the question of when $\Box^{-1}$ maps $L^1_t L^2_x$ to $L^p_t L^{2p}_x$. This is the crucial calculation. In Lemma 6.8.6, take $r = 2p$ to find

$$\frac{1}{q} \;+\; \frac{d}{2p} \;=\; \frac{d-2}{2}\,,$$

so,

$$\frac{1}{q} \;=\; \frac{d-2}{2} - \frac{d}{2p} \;=\; \frac{p(d-2)-d}{2p}\,, \qquad q \;=\; p\left(\frac{2}{p(d-2)-d}\right).$$

We want $q \ge p$, that is

$$\frac{2}{p(d-2)-d} \;\ge\; 1\,, \quad \Leftrightarrow \quad p(d-2)-d \;\le 2 \quad \Leftrightarrow \quad p \;\le\; \frac{d+2}{d-2}\,.$$

The critical case is that of equality, and the subcritical case that we treat is the one with strict inequality. For $d = 3$ the critical power is $p = 5$ and for $d = 4$ it is $p = 3$. In the subcritical case the operator has small norm for $T << 1$.

The strategy of the proof is to write the solution u as a perturbation of the solution of the linear problem, at least for small times. Define u_0 to be the solution of

$$\Box u_0 \;=\; 0\,, \qquad u_0(0) \;=\; f, \quad \frac{\partial u_0}{\partial t}(0) \;=\; g\,. \tag{6.8.17}$$

Write

$$u \;=\; u_0 \;+\; v \tag{6.8.18}$$

with the hope that v will be small at least for t small.

Lemma 6.8.7. *If $u = u_0 + v$ with $v \in L^p_t L^{2p}_x\big([0,T]\big)$ satisfying*

$$v \;=\; \pm\Box^{-1} F_p(u_0 + v)\,. \tag{6.8.20}$$

then

$$u \;\in\; C([0,T]\,;\,H^1(\mathbb{R}^d)) \;\cap\; C^1([0,T]\,;\,L^2(\mathbb{R}^d)) \;\cap\; L^p_t L^{2p}_x([0,T]) \tag{6.8.21}$$

satisfies

$$\Box u \;\pm\; F_p(u) \;=\; 0\,, \qquad u(0) = f, \quad u_t(0) = g\,, \tag{6.8.22}$$

Conversely, if u satisfies (6.8.21)-(6.8.22) then $v := u - u_0 \in L^p_t L^{2p}_x([0,T])$ and satisfies (6.8.21)

Proof. The Strichartz inequality implies that $u_0 \in L^p_t L^{2p}_x$ and by hypothesis the same is true of v. Therefore $u_0 + v$ belongs to $L^p_t L^{2p}_x$ so $F_p(u_0 + v) \in L^1_t L^2_x$.

Therefore $v = \pm \Box^{-1} F_p$ is $C(H^1) \cap C^1(L^2)$. The differential equation and initial condition for v are immediate.

The converse is similar, not used below, and left to the reader. ∎

Proof of Theorem 6.8.4. For $K > 0$ arbitrary but fixed, we prove unique local solvability with continuous dependence for $0 \le t \le T$ with T uniform for all data f, g with

$$\|f\|_{H^1} \;+\; \|g\|_{L^2} \;\le\; K\,.$$

Choose $R = R(K)$ so that for such data,

$$\|u_0\|_{L^p_t L^{2p}_x([0,1])} \;\le\; \frac{R}{2}\,.$$

Define

$$B \;=\; B(T) \;:=\; \Big\{ v \in L^p_t L^{2p}_x([0,T]) \;:\; \|v\|_{L^p_t L^{2p}_x([0,T])} \;\le R \Big\}\,.$$

We show that for $T = T(K)$ sufficiently small, the map $v \mapsto \Box^{-1} F_p(u)$ is a contraction from B to itself.

This is a consequence of three facts.

1. Lemma 6.8.5 shows that F_p is uniformly lipschitzean from B to $L^1_t L^2_x([0,T])$ uniformly for $0 < T \le 1$.

2. Lemma 6.8.6 together with subcriticality shows that there is an $r > p$ so that $\Box^{-1}$ is uniformly lipshitzean from $L^1_t L^2_x$ to $L^r_t L^{2p}_x$ uniformly for $0 < T < 1$.

3. The injection $L^r_t L^{2p}_x \mapsto L^p_t L^{2p}_x$ has norm which tends to zero as $T \to 0$.

This is enough to carry out the existence parts of Theorem 6.8.4.

If there are two solutions u, v with the same initial data, compute

$$\Box(u - v) \;=\; G(u,v)(u - v)\,.$$

Lemma 6.8.6 together with subcriticality shows that with r slightly larger than p,

$$\|u - v\|_{L^r_t L^{2p}_x} \;\le\; C\, \|G(u,v)(u - v)\|_{L^1_t L^2_x} \;\le\; C\, \|u - v\|_{L^p_t L^{2p}_x}\,.$$

Use this estimate for $0 \le t \le T << 1$ noting that Hölder's inequality shows that for $T \to 0$,

$$\|u - v\|_{L^p_t L^{2p}_x} \;\le\; C\, T^\rho\, \|u - v\|_{L^r_t L^{2p}_x} \;\le\; C\, T^\rho \|u - v\|_{L^p_t L^{2p}_x}\,, \qquad \rho > 0\,,$$

to show that the two solutions agree for small times. Thus the set of times where the solutions agree is open and closed proving uniqueness.

To prove the energy law note that $F_p(u) \in L^1_t L^2_x$ so the linear energy law shows that

$$\int \frac{|u_t|^2 + |\nabla_x u|^2}{2}\, dx\Big|_{t=0}^{t} \;=\; \mp \int_0^t \int u_t\; F_p(u)\; dx\, dt\,. \tag{6.8.23}$$

Now

$$u_t \;\in\; L^\infty_t L^2_x\,, \quad \text{and} \quad F_p(u) \in L^1_t L^2_x\,.$$

Hölder's inequality shows that

$$\int |u_t\; F_p(u)| dx \;\le\; \|u_t(t)\|_{L^2_x}\; \|F_p(u(t)\|_{L^2_x}\,.$$

The latter is the product of a bounded and an integrable function so

$$\forall T, \quad u_t\, F_p(u) \;\in\; L^1([0,T] \times \mathbb{R}^d)\,.$$

Let

$$w \;:=\; \frac{|u|^{p+1}}{p+1}\,.$$

Since p is subcritical, one has for some $0 < \epsilon$,

$$\|w(t)\|_{L^1_x} \;\le\; C\|u(t)\|_{H^{1-\epsilon}(\mathbb{R}^d)} \;\in\; L^\infty([0,T])\,.$$

In particular $w \in L^1([0,T] \times \mathbb{R}^d)$ and the family $\{w(t)\}_{t\in[0,T]}$ is precompact in $L^1_{\rm loc}$.
Formally differentiating yields

$$w_t \;=\; u_t F_p(u) \;\in\; L^1([0,T] \times \mathbb{R}^d)\,. \tag{6.8.24}$$

Using the above estimates, it is not hard to justify (6.8.24).
It then follows that $w \in C([0,T]\,;\, L^1(\mathbb{R}^d))$ and

$$\int w(t,x)\; dx\Big|_{t=0}^{t=T} \;=\; \int_0^T \int u_t\; F_p(u) dx\, dt\,.$$

Together with (6.8.23) this proves the energy identity.

Once the energy law is known, one concludes global solvability in the attractive case since the blow up criterion (6.8.11) is ruled out by energy conservation. ∎

Chapter 7. One Phase Nonlinear Geometric Optics

In this chapter we construct asymptotic expansions which are nonlinear analogues of the Lax construction. There are two important nonlinear effects which must be understood in order to arrive at the appropriate *ansatz*.

§7.1. Amplitudes and harmonics.

For linear equations, any solution may be multiplied by a constant to yield another solution. This is not the case for nonlinear equations. If one studies short wavelength oscillatory solutions, the propagation and interactions depend crucially on the amplitudes. The easiest case to understand, and therefore a natural starting point, is small oscillations. For that we perform a (regular) perturbation analysis.

Consider the semilinear equation

$$L(y,\partial)\,u \;+\; F(u) \;=\; 0\,, \tag{7.1.1}$$

with nonlinear function satisfying

$$F(0)=0\,,\quad F'(0)=0\,. \tag{7.1.2}$$

Suppose that $a(\epsilon,y)\,e^{i\phi(y)/\epsilon}$ is a Lax solution as in §5.4 and that a has compact support for each t. Consider the semilinear initial value problem with the initial data

$$g(\epsilon,x)=\epsilon^m\,a(\epsilon,0,x)\,e^{i\phi(0,x)/\epsilon}\,. \tag{7.1.3}$$

The power m scales the amplitude as a function of the wavelength. The larger is m the smaller is the data. The initial data is bounded in $H^s(\mathbb{R}^d)$ if and only if $s\le m$.

If $m>d/2$ then the data converges to zero in $H^s(\mathbb{R}^d)$ for all $s\in]d/2,m[$. The perturbation theory of §6.5, proves that solutions exist on an ϵ independent neighborhood and are given by a Taylor series,

$$u(\epsilon,y)\;\sim\;\sum_{j=1}^{\infty} M_j(g(\epsilon,x))\;:=\;\sum_j u_j(\epsilon,x),$$

with u_j a j-linear function of g hence $\|u_j\|_{H^s(\mathbb{R}^d)}=O(\|g\|^j_{H^s(\mathbb{R}^d)})$. The leading u_j are determined by equations (6.5.4) through (6.5,6).

Theorem 6.3.1 proves existence on a domain independent of ϵ. For $m\le d/2$, the theorem guarantees existence only on a domain which shrinks with ϵ because the H^s norm of the data grows to ∞ for all $s>d/2$. We will see that for $m\ge 0$, there is, nevertheless, existence on an ϵ independent domain. The simple explicitly solvable example

$$\partial_t u(\epsilon,y)=u(\epsilon,y)^2\,,\qquad u(\epsilon,0,x)=\epsilon^m\,e^{ix.\xi/\epsilon}$$

shows that the domain may shrink to zero for $m<0$.

Exercise 7.1.1. *Verify.*

Equations (6.5.4)-(6.5.5) show that the two leading terms in perturbation theory are determined by,

$$L\,u_1\;=\;0\,,\qquad L\,u_2\;+\;F_2(0)\big(u_1,u_1\big)\;=\;0\,, \tag{7.1.4}$$

with initial conditions,

$$u_1(\epsilon, 0, x) \;=\; \epsilon^m\, a(\epsilon, 0, x)\, e^{i\phi(0,x)/\epsilon}\,, \qquad u_2(0,x) \;=\; 0\,. \tag{7.1.5}$$

Equations (7.1.4) and (7.1.5) show that as $\epsilon \to 0$, u_1 is given asymptotically by the Lax solution $\epsilon^m\, a(\epsilon, y)\, e^{i\phi(y)/\epsilon}$. Once u_1 is known the next term, u_2 can be found. And so on.

To see the form of u_2, it is crucial to consider the source term. It is a quadratic expression in u_1. The term u_1 oscillates with phase $\phi(y)/\epsilon$. Squaring such a term yields a source oscillating with phase $2\phi(y)/\epsilon$. The square of the complex conjugate, which is a second example of a smooth quadratic expression, yields a phase $-2\phi(y)/\epsilon$. Finally an expression in the product of u with its conjugate yields a nonoscillatory source. The source term has the form

$$\epsilon^{2m}\left(c_{-2}(y)\, e^{-i2\phi(y)/\epsilon} + c_0(y) + c_{+2}(y) e^{i2\phi(y)/\epsilon}\right) \;+\; O(\epsilon^{2m+1})\,. \tag{7.1.6}$$

From Lax's Theorem with oscillatory source, the oscillatory parts of this source yields terms of the form

$$\sum_{\pm}\; \epsilon^{2m}\left(a_{\pm 2}(y) + O(\epsilon)\right) e^{\pm i2\phi(y)/\epsilon}$$

in the solution $u_2(\epsilon, y)$.

The key observation is that the Taylor expansion begins with a $O(\epsilon^m)$ term which is linear in the initial data, and, is equal to the Lax solution. The next term, quadratic in the initial data, is of order ϵ^{2m} and has terms oscillating with the new phases $\pm 2\phi(y)/\epsilon$. It may also have nonoscillating terms of order ϵ^{2m}. The cubic and higher order terms in the Taylor expansion are of order ϵ^{jm} for integer j and will have terms oscillating with phases including higher integer multiples of $\pm\phi(y)/\epsilon$.

This generation and interaction of harmonics is one of the key signatures of nonlinear problems. Note that the wavelength of the j^{th} harmonic is $1/j$ times the original wavelength. Thus the interaction also is an interaction between different length scales. A classical experiment from the early sixties involved passing monochromatic red laser light through glass and observing the blue harmonic in the output. This was the birth of Nonlinear Optics. At the energies of that experiment, a small data perturbation theory like that just sketched is appropriate. Such an analysis can be found in the classic text of Nobel laureat Bloembergen.

Though this computation is so far only justified for $m > d/2$, it is an interesting indication that something better is true. Formally, the expansion seems to work provided only that $m > 0$, in which case the supposedly higher order corrections are indeed higher order in ϵ. In fact, using local existence results tailored to oscillatory data as in §8, the expansions can be justified for $m > 0$ on an arbitrary but fixed interval of time. On the other hand, for $m < 0$ we know that the domain of existence may shrink.

A fundamental lesson to be learned is that for $m > 0$ linear phenomena are accompanied by *creation of harmonics at higher order in* ϵ. This leads to correction terms in the solution which have amplitudes with higher powers of ϵ and phases which are integer multiples of $\phi(y)/\epsilon$. The higher is m, the smaller is the initial data and the greater is the gap between the amplitudes of the principal term and the harmonics. Equivalently, the smaller is m, the larger are the data, and the more important are the nonlinear effects.

There is another important lesson. The leading nonlinear term is of order ϵ^{2m} while the Lax solution enters at order ϵ^m. As $m \to 0$, these orders approach each other. This leads the courageous to suspect that there may be something interesting occurring when $m = 0$ in which case the harmonics should appear in the principal term. This in fact is the case. For $m = 0$, oscillations can be described on an ϵ independent domain, and the leading term in the expansion involves a nonlinear interaction among oscillations with phases $j\phi(y)/\epsilon$ for all $j \in \mathbb{Z}$. This critical scaling of the amplitudes is called *nonlinear geometric optics*, or *weakly nonlinear geometric optics* depending on the author. For this scaling the nonlinear terms are not negligible for the leading order. For this reason we say that the time of nonlinear interaction is ~ 1. For the same amplitudes one can show that the nonlinear terms can be neglected for times $o(1)$ as $\epsilon \to 0$.

Nonlinear geometric optics described here is more complicated than but descendant from earlier work on pulses of width ϵ and height one in spatial dimension 1. A description of the pulses and the relation to wave trains can be found in [Hunter-Majda-Rosales, Studies in Applied Math, 75(1986)] and in the survey article of [Majda]. Wave trains are blessed with interesting nonlinear interactions which go under the name of *resonance*. Generation of harmonics is the simplest case. Resonance for the $m = 0$ scaling of geometric optics are also described in the articles just cited. Resonance phenomena are introduced in §9.

§7.2. More on the generation of harmonics.

Here are three ordinary differential equation calculations aimed at making you more familiar with the creation of harmonics.

Exercise 7.2.1. *Consider the solution $x(\epsilon, t)$ of the nonlinear initial value problem*

$$\frac{d^2x}{dt^2} + \omega^2 x + x^2 = 0\,, \qquad x|_{t=0} = \epsilon\,, \quad \frac{dx}{dt}\Big|_{t=0} = 0\,.$$

Then x is an analytic function of ϵ, t on its domain of existence. Compute the first three terms in the Taylor expansion

$$x(\epsilon, t) = a_0(t) + \epsilon a_1(t) + \epsilon^2 a_2(t) + \cdots\ .$$

Note the presence of harmonics when they appear, and the amplitude of the harmonics.

In the last exercise, the harmonics appeared in a regular perturbation expansion of small solutions to a nonlinear equation. An entirely equivalent problem is the expansion of solutions of fixed amplitude with a weak nonlinearity.

Exercise 7.2.2. *Consider the solution $x(\epsilon, t)$ of the weakly nonlinear initial value problem*

$$\frac{d^2x}{dt^2} + \omega^2 x + \epsilon x^2 = 0\,, \qquad x|_{t=0} = 1\,, \quad \frac{dx}{dt}\Big|_{t=0} = 0\,.$$

Then x is an analytic function of ϵ, t on its domain of existence. Compute the first two terms in the Taylor expansion

$$x(\epsilon, t) = a_0(t) + \epsilon a_1(t) + \epsilon^2 a_2(t) + \cdots\ .$$

Note the presence of harmonics when they appear, and the amplitude of the harmonics.

Finally, here is an example of the generation of harmonics for forced oscillations.

Exercise 7.2.3. *Consider the solution $x(\epsilon, t)$ of the nonlinear initial value problem*

$$\frac{d^2x}{dt^2} + x + x^2 = \epsilon \cos\beta t\,, \qquad x|_{t=0} = 0\,, \quad \frac{dx}{dt}\Big|_{t=0} = 0\,, \qquad \beta \neq 0, \pm 1\,.$$

Then x is an analytic function of ϵ, t on its domain of existence. Compute the first three terms in the Taylor expansion

$$x(\epsilon, t) = a_0(t) + \epsilon a_1(t) + \epsilon^2 a_2(t) + \cdots\ .$$

Note the presence of harmonics when they appear, and the amplitude of the harmonics.

§7.3. Formulating the ansatz.

The results of §7.1, lead us to consider semilinear initial value problems with initial data of the form $a(\epsilon, 0, x)\, e^{i\phi(0,x)/\epsilon}$ which are initial data of a Lax solution in the linear case. The key fact is that the amplitude is of order ϵ^0. For this amplitude one expects harmonics to be present in the leading ϵ^0 term and for these harmonics to interact. We will describe these phenomena.

The computations suggest that the solution will have oscillations with all the phases $n\phi(y)/\epsilon$. Thus the principal term is expected to be at least as complicated as the sum of leading terms one for each harmonic. The amplitude of the $n^{\rm th}$ harmonic is denoted

$$a_0(n, \epsilon, y) \;\sim\; a_0(n, y) + \epsilon\, a_1(n, y) + \cdots \,. \tag{7.3.1}$$

It seems that the natural thing to do is to derive dynamic equations for the infinite set of amplitudes $a_0(n, y)$ which must include both the linear hyperbolic propagation properties given by rays and transport equations for each $a_0(n, y)$ and also nonlinear interaction terms which express at least the idea that if one starts with $a_0(1, y) \neq 0$ and all others vanishing then the other modes will tend to be illuminated.

There is a very effective method for managing this infinity of unknowns. The expected form for the leading terms is

$$\sum_{n\in\mathbb{Z}} a(n, y)\, e^{in\phi(y)/\epsilon} \,.$$

The sum on n suggests the Fourier series

$$U_0(y, \theta) \;:=\; \sum_{n=-\infty}^{\infty} a_0(n, y)\, e^{in\theta} \,. \tag{7.3.2}$$

The leading terms takes the elegant form $U_0(y, \phi(y)/\epsilon)$. The nonoscillatory terms are present from the $n = 0$ term. The function U_0 is periodic in θ and the amplitudes $a_0(n, y)$ are the Fourier coefficients of U. Knowing U is equivalent to knowing the $a_0(n, y)$ for all $n \in \mathbb{Z}$.

Adding correctors we *seek asymptotic solutions of first order semilinear symmetric hyperbolic systems in the form*

$$u(\epsilon, y) = U(\epsilon, y, \phi(y)/\epsilon) \tag{7.3.3}$$

where $U(\epsilon, y, \theta)$ is periodic in θ and has asymptotic expansion,

$$U(\epsilon, y, \theta) \sim \sum_{j=0}^{\infty} \epsilon^j\, U_j(y, \theta) \,. \tag{7.3.4}$$

The leading term, $U_0(y, \phi(y)/\epsilon)$ presents two scales. If $U_0(y, \theta)$ and $\phi(y)$ vary on the length scale 1, then the leading term varies on the scale 1, and the scale ϵ. The expansion (7.3.3) is called a two scale or multiscale expansion. In the case of ordinary differential equations, where there is only one independent variable, often called time, such expansions are often called two timing, after the presence of two time scales.

Using an *ansatz* containing a principal term as in (7.3.2) is a classical procedure in applied mathematics. Our approach here can be viewed as originating in the articles of Choquet-Bruhat and Hunter-Keller. In this one phase problem, the only resonances are among the harmonics, and in this case correctors as in (7.3.2) can be constructed. The multiphase theory with resonance must often content itself with leading order asymptotics only. In chapters 9-11, multiphase examples with correctors of all orders are considered.

§7.4. Equations for the profiles.

Once the *ansatz* (7.3.3-7.3.4) is formulated, the key question is whether it is possible to find profiles $U_j(y,\theta)$ so that

$$L(y,\partial_y)\,U(\epsilon,y,\phi(y)/\epsilon)+F(y,U(\epsilon,y,\phi(y)/\epsilon)\,)\ \sim\ 0\,. \tag{7.4.1}$$

Since

$$\partial_j\Big(U(\epsilon,y,\phi(y)/\epsilon)\Big)\ =\ \frac{\partial U}{\partial y}\Big|_{\epsilon,y,\phi(y)/\epsilon}\ +\ \frac{\partial_j\phi}{\epsilon}\,\frac{\partial U}{\partial\theta}\Big|_{\epsilon,y,\phi(y)/\epsilon}\ =\ \Big(\frac{\partial}{\partial y}\ +\ \frac{\partial_j\phi}{\epsilon}\,\frac{\partial}{\partial\theta}\Big)U\big|_{\theta=\phi(y)/\epsilon},$$

one has,

$$L(y,\partial_y)\,U(\epsilon,y,\phi(y)/\epsilon)\ =\ \Big[L\Big(y,\partial_y+\frac{d\phi(y)}{\epsilon}\,\frac{\partial}{\partial\theta}\Big)\,U(\epsilon,y,\theta)\Big]_{\theta=\phi(y)/\epsilon}.$$

Therefore the left hand side of (7.4.1) is equal to

$$W(\epsilon,y,\phi(y)/\epsilon)\ =\ W(\epsilon,y,\theta)\Big|_{\theta=\phi(y)/\epsilon}$$

with

$$W(\epsilon,y,\theta):=\frac{1}{\epsilon}\,L(y,d\phi(y))\,\frac{\partial}{\partial\theta}\,U(\epsilon,y,\theta)+L(y,\partial_y)\,U(\epsilon,y,\theta)+F\big(U(\epsilon,y,\theta)\big)\,.$$

The profile W is periodic in θ.

The middle term has asymptotic expansion,

$$L(y,\partial_y)\,U(\epsilon,y,\theta)\ \sim\ \sum\epsilon^j\,L(y,\partial_y)\,U_j(y,\theta)\,.$$

Taylor expansion about U_0 yields

$$F(y,U_0+\epsilon U_1+\cdots)\ \sim\ F(y,U_0)+\epsilon\,F_u(y,U_0)\,U_1+\text{h.o.t.} \tag{7.4.2}$$

The linear terms in U_1 are real linear and not necessarily complex linear, since F is assumed to be smooth but not necessarily holomorphic. These two expansions show that

$$W(\epsilon,y,\theta)\sim\sum_{j=-1}^{\infty}\epsilon^j\,W_j(y,\theta)=\epsilon^{-1}W_{-1}(y,\theta)+W_0(y,\theta)+\epsilon^1\,W_1(y,\theta)+\cdots\ . \tag{7.4.3}$$

The first terms are given by

$$W_{-1}(y,\theta)\ =\ L_1(y,d\phi(y))\,\partial_\theta U_0\,. \tag{7.4.4}$$

$$W_0(y,\theta)\ =\ L_1(y,d\phi(y))\,\partial_\theta\,U_1+L(y,\partial_y)\,U_0+F(y,U_0(y,\theta))\,. \tag{7.4.5}$$

$$W_1(y,\theta)=L_1(y,d\phi(y))\,\partial_\theta\,U_2+L(y,\partial_y)\,U_1+F_u(y,U_0)\,U_1\,. \tag{7.4.6}$$

$$W_2=L_1(y,d\phi(y))\,\partial_\theta\,U_3+L(y,\partial_y)\,U_2+F_u(y,U_0)\,U_2+F_{uu}(U_0)(U_1,U_1)\,. \tag{7.4.7}$$

The general case if of the form

$$W_j = L_1(y, d\phi(y))\, \partial_\theta\, U_{j-1} + L(y, \partial_y)\, U_j + F_u(y, U_0)\, U_j + G(U_0, \ldots, U_{j-1})\,, \tag{7.4.8}$$

where $G(U_0, \ldots, U_{j-1})$ denotes a nonlinear function of the preceding profiles. The G term and the $F_u U_j$ terms are are the $O(\epsilon^j)$ part of the Taylor expansion (7.4.2)

We will choose the U_j so that all the W_j vanish identically.

The W_{-1} is $O(\epsilon^{-1})$, and comes from the terms in (7.4.1) where the y derivatives fall on the $\phi(y)/\epsilon$ part.

In order for there to be nontrivial oscillations, one must have $\partial_\theta U_0 \neq 0$ so the first constraint we place on the expansion is that the matrix $L_1(y, d\phi(y))$ have nontrivial kernel. Equivalently, ϕ must satisfy the familiar eikonal equation

$$\det\, L_1(y, d\phi(y)) = 0\,. \tag{7.4.9}$$

Setting $W_{-1} = 0$ then yields the equation

$$U_0 \;\in\; \ker L_1(y, d\phi(y))\, \frac{\partial}{\partial\theta}\,. \tag{7.4.10}$$

Setting $W_0 = 0$ yields an equation which mixes U_0 and U_1. As in the linear case, information about U_0 is contained in the assertion

$$L(y, \partial_y)\, U_0(y, \theta) + F(y, U_0(y, \theta)) \;\in\; \text{range}\Big(\, L_1(y, d\phi(y))\, \frac{\partial}{\partial\theta}\,\Big)\,. \tag{7.4.11}$$

Equations (7.4.10) and (7.4.11) are our first form of the profile equations of nonlinear geometric optics. Written this way, it is not at all clear that they determine U_0 from its initial data. They are open invitations to study the action of the operator $L_1(y, d\phi(y))\partial_\theta$ on periodic functions of θ.

We suppose that (7.4.9) satisfied, and in addition that the constant rank hypothesis from §5.4 is satisfied on $\overline{\Omega}$ where Ω is open in $\mathbb{R}^{1+d}$. Denote by $\pi(y)$ orthogonal projection of $\mathbb{C}^N$ onto the kernel and $Q(y)$ the partial inverse. They are smooth thanks to Proposition 5.3.4.

The operator $L_1(y, d\phi(y))\partial_\theta$ maps $\mathcal{D}'(\Omega \times S^1)$ to itself with the subspace $C^\infty(\Omega \times S^1)$ also mapped to itself. The kernel and image can be computed by expanding in Fourier series in θ,

$$V(y, \theta) \;=\; \sum_{n=-\infty}^{\infty} V_n(y)\, e^{in\theta}\,. \tag{7.4.12}$$

When V is a distribution, the coefficient $V_n \in \mathcal{D}'(\Omega)$ is defined by

$$\big\langle V_n, \psi(y) \big\rangle \;:=\; \Big\langle V\,,\, \psi(y)\, \frac{e^{-in\theta}}{\sqrt{2\pi}} \Big\rangle, \qquad \psi \;\in\; C_0^\infty(\Omega)\,.$$

$V_0(y)$ is the nonoscillating contribution, and $(V - V_0)(y, \phi/\epsilon)$ is the oscillating part. One has

$$L_1(y, d\phi(y))\, \partial_\theta\, V \;=\; \sum L_1(y, d\phi(y))\, in\, V_n(y)\, e^{in\theta}\,. \tag{7.4.13}$$

The kernel consists of functions such that for $n \neq 0$, V_n takes values in the kernel of $L_1(y, d\phi(y))$. Equivalently,

$$V \subset \ker L_1(y, d\phi)\, \partial_\theta \quad \Longleftrightarrow \quad \forall\, n \geq 1, \quad \pi(y)\, V_n(y) \;=\; V_n(y)\,. \tag{7.4.14}$$

Formula (7.4.13) shows that the image of $L_1(y, d\phi(y))\partial_\theta$ consists of those Fourier series whose constant term vanishes, and whose other coefficients lie in the image of $L_1(y, d\phi(y))$. Equivalently,

$$V \subset \operatorname{range} L_1(y, d\phi)\, \partial_\theta \quad \Longleftrightarrow \quad V_0 = 0\,, \quad \text{and,} \quad \forall\, n \geq 1,\ (I - \pi(y))\, V_n(y) = V_n(y)\,. \tag{7.4.15}$$

Define a projection operator $\mathbf{E}$ on Fourier series by

$$\mathbf{E} \sum_{n=-\infty}^{\infty} V_n(y)\, e^{in\theta} \;:=\; V_0 \;+\; \pi(y) \sum_{n \neq 0} V_n(y)\, e^{in\theta}\,. \tag{7.4.16}$$

Then

$$\mathbf{E}\, V \;=\; V_0 + \pi(y)\big(V - V_0\big) \;=\; V_0 + \pi(y)\Big(V \;-\; \frac{1}{2\pi}\int_0^{2\pi} V(y,\theta)\; d\theta\Big)\,.$$

For each y, $\mathbf{E}$ acts as an orthogonal projection in $L^2(S^1)$. It follows that $\mathbf{E}$ is an orthogonal projection on $L^2(B \times S^1)$ for B a subset of $\{t = \mathit{const}\}$ or a subset of $\mathbb{R}^{1+d}$.
Formulas (7.4.15) and (7.4.16) show that

$$V \subset \ker L_1(y, d\phi)\, \partial_\theta \quad \Longleftrightarrow \quad \mathbf{E}\, V \;=\; 0\,, \tag{7.4.17}$$

and

$$V \subset \operatorname{range} L_1(y, d\phi)\, \partial_\theta \quad \Longleftrightarrow \quad (I - \mathbf{E})V \;=\; V\,. \tag{7.4.18}$$

Thus $\mathbf{E}$ projects onto the kernel of $L_1(y, d\phi)\partial_\theta$ along its range. The operators satisfy

$$\Big(L_1(y, d\phi(y))\partial_\theta\Big)\, \mathbf{E} \;=\; \mathbf{E}\,\Big(L_1(y, d\phi(y))\partial_\theta\Big) \;=\; 0\,, \tag{7.4.19}$$

and

$$\Big(L_1(y, d\phi(y))\partial_\theta\Big)\,(I - \mathbf{E}) \;=\; (I - \mathbf{E})\,\Big(L_1(y, d\phi(y))\partial_\theta\Big) \;=\; L_1(y, d\phi(y))\partial_\theta\,. \tag{7.4.20}$$

Define the partial inverse $\mathbf{Q}$ of the operator $L_1(y, d\phi(y))\partial_\theta$, by

$$\mathbf{Q}\Big(\sum V_n(y)\, e^{in\theta}\Big) \;:=\; Q(y) \sum_{n \neq 0} \frac{1}{in}\, V_n(y)\, e^{in\theta}\,, \tag{7.4.21}$$

where $Q(y)$ is the partial inverse of $L_1(y, d\phi(y))$ defined in (5.3.8). Then

$$\mathbf{E}\,\mathbf{Q} = \mathbf{Q}\mathbf{E} = 0, \quad \text{and} \quad \mathbf{Q}\,\Big(L_1(y, d\phi(y))\partial_\theta\Big) \;=\; \Big(L_1(y, d\phi(y))\partial_\theta\Big)\,\mathbf{Q} \;=\; I - \mathbf{E}\,. \tag{7.4.22}$$

Equation (7.4.10) is equivalent to

$$\mathbf{E}\, U_0 \;=\; U_0\,, \tag{7.4.23}$$

This equation shows that the oscillating part of U_0 satisfies the familiar polarization from §5.

In the same way, equation (7.4.11) is equivalent to,

$$\mathbf{E}\Big(\,L(y,\partial_y)\,U_0(y,\theta) + F(y,U_0(y,\theta))\Big) \;=\; 0\,. \tag{7.4.24}$$

The pair of equations (7.4.23), (7.4.24) is analogous in form to the pair of equations (5.3.10) and (5.3.11) which determined a_0. Equations (7.4.23-24) hold if and only if

$$W_{-1} = 0\,, \qquad \text{and,} \qquad \mathbf{E}\,W_0 = 0\,. \tag{7.4.25}$$

A note about our strategy here. Each equation $W_j = 0$ is equivalent to a pair of equations

$$W_j \;=\; 0 \qquad \Longleftrightarrow \qquad \mathbf{E}\,W_j \;=\; 0\,, \quad \text{and} \quad (I-\mathbf{E})\,W_j \;=\; 0\,.$$

The second equation if often transformed using,

$$(I-\mathbf{E})\,W_j \;=\; 0 \qquad \Longleftrightarrow \qquad \mathbf{Q}\,W_j = 0\,.$$

The equations for the profiles U_j are found by induction. Suppose that $j \ge 1$ and that $U_0, \dots, U_{j-1}$ have been determined so that

$$W_{-1} = \cdots = W_{j-1} = 0, \qquad \text{and} \qquad \mathbf{E}W_j = 0\,. \tag{7.4.26}$$

The equations determining U_j are then equivalent to,

$$(I-\mathbf{E})\,W_{j-1} = 0, \qquad \text{and} \qquad \mathbf{E}\,W_j = 0\,. \tag{7.4.27}$$

To illustrate the procedure, we find the profile equations for U_1. Equation (7.4.5) shows that $(I-\mathbf{E})W_0 = 0$ if and only if

$$(I-\mathbf{E})\,L_1(y,d\phi(y))\,\partial_\theta\,U_1 = -(I-\mathbf{E})\,\Big(\,L(y,\partial_y)\,U_0 + F(u,U_0)\Big) := F_0(y,U_0)\,, \tag{7.4.28}$$

where the right hand side, denoted F_0, is a function of the profile U_0 and its derivatives which are assumed known. The dependence on the derivatives is not indicated in the notation, since for the sequel it is not important just how many derivatives occur in the terms F_j. This equation determines U_1 modulo the kernel of the operator $(I-\mathbf{E})\,L_1(y,d\phi(y))\partial_\theta$, which is equal to the kernel of $L_1(y,d\phi(y))\partial_\theta$.
Multiplying by $\mathbf{Q}$ shows that equation (7.4.28) is equivalent to,

$$(I-\mathbf{E})\,U_1 = -\mathbf{Q}\,F_0(y,U_0)\,. \tag{7.4.29}$$

The determination of $\mathbf{E}\,U_1$ comes from setting $\mathbf{E}\,W_1 = 0$. Multiplying (7.4.6) by $\mathbf{E}$ eliminates the $\partial_\theta U_2$ term, and yields the second equation for the profile U_1,

$$\mathbf{E}\Big(\,L(y,\partial_y)\,U_1(y,\theta) + F_u(y,U_0)\,U_1(y,\theta)\Big) = 0\,. \tag{7.4.30}$$

The pattern is now established. Setting $(I-\mathbf{E})W_1 = 0$ yields

$$(I-\mathbf{E})\,U_2 \;=\; F_1(y,U_0,U_1)\,, \tag{7.4.31}$$

and the equation $\mathbf{E}W_2 = 0$ yields

$$\mathbf{E}\Big(L(y,\partial_y)\, U_2(y,\theta) + F_u(y,U_0)\, U_2(y,\theta) + F_{uu}(U_1,\overline{U}_1)\Big) = 0\,. \tag{7.4.32}$$

Here F_{uu} is the is a order two term in (7.4.2) so is a symmetric quadratic form in in U_1.

Continuing in this fashion yields for all $j \ge 1$ a pair of equations

$$(I - \mathbf{E})\, U_j = F_{j-1}(y, U_0, U_1, \dots, U_{j-1})\,, \tag{7.4.33}$$

and

$$\mathbf{E}\Big(L(y,\partial_y)\, U_j(y,\theta) + F_u(y,U_0)\, U_j(y,\theta) + G_j(y,U_0,\dots,U_{j-1})\Big) = 0 \tag{7.4.34}$$

which are equivalent to (7.4.27). The right hand side of (7.4.33) is a shorthand hiding the fact that it also depends on derivatives of the previously determined profiles $U_0, \dots, U_{j-1}$.

Theorem 7.4.1. *Suppose that $\phi \in C^\infty(\Omega)$ satisfies the eikonal equation (7.19) with nowhere vanishing differential and* $\dim\ker L(y, d\phi(y))$ *independent of y. In addition suppose $U_j \in C^\infty(\Omega \times \mathbb{T}^1)$ are profiles such that the principal profile U_0 satisfies (7.3.23-24) and the for $j \ge 1$ the profiles satisfy (7.4.33-34). If $U(\epsilon, y, \theta) \sim \sum_{j=0}^\infty \epsilon^j\, U_j(y,\theta)$ in $C^\infty(\Omega \times \mathbb{T}^1)$ and $u^\epsilon(y) := U(\epsilon, y, \phi(y)/\epsilon)$, then*

$$L(y,\partial_y)\, u^\epsilon + F(u^\epsilon) \ \sim\ 0\,, \qquad \text{in} \quad C^\infty(\Omega)\,.$$

Proof. The equations for the profiles are equivalent to solving $W_j = 0$ for all j. Thus if the profiles U_j satisfy the profile equations, and, $U(\epsilon, y, \theta) \ \sim\ \sum_{j=0}^\infty \epsilon^j\, U_j(y,\theta)$, then

$$L(y,\partial_y)\, U(\epsilon, y, \phi(y)/\epsilon) + F(y, U((\epsilon, y, \phi(y)/\epsilon))\,) \ \sim\ 0\,. \qquad \blacksquare$$

Remarks. 1. The equation for the principal profile U_0 is nonlinear in U_0 whereas the equations for the higher profiles U_j with $j \ge 1$, are $\mathbb{R}$-linear in U_j.

2. It is not at all obvious that the profile equation have solutions. We must prove analogues of Theorem 5.3.5.

§7.5. Solving the profile equations.

This subsection shows that the equations derived above determine the profiles U_j from suitable initial data. Once this is done, the asymptotic expansion is constructed yielding an approximate solution with infinitely small residual. In Chapter 8 it is proved that the approximate solution is asymptotic to the exact solution which has the same initial data.

To see that U_0 is determined from its initial data, start with the fact that (7.4.23) and (7.4.24) together imply that

$$\mathbf{E}\, L(y,\partial_y)\, \mathbf{E}U_0 + \mathbf{E}\, F(y, \mathbf{E}U_0(y,\theta)) = 0\,. \tag{7.5.1}$$

Applying $(I - \mathbf{E})\, L(y,\partial_y)$ to (7.4.23) yields

$$(I - \mathbf{E})\, L(y,\partial_y)\, (I - \mathbf{E})\, U_0 = 0\,. \tag{7.5.2}$$

Adding these two equations yields

$$(I - \mathbf{E})\, L(y,\partial_y)\, (I - \mathbf{E})\, U_0 \ +\ \mathbf{E}\, L(y,\partial_y)\, \mathbf{E}\, U_0 \ +\ \mathbf{E}\, F(y, \mathbf{E}\, U_0(y,\theta)) \ =\ 0\,, \tag{7.5.3}$$

an analogue of (5.3.16).

Define the linear operator

$$\mathbf{L} \ := \ \sum_{\mu=0}^{d} \mathbf{A}_\mu\, \partial_\mu \ + \ \mathbf{B}\,, \tag{7.5.4}$$

where the coefficients are the operators

$$\mathbf{A}_\mu \ := \ (I-\mathbf{E})\, A_\mu(y)\, (I-\mathbf{E}) + \mathbf{E}\, A_\mu(y)\, \mathbf{E}\,, \qquad \mathbf{B} \ := \ (I-\mathbf{E})\, B(y)\, (I-\mathbf{E}) + \mathbf{E}\, B(y)\, \mathbf{E}\,. \tag{7.5.5}$$

The unknown U_0 is a $\mathbb{C}^N$ valued function of t, x, θ with $\theta \in S^1$. The notation is chosen so that $\mathbf{L}$ looks like a differential operator. Some care must be taken since the coefficient operators are not simple matrix multiplications. However, the idea behind the basic energy estimate for symmetric hyperbolic operators extends nearly immediately to $\mathbf{L}$.

First of all the operators $\mathbf{A}_\mu$ are selfadjoint in $L^2(\omega \times S^1)$ because A_μ and $\mathbf{E}$ are. Since $\mathbf{E}$ commutes with differentiation one has

$$[\mathbf{A}_\mu, \partial] \ = \ (I-\mathbf{E})\, (\partial A_\mu(y))\, (I-\mathbf{E}) + \mathbf{E}\, (\partial A_\mu(y))\, \mathbf{E}\,.$$

This is bounded on any $L^2(\omega \times S^1)$ with norm bounded independent of ω.

The operator $\mathbf{A}_0$ is strictly positive. If ω is arbitrary, and $(\,\cdot\,,\cdot\,)$ denotes the $L^2(\omega \times S^1)$ scalar product, one has

$$(V, \mathbf{A}_0 V) \ = \ \big(V\,,\, (I-\mathbf{E})(A_0(I-\mathbf{E})V + \mathbf{E}A_0\mathbf{E}V\big) \ = \ \big((I-\mathbf{E})V\,,\, A_)(I-\mathbf{E})V\big) + \big(\mathbf{E}V\,,\, A_0\mathbf{E}V\big).$$

If C is a lower bound for A_μ, this is

$$\geq \ C\Big(\|(I-\mathbf{E})V\|_{L^2}^2 + \|\mathbf{E}V\|_{L^2}^2\Big) \ = \ C\,V\|_{L^2}^2\,,$$

proving strict positivity with a bound indpependent of ω. The same argument proves that if $\sum \eta_\mu A_\mu \geq 0$ then

$$\sum \eta_\mu \Big((I-\mathbf{E})A_0(I-\mathbf{E}) + \mathbf{E}A_0\mathbf{E}\Big) \ \geq \ 0\,.$$

In order to treat phases which need not be globally defined, this argument needs to be localized to domains of the form $\Omega \times S^1$ where Ω denotes a domain of determinacy for $L(y, partial)$ as in Corollary 2.3.7 or more generally as in §2.6. In either case the outward conormals η to the lateral boundaries satisfy $\sum \eta_\mu A_\mu \geq 0$ asserting that the flux corresponding to the energy density (A_0, u, u) is outward. This is reasonable for a domain that is not influenced by what goes on outside.

With these elements in place we can derive an L^2 energy estimate for the operator $\mathbf{L}$ in the set $\Omega \times S^1$. By definition,

$$\big(\mathbf{L}\,V\,,\, V\big)_{L^2(\Omega_t \times S^1)} \ = \ \big(\mathbf{E}\,L\,\mathbf{E}\,V\,,\, V\big) \ + \ \big((I-\mathbf{E})\,L\,(I-\mathbf{E})\,V\,,\, V\big)\,.$$

With $\omega = \Omega_t$ one finds

$$\big(\mathbf{L}\,V\,,\, V\big)_{L^2(\Omega_t \times S^1)} \ = \ \big(\,L\,\mathbf{E}\,V\,,\, \mathbf{E}\,V\big) \ + \ \big(L\,(I-\mathbf{E})\,V\,,\, (I-\mathbf{E})\,V\big)\,.$$

In this last expression θ plays the role of a parameter, and one has the form $(L\cdot\,,\,\cdot)$ applied to $\mathbf{E}V$ and to $(I-\mathbf{E})V$.

Using the energy flux is outward at the boundary, the energy balance identity (2.3.1) , implies that for $W \in C^1(\Omega_t \times S^1)$,

$$2\mathrm{Re}\big(L\,W\,,\,W\big)_{L^2(\Omega_t\times S^1)} \;\geq\; \big(A_0\,W\,,\,W\big)_{L^2(\Omega(t)\times S^1)}\Big|_0^t \;-\; \big(ZW\,,\,W\big)_{L^2(\Omega_t\times S^1)}\,. \tag{7.5.7}$$

Applying this with $W = \mathbf{E}\,V$ and $(I-\mathbf{E})W$ and adding the results shows that for $U \in C^1(\Omega_t \times S^1)$,

$$\big(U(t),\mathbf{A}_0\,U(t)\big)_{L^2(\Omega(t)\times S^1)}\Big|_0^t \;\leq\; 2\mathrm{Re}\;(U(t),\mathbf{L}\,U(t))_{L^2(\Omega_t\times S^1)} + C\,(U(t)\,,\,U(t)\,)_{L^2(\Omega_t\times S^1)}\,. \tag{7.5.8}$$

Since $\mathbf{A}_0$ is a strictly positive operator, (7.5.7) implies

$$\|U(t)\|_{L^2(\Omega(t)\times S^1)} \leq C(L,\phi)\,\Big(\,\|U(0)\|_{L^2(\Omega(0)\times S^1)} + \int_0^t \,\|\big(\mathbf{L}\,U(\sigma)\big)\|_{L^2(\Omega(\sigma)\times S^1)}\,d\sigma\Big)\,. \tag{7.5.9}$$

A commutation argument like that in §2.1 yields the more general estimate for $s \in \mathbb{N}$

$$\|U(t)\|_{H^s(\Omega(t)\times S^1)} \leq C(s,L,\phi)\,\Big(\,\|U(0)\|_{H^s(\Omega(0)\times S^1)} + \int_0^t \,\|\big(\mathbf{L}\,U(\sigma)\big)\|_{H^s(\Omega(\sigma)\times S^1)}\,d\sigma\Big)\,. \tag{7.5.10}$$

Replacing derivatives by difference quotients leads then to a convergent sequence of approximating equations which can be used to prove the following linear existence theorem.

Theorem 7.5.1. *If $s \in \mathbb{N}$, $g \in H^s(\Omega(0) \times S^1)$, and $f \in L^1(\Omega \times S^1)$ satisfies*

$$\int_0^T \,\Big(\,\int_{\Omega(t)\times S^1} \sum_{|\alpha|\leq s} |\partial^\alpha_{x,\theta} f(t,x,\theta)|^2\,dxd\theta\Big)^{1/2}\;dt < \infty\,, \tag{7.5.11}$$

then there is a unique $U \in L^2(\Omega \times S^1)$ satisfying

$$\mathbf{L}\,U = f\,, \qquad \text{and} \qquad U|_{t=0} = g\,. \tag{7.5.12}$$

The solution satisfies the estimate (7.5.10). If $g \in C^\infty(\Omega(0) \times S^1)$ and $f \in C^\infty(\Omega \times S^1)$, then $U \in C^\infty(\Omega \times S^1)$.

To treat nonlinear problems as in §6, note that for $s > (d+1)/2$, Schauder's Lemma implies that the map

$$U(t) \mapsto \mathbf{E}\,F(y, \mathbf{E}\,U(t))$$

is a locally Lipshitzean map of $H^s(\Omega(t) \times S^1)$ to itself, uniformly for $0 \leq t \leq T$.

Standard Picard iteration,

$$\mathbf{L}\,U^{\nu+1} + \mathbf{E}\,F(y, \mathbf{E}\,U^\nu) = 0\,, \qquad U^{\nu+1}|_{t=0} = g \tag{7.5.13}$$

as in §6 leads to the basic nonlinear local existence theorem. Existence is proved on $\Omega_T \times S^1$ where $\Omega_T \;:=\; \Omega \cap \{0 \leq t \leq T\}\,.$

Theorem 7.5.2 Local Solvability of the Principal Profile Equation. *If $(d+1)/2 < s \in \mathbb{N}$ and $g_0 \in H^s(\Omega(0)\times S^1)$, then there is a $0 < T$ and unique $U_0 \in C(\Omega_T \times S^1)$ satisfying (7.5.3) together with the initial condition $U_0(0,\cdot) = g$. If $g_0 \in C^\infty(\Omega(0)\times S^1)$ then $U_0 \in C^\infty(\Omega_T \times S^1)$.*

It is important to note that what is solved here is equation (7.5.3) which follows from the desired equations (7.4.23-24). Thus we have shown that the latter equations determine uniquely U_0 but we have not yet shown that there exists a U_0 satisfying (7.4.23-24). Clearly if the initial data g do not satisfy (7.4.23), then there is no chance for that equation.

Lemma 7.5.3. *If in addition to the hypotheses of the Theorem 7.5.2, $\mathbf{E}\, g = g$, then the resulting solution U_0 satisfies (7.4.23-24).*

Proof. As in the analysis of §5.4 an important first step is to observe that the left hand side of (7.5.3) is the sum of two orthogonal parts so that equation (7.5.3) implies that both vanish. Equivalently, multiplying (7.5.3) by $\mathbf{E}$ shows that U_0 satisfies the pair of equations

$$(I-\mathbf{E})\, L(y,\partial_y)\,(I-\mathbf{E})\, U_0 = 0\,, \quad \text{and} \quad \mathbf{E}\, L(y,\partial_y)\,\mathbf{E}\, U_0 + \mathbf{E}\, f(y,\mathbf{E}\, U_0(y,\theta)) = 0\,. \tag{7.5.14}$$

It follows that $\mathbf{E}\, U_0$ also satisfies both of these equations, and therefore equation (7.5.3). Since $\mathbf{E}\, U_0$ has the same initial data as U_0, it follows by uniqueness of solutions of the initial value problem for (7.5.3) that $\mathbf{E}\, U_0 = U_0$, which is equation (7.4.23).

Finally, (7.5.3) and the second equation of (7.5.14) imply (7.5.4). ∎

Fix $0 < T$ as in the Existence Theorem 7.5.3. Then the higher order profiles can be found on $\Omega_T \times S^1$ so as to satisfy (7.4.33-34). The argument is as follows. In (7.4.34) write

$$U_j = \mathbf{E}\, U_j + (I-\mathbf{E})\, U_j = \mathbf{E}\, U_j + F_{j-1}$$

where (7.4.33) is used in the last equality. This yields an equation of the form

$$\mathbf{E}\, L(y,\partial_y)\,\mathbf{E}\, U_j + \text{linear in } \ \mathbf{E}\, U_j = \text{known}\,.$$

To this equation add $(I-\mathbf{E})\, L(y,\partial_y)$ applied to (7.4.33) to find an equation for

$$C = \mathbf{E} U_j \tag{7.5.15}$$

of the form

$$\mathbf{L}\, C + \text{linear in } C = \text{known}\,. \tag{7.5.16}$$

This linear equation determines C from its initial data. Imitating arguments which by now should be familiar one shows that if the solution C satisfies $\mathbf{E}\, C = C$ at $t = 0$ then it does so throughout $\Omega_T \times S^1$ and that $U_j := C + \mathbf{Q}\, F_{j-1}$ satisfies the two profile equations (7.4.33-34).

Exercise 7.5.1. *Flesh out the details of this argument.*

Theorem 7.5.4. [Joly-Rauch]. *Suppose that $g_j = \mathbf{E}\, g_j \in C^\infty(\Omega(0)\times S^1)$, and that $T > 0$ is chosen as Theorem 7.5.2. Then there are uniquely determined profiles $U_j(y,\theta) \in C^\infty(\Omega_T \times S^1)$ with*

$$\mathbf{E}\, U_j\big|_{t=0} = g_j \quad \text{on} \quad \Omega(0)\times S^1 \tag{7.5.17}$$

and satisfying the profile equations (7.4.23)-(7.4.24) and (7.4.33)-(7.4.34). If

$$U(\epsilon,y,\theta) \ \sim \ \sum_{j=0}^{\infty} \epsilon^j\, U_j(y,\theta) \quad \text{in} \ \ C^\infty(\Omega_T \times S^1)\,, \quad \text{and} \quad u^\epsilon(y) := U(\epsilon,y,\phi(y)/\epsilon)\,, \tag{7.5.18}$$

then

$$L(y, \partial_y)\, u^\epsilon + f(y, u^\epsilon) \;\sim\; 0 \qquad \text{in} \quad C^\infty(\Omega_T)\,. \tag{7.5.19}$$

This completes the construction of an infinitely accurate family of approximate solutions u^ϵ. One point of view toward this, and that expressed in most science texts, is that the partial differential equations involve parameters which are only known approximately so an infinitely accurate approximation is for all practical purposes as good as an exact solution.

Hadamard offered a deeper appreciation of this remark. He observes that since there are uncertainties in the equations and data, in order for the equations to lead to well defined predictions, it is crucial that the predictions be unchanged or only very slightly changed when the equations and data are changed within the limits of the uncertainties. This lead to his notion of *well posed problems*.

In our case, the point of view of Hadamard leads to the question of showing that a pair of infinitely accurate approximate solutions with infinitely close initial data are in fact close. This does not follow from the basic existence theorem of section 6, because the approximate solutions tend to infinity in the configuration space H^s with $s > d/2$ and the sensitivity of the equation to perturbations grows for large data. One approach to circumventing this is to find a different configuration space in which a good existence theory is available and in which the approximate solutions do not grow. In the case $d = 1$, L^∞ does the trick. For higher dimensions, the space of bounded stratified solutions introduced by [Rauch and Reed, 1989] works and is the heart of the proof in [Joly-Rauch, 1992]. In §8 we give a different proof borrowing ideas from [Gues, 1992] and [Donnat, 1994].

Chapter 8. Stability for One Phase Nonlinear Geometric Optics

In the last section profiles $U_j(y,\theta)$, periodic in θ were constructed so that if

$$U(\epsilon, y, \theta) \sim \sum_{j=0}^{\infty} \epsilon^j \, U_j(y,\theta)\,, \tag{8.0.1}$$

then

$$u^\epsilon(y) \;:=\; U(\epsilon, y, \phi(y)/\epsilon) \tag{8.0.2}$$

satisfies,

$$L(y,\partial)\, u^\epsilon + F(y, u^\epsilon) \;\sim\; 0\,. \tag{8.0.3}$$

Denote by $v^\epsilon(y)$ the exact solution of

$$L(y,\partial)\, v^\epsilon + F(y, v^\epsilon) \;=\; 0\,, \qquad v^\epsilon\big|_{t=0} = u^\epsilon\big|_{t=0}\,. \tag{8.0.4}$$

To show that the asymptotic expansion is correct amounts to showing that

$$u^\epsilon(y) \;\sim\; v^\epsilon(y) \tag{8.0.5}$$

The difference between the equations defining the exact and approximate solutions is an infinitely small source term on the right hand side of (8.0.4). The task is to show that this small source can only lead to small changes in the solution. This is a stability problem. The techinical challenge is that the stability is needed near a family of solutions u^ϵ which though bounded in L^∞ is unbounded in the natural H^s spaces on which the time evolution is well behaved.

An important part of the proof is that the exact solution v^ϵ exists on an ϵ independent time interval. Since the $H^s(\mathbb{R}^d)$ norm of the initial data grows infinitely large for any $s > 0$ this is not obvious. It is nearly as hard to prove this existence as to prove the asymptotic equality (8.0.5).

§8.1. The $\mathbf{H}^{\mathbf{s}}_\epsilon(\mathbb{R}^{\mathbf{d}})$ norms.

A key to the analysis is the introduction of ϵ dependent Sobolev norms. The asymptotic solution has the form (8.0.2). The derivatives grow as ϵ decreases, but the operator $\epsilon\partial$ applied to the asymptotic solution is bounded independent of ϵ. This suggests that one estimates $(\epsilon\partial)^\alpha$ applied to the exact solution. This strategy was introduced by O. Gues [1993, 1992] to study the quasilinear version of the one phase theorems. It is also a centerpiece of the semiclassical limit in quantum mechanics where operators in $\hbar\partial$ take center stage.

Definition. *For $s \in \mathbb{Z}$, $0 < \epsilon \le 1$, and $w \in H^s(\mathbb{R}^d)$ define the $H^s_\epsilon(\mathbb{R}^d)$ norm by*

$$\|w\|^2_{H^s_\epsilon(\mathbb{R}^d)} := \sum_{|\alpha| \le s} \|\, (\epsilon\partial_x)^\alpha \, w\,\|^2_{L^2(\mathbb{R}^d)}\,. \tag{8.1.1}$$

A family w^ϵ is bounded in $H^s_\epsilon(\mathbb{R}^d)$ when

$$\sup_{0<\epsilon\le 1} \|w^\epsilon\|_{H^s_\epsilon(\mathbb{R}^d)} \;<\; \infty\,.$$

Example. For t fixed, the family $u^\epsilon(y)$ defined in (2) is bounded in $H^s_\epsilon(\mathbb{R}^d)$ provided that the support of $U(\epsilon, t, x, \theta)$ is bounded in x and contained in the domain of definition of ϕ.

The norm in $H^s_\epsilon(\mathbb{R}^d)$ is equivalent to the norm whose square is equal to

$$\int_{\mathbb{R}^d} (1 + |\epsilon\,\xi|^2)^s\,|\hat{u}(\xi)|^2\;d\xi\,, \tag{8.1.2}$$

which shows how the definition is generalized to noninteger s.

The $H^s_\epsilon(\mathbb{R}^d)$ Sobolev inequalities are immediate consequences of the following scaling identity,

$$v(x)\ :=\ w(\epsilon x)\quad\Longrightarrow\quad \partial_x v\ =\ (\epsilon\partial_x w)(\epsilon x)\,. \tag{8.1.3}$$

Thus,

$$\partial_x^\alpha v\ =\ \big((\epsilon\partial_x)^\alpha w\big)\,(\epsilon x)\,, \tag{8.1.4}$$

so

$$\|\partial_x^\alpha v\|^2_{L^2(\mathbb{R}^d)}\ =\ \int\,\big|(\epsilon\partial_x^\alpha)w\,(\epsilon x)\big|^2\;dx\,.$$

The change of variable $X := \epsilon x$ shows that this is equal to

$$\int\,|\,(\epsilon\partial_x)^\alpha w(X)\,|^2\;\epsilon^{-d}\;dX\ =\ \epsilon^{-d}\,\|\,(\epsilon\partial_x)^\alpha\,w\,\|^2_{L^2(\mathbb{R}^d)}\,.$$

Summing shows that

$$\epsilon^{d/2}\,\|v\|_{H^s(\mathbb{R}^d)}\ =\ \|w\|_{H^s_\epsilon(\mathbb{R}^d)}\,. \tag{8.1.5}$$

Using this one finds the following embedding of $H^s_\epsilon(\mathbb{R}^d)$ in L^∞. For $s > d/2$,

$$\|w\|_{L^\infty(\mathbb{R}^d)} = \|v\|_{L^\infty(\mathbb{R}^d)} \le C(s,d)\,\|v\|_{H^s(\mathbb{R}^d)} = \epsilon^{-d/2}\,C(s,d)\,\|w\|_{H^s_\epsilon(\mathbb{R}^d)}\,. \tag{8.1.6}$$

Similarly for smooth $F(w)$ which vanish for $w = 0$ one has an $H^s_\epsilon(\mathbb{R}^d)$ version of Schauder's Lemma for $s > d/2$. The starting point is the estimate $\|\,F(w)\,\|_{H^s(\mathbb{R}^d)}\ \le\ H(\|v\|_{H^s(\mathbb{R}^d)})$ with a nonlinear function H. Then,

$$\begin{aligned}\|\,F(w)\,\|_{H^s_\epsilon(\mathbb{R}^d)}\ &=\ \epsilon^{d/2}\,\|\,F(v)\,\|_{H^s(\mathbb{R}^d)}\\ &\le \epsilon^{d/2}\,H(\|v\|_{H^s(\mathbb{R}^d)})\ =\ \epsilon^{d/2}\,H(\epsilon^{-d/2}\,\|w\|_{H^s_\epsilon(\mathbb{R}^d)}\,)\,.\end{aligned} \tag{8.1.7}$$

The negative power of ϵ in the the argument of H is intolerable.

Moser's inequality is much better behaved,

$$\begin{aligned}\|\,F(w)\,\|_{H^s_\epsilon(\mathbb{R}^d)}\ &=\ \epsilon^{d/2}\,\|\,F(v)\,\|_{H^s(\mathbb{R}^d)}\\ &\le\ \epsilon^{d/2}\,G(\|v\|_{L^\infty})\,\|v\|_{H^s(\mathbb{R}^d)}\ =\ G(\|w\|_{L^\infty})\,\|w\|_{H^s_\epsilon(\mathbb{R}^d)}\ .\end{aligned} \tag{8.1.8}$$

The cancellation of powers of $\epsilon^{\pm d/2}$ in the last inequality shows that *the Moser inequality for $H^s_\epsilon(\mathbb{R}^d)$ is independent of* ϵ.

In the justification of the asymptotic expansion, it is crucial to estimate the difference $F(u)-F(v)$ when u and v are close. In our application, the function u is our approximate solution and v is the exact solution. In this way, more is known of u than of v. There is sup norm control of the operators $\epsilon\partial$ applied to u which will be used to get L^2 control on these operators applied to v.

Lemma 8.1.1. *For any $R>0$ and s there is a constant $C = C(F,R,s)$ so that if u satisfies*

$$\|(\epsilon\partial)^\alpha u\|_{L^\infty(\mathbb{R}^d)} \;\le\; R \qquad \text{for all } \;|\alpha|\le s\,,$$

and w satisfies the weaker estimate,

$$\|w\|_{L^\infty(\mathbb{R}^d)} \;\le\; R\,,$$

then for $0<\epsilon$,

$$\|F(y,u+w)-F(y,u)\|_{H^s_\epsilon(\mathbb{R}^d)} \;\le\; C\,\|w\|_{H^s_\epsilon(\mathbb{R}^d)}\,.$$

Proof. To simplify the exposition suppose that F does not depend on y. It suffices to prove the assertion with $\epsilon=1$ since both sides of the inequality scale as $\epsilon^{d/2}$.

To prove the assertion for $\epsilon=1$ write,

$$F(u+w)-F(w) \;=\; \Big(\int_0^1 F'(u+\sigma w)\; d\,\sigma\Big) w \;:=\; \mathcal{G}(u,w)\,w\,. \tag{8.1.9}$$

Expanding $\partial^\nu(\mathcal{G}(u,w)\,w)$ using Leibniz' rule yields a finite number of terms of the form

$$H_{\alpha,\beta}(u,w)\;(\partial^{\alpha_1}u)\cdots(\partial^{\alpha_m}u)\;(\partial^{\beta_1}w)\cdots(\partial^{\beta_n}w)$$

with $\sum\alpha_k+\sum\beta_l=\nu$. The product of the first $m+1$ factors has sup norm bounded by $C(R)$. The proof of Moser's inequality shows that the product of the last n has $L^2(\mathbb{R}^d)$ norm bounded by $C(R)\|w\|_{H^s(\mathbb{R}^d)}$. This completes the poof. ∎

Exercise 8.1.1. *Prove the Lemma for F which depend on y.*

For applications where the phase is only locally defined one must work locally. As in Chapter 7, we suppose that ϕ is defined on a domain of determinacy, Ω, and denote,

$$\Omega_T \;:=\; [0,T]\cap\Omega, \qquad \text{and,} \qquad \Omega(t):=\{x\,:\,(t,x)\in\Omega\}\,.$$

The usual reflection operators construct a linear extension operators $v\to E(t)v$ from $H^s(\Omega(t))$ to $H^s(\mathbb{R}^d)$ so that $Ev=v$ on $\Omega(t)$. To study H^s_ϵ by scaling one needs extensions from $\Omega(t)/\epsilon$ to $\mathbb{R}^d$ for $0<\epsilon<1$. These are domains which are increasingly regular. The standard constructions yield extension operators with norms bounded independently of $0<\epsilon\le 1$, $0\le t\le T$ under very mild regularity assumptions on Ω. The next exercise recalls the construction for balls. An analogous construction works for half spaces and then via coordinate charts for regular Ω.

Exercise 8.1.2. *For $s=0$ extending v to vanish outside B works. For $s=1$ and $|x|>1$ denote by $R(x):=x/|x|^2$ the reflected point in the unit sphere. Choose a smooth function χ which is equal to 1 on a neighborhood of 1 and vanishes ouside $]1/2,3/2[$. Show that setting $Ev(x):=\chi(|x|)\,v(R(x))$ for $|x|>1$ works for $s=1$. For larger s construct an appropriate extension operator by setting*

$$Ev \;:=\; \sum_{j=1}^{s} c_j\, v(R(x_j))\,, \qquad x_j \;:=\; \Big(1+2^j(|x|-1)\Big)\,\frac{x}{|x|}\,.$$

The key is the choice of the constants c_j so that $s-1$ derivatives match at the boundary of the ball. This elegant idea is called Lions reflection after J.L. Lions.

Assumption. *Assume that uniformly bounded extension operators from $H^s_\epsilon(\Omega(t))$ to $H^s_\epsilon(\mathbb{R}^d)$ exist for $0 < \epsilon \le 1$ and $0 \le t \le T$.*

Lemma 8.1.2. *When the assumption holds, Lemma 8.1.1 holds with $\mathbb{R}^d$ replaced by $\Omega(t)$ with constant independent of $0 < \epsilon \le 1$ and $0 \le t \le T$.*

Exercise 8.1.3. *Write out the details of the proof using the extension operators..*

§8.2. $\mathbf{H^s_\epsilon}$ estimates for linear symmetric hyperbolic systems.

In addition to the estimates of the last section, the analysis relies on the fact that linear hyperbolic systems propagate the $H^s_\epsilon(\mathbb{R}^d)$ norms. This fact depends on the basic linear estimate and commutation identities between the operator L and the operators $(\epsilon\partial)^\alpha$. The argument is entirely analogous to the commutation arguments in §1.1 and §2.1. Square brackets are used to denote the commutator.

Introducing the new variable $\underline{u} := A_0^{-1/2} u$ and multiplying the resulting system for $\underline{u}$ by $A_0^{-1/2}$ yields a semilinear equation of the same form as before with new coefficient matrices $\underline{A}_\mu := A_0^{-1/2} A_\mu A_0^{-1/2}$. In particular, the coefficient of the time derivative is equal to the identity matrix. Thus, without loss of generality we may suppose that $A_0 = I$.

Lemma 8.2.1. *If $A_0 = I$, then for any $\alpha \in \mathbb{N}^d$ there are matrix valued functions $C_{\alpha\beta}(\epsilon, y)$ with uniformly bounded derivatives on $]0,1] \times \mathbb{R}^{1+d}$ so that*

$$[L(y,\partial_y)\,,\,(\epsilon\partial_x)^\alpha\,] = \sum_{|\beta|\le|\alpha|} C_{\alpha\beta}(\epsilon,y)\,(\epsilon\partial_x)^\beta\,.$$

Remark. If A_0 depended on time, there would be time derivatives in the commutators. It is to avoid these, that we transform to the case $A_0 = I$.

Proof. The proof is by induction on $|\alpha|$. For $|\alpha| = 1$ compute

$$[L(y,\partial_y)\,,\,\epsilon\partial_j] = -\sum_k (\partial_j A_k)\,\epsilon\partial_k + \epsilon(\partial_j B)$$

Suppose next that $m \ge 1$, and the result is true for derivatives of length less than or equal to m. A differention of length $m+1$ is of the form $\epsilon\partial_j(\epsilon\partial_x)^\alpha$ with $|\alpha| = m$. Then

$$\begin{aligned} L\,\epsilon\partial_j\,(\epsilon\partial_x)^\alpha - \epsilon\partial_j\,(\epsilon\partial_x)^\alpha\,L =& [L,\epsilon\partial_j]\,(\epsilon\partial_x)^\alpha + \epsilon\partial_j\,L\,(\epsilon\partial_x)^\alpha - \epsilon\partial_j\,(\epsilon\partial_x)^\alpha\,L \\ =& [L,\epsilon\partial_j]\,(\epsilon\partial_x)^\alpha + \epsilon\partial_j\,[L,(\epsilon\partial_x)^\alpha]\,. \end{aligned}$$

Using the inductive hypothesis to express the commutators, the result follows. ∎

Theorem 8.2.2. *If $A_0 = I$ then for any $s \in \mathbb{N}$ and $T \in]0,\infty[$ there is a constant $C = C(s,T,L)$ so that for all $0 \le \underline{t} \le T$, and $u \in C([0,\underline{t}]\,;\,H^s(\mathbb{R}^d))$ with $Lu \in L^1([0,\underline{t}]\,;\,H^s(\mathbb{R}^d))$,*

$$\|u(\underline{t})\|_{H^s_\epsilon(\mathbb{R}^d)} \le C\left(\|u(0)\|_{H^s_\epsilon(\mathbb{R}^d)} + \int_0^{\underline{t}} \|(Lu)(\sigma)\|_{H^s_\epsilon(\mathbb{R}^d)}\,d\sigma\right). \tag{8.2.1}$$

Proof. For $|\alpha| \le s$ use the commutation lemma to write

$$L\,(\epsilon\partial_x)^\alpha u = (\epsilon\partial_x)^\alpha Lu + \sum C_{\alpha\beta}(\epsilon, y)\,(\epsilon\partial_x)^\beta u\,. \tag{8.2.2}$$

The basic linear estimate (2.1.18) then implies that for any $0 \le t \le \underline{t}$,

$$\begin{aligned} \|(\epsilon\partial_x)^\alpha u(t)\|_{L^2(\mathbb{R}^d)} \le C\Big(&\|(\epsilon\partial_x)^\alpha u(0)\|_{L^2(\mathbb{R}^d)} \\ &+ \int_0^t \big\{\|(\epsilon\partial_x^\alpha)(Lu)(\sigma)\|_{L^2(\mathbb{R}^d)} + \|u(\sigma)\|_{H^s_\epsilon(\mathbb{R}^d)}\big\}\,d\sigma\Big). \end{aligned} \tag{8.2.3}$$

Summing over all $|\alpha| \le s$ yields with a new constant

$$\|u(t)\|_{H^s_\epsilon(\mathbb{R}^d)} \le C\Big(\,\|u(0)\|_{H^s_\epsilon(\mathbb{R}^d)} + \int_0^t \big\{\|(Lu)(\sigma)\|_{H^s_\epsilon(\mathbb{R}^d)} + \|u(\sigma)\|_{H^s_\epsilon(\mathbb{R}^d)}\big\}\,d\sigma\Big)\,. \tag{8.2.4}$$

Gronwall's inequality completes the proof. ∎

The following local estimate is sufficient for our needs. The proof is exactly like the proof of the estimate in $H^s(\mathbb{R}^d)$.

Theorem 8.2.3. *If Ω_T is defined by (8.2.5) and $s \in \mathbb{N}$ there is a constant $C = C(s, L, \Omega)$ so that for all $u \in C^\infty(\Omega_T)$ and all $t \in [0, T]$*

$$\|u(t)\|_{H^s_\epsilon(\Omega(t))} \le C\Big(\,\|u(0)\|_{H^s_\epsilon(\Omega(0))} + \int_0^t \|Lu(\sigma)\|_{H^s_\epsilon(\Omega(\sigma))}\,d\sigma\Big)\,. \tag{8.2.5}$$

§8.3. Justification of the asymptotic expansion.

Theorem 8.3.1 [Joly-Rauch, 1992]. *Suppose that the phase ϕ and smooth profiles $U_j(y,\theta)$ satisfy the profile equations on the domain of determinacy Ω_T as above, and, the approximate solution u^ϵ is defined by (8.0.2) with $U(\epsilon, y, \theta) \sim \sum \epsilon^j\, U_j(y,\theta)$ in $C^\infty(\Omega_T \times S^1)$. Then for ϵ small the exact solution v^ϵ defined in (8.0.4) exists and is smooth on Ω_T and*

$$v^\epsilon \;\sim\; u^\epsilon \qquad \text{in} \quad C^\infty(\Omega_T)\,.$$

This result has nothing to do with the form of the profile equations and the algorithm to construct the approximate solutions. It is a special case of a stability result about families of approximate solutions with bounded $\epsilon\partial$ derivatives.

Theorem 8.3.2. (Gues, Donnat). *Suppose that $u^\epsilon \in C^\infty(\Omega_T)$ is $\epsilon\partial$ bounded in the sense that for all $\alpha \in \mathbb{N}^{d+1}$*

$$\sup_{0<\epsilon<1} \big\|\big(\epsilon\,\partial\big)^\alpha u^\epsilon\big\|_{L^\infty(\Omega_T)} < \infty.$$

Suppose that it is an infinitely accurate family of approximate solutions in the sense that,

$$L(u^\epsilon) + F(u^\epsilon) \;\sim\; 0 \quad \text{in} \quad C^\infty(\Omega_T)\,.$$

Then for ϵ small the exact solution v^ϵ defined in (8.0.4) exists and is smooth on Ω_T and

$$v^\epsilon \;\sim\; u^\epsilon \qquad \text{in} \quad C^\infty(\Omega_T)\,. \tag{8.3.1}$$

Proof. Fix $d/2 < s \in \mathbb{N}$. The local existence theorem implies either the existence of a smooth solution v^ϵ on Ω_T or the existence of a $T^*(\epsilon) \le T$ so that v^ϵ is smooth on

$$\Omega_* \ := \ \Omega \cap \{0 \le t < T^*\}\,, \tag{8.3.2}$$

and v^ϵ blows up at $T^*(\epsilon)$,

$$\lim_{t \to T^*(\epsilon)} \|v^\epsilon(t)\|_{H^s_\epsilon(\Omega(t))} \ = \ \infty\,. \tag{8.3.3}$$

We show that for ϵ sufficiently small, the second alternative does not occur and that (8.3.1) holds. The $\epsilon\partial$ boundedness of u^ϵ implies that there is an $R > 0$ so that for $0 < \epsilon \le 1$ and $|\alpha| \le s$

$$\sup_{0 \le t \le T} \Big(\|(\epsilon\partial_x)^\alpha u^\epsilon\|_{L^\infty(\Omega(t))} \Big) \ \le \ R/2\,. \tag{8.3.4}$$

Denote by $r^\epsilon(y)$ the residual in the equation for the approximate solution,

$$L(y, \partial_y)\, u^\epsilon + F(y, u^\epsilon) \ := \ r^\epsilon\,. \tag{8.3.5}$$

By hypothesis,

$$r^\epsilon(y) \ \sim \ 0 \qquad \text{in} \quad C^\infty(\Omega_T)\,. \tag{8.3.6}$$

Introduce the error

$$w^\epsilon \ := \ v^\epsilon - u^\epsilon\,. \tag{8.3.7}$$

An initial value problem for the error is derived by subtracting (8.3.5) from (8.0.4). Suppressing the y dependence of F this yields,

$$L\, w^\epsilon + F(u^\epsilon + w^\epsilon) - F(u^\epsilon) \ = \ -r^\epsilon\,, \quad \text{on} \quad \Omega_*\,, \tag{8.3.8}$$

$$w^\epsilon\big|_{t=0} \ = \ 0\,. \tag{8.3.9}$$

Estimate (8.2.5) gives a C independent of ϵ and t so that for $0 \le t < T^*(\epsilon)$,

$$\|w^\epsilon(t)\|_{H^s_\epsilon(\Omega(t))} \le C\Big(\int_0^t \|F(u^\epsilon + w^\epsilon) - F(u^\epsilon)\|_{H^s_\epsilon(\Omega(\sigma))}\, d\sigma + \int_0^t \|r^\epsilon\|_{H^s_\epsilon(\Omega(\sigma))}\, d\sigma \Big).$$

So long as

$$\sup_{0 \le \sigma \le t} \Big(\|w^\epsilon\|_{L^\infty(\Omega(\sigma))} \Big) \ \le \ R/2\,, \tag{8.3.10}$$

Lemma 8.1.1 yields with new C,

$$\|w^\epsilon(t)\|_{H^s_\epsilon(\Omega(t))} \ \le \ C\Big(\int_0^t \|w^\epsilon\|_{H^s_\epsilon(\Omega(\sigma))}\, d\sigma + \int_0^t \|r^\epsilon\|_{H^s_\epsilon(\Omega(\sigma))}\, d\sigma \Big). \tag{8.3.11}$$

The first application of this estimate is to show that $T^*(\epsilon) = T$ for ϵ small. If not, then since $w^\epsilon(0) = 0$ and $\|w^\epsilon\|_{H^s(\Omega(t))} \to \infty$ as $t \nearrow T^*(\epsilon)$, there is a smallest $\underline{t} \in]0, T[$ so that

$$\|w^\epsilon\|_{H^s_\epsilon(\Omega(\underline{t}))} \ + \ \|w^\epsilon\|_{L^\infty(\Omega(\underline{t}))} \ = \ R/2\,, \tag{8.3.12}$$

Then from the definition of $\underline{t}$, (8.3.11) holds for $0 \le t \le \underline{t}$ and Gronwall's inequality implies that

$$\sup_{0\le t<\underline{t}} \|w^\epsilon(t)\|_{H^s_\epsilon(\Omega(t)} \ \le\ C' \int_0^T \|r^\epsilon\|_{H^s_\epsilon(\Omega(\sigma))}\, d\sigma \ \le\ C_{n,s}\,\epsilon^n\,. \tag{8.3.13}$$

The H^s_ϵ Sobolev inequality (8.1.6) implies that for $0 \le t \le \underline{t}\,$,

$$\sup_{0\le t<\underline{t}} \|w^\epsilon\|_{L^\infty(\Omega(t))} \ \le\ C_{n,s}\,\epsilon^{n-d/2}\,. \tag{8.3.14}$$

Equations (8.3.12-13) imply that there is an $\epsilon_0 > 0$ so that for $\epsilon \le \epsilon_0\,$,

$$\sup_{0\le t\le\underline{t}} \ \|w^\epsilon\|_{H^s_\epsilon(\Omega(t))} + \|w^\epsilon\|_{L^\infty(\Omega(t))} \ \le\ R/4\,.$$

For $t = \underline{t}$, this contracdicts (8.3.12). It follows that v^ϵ is smooth on Ω_T and (8.3.10) holds with $t = T$.

Therefore, (8.3.13-14) holds with $\underline{t} = T$. Since $n > s > d/2$ are abitrary, it follows that

$$\forall\,\epsilon \le \epsilon_0 \ \ \forall\,n, \ \ \forall 0 \le t \le T, \ \ \exists\,C, \quad \|w^\epsilon\|_{H^s_\epsilon(\Omega(t))} + \|w^\epsilon\|_{L^\infty(\Omega(t))} \ \le\ C\,\epsilon^n\,. \tag{8.3.15}$$

Estimate (8.3.15) is nearly equivalent to $w^\epsilon \sim 0$. What is missing is an analogous estimate for the time derivatives.

Express

$$\partial_t w \ =\ -\sum A_j \partial_j w + \mathcal{G}(u^\epsilon, w^\epsilon)\, w^\epsilon \ -\ r^\epsilon\,.$$

The H^s_ϵ Moser inequality shows that $\mathcal{G}(u^\epsilon, w^\epsilon)$ is bounded in $H^s_\epsilon(\Omega(t))$ since both u^ϵ and w^ϵ are uniformly bounded. Therefore (8.3.15) implies the case $j = 1$ of

$$\forall\,j, \ \ \forall\,s, \ \ \forall\,n, \ \ \forall\,\epsilon < \epsilon_0, \qquad \sup_{0\le t\le T} \ \|\partial_t^j w^\epsilon(t)\|_{H^s_\epsilon(\Omega(t))} \ \le\ C_{n,s,j}\,\epsilon^n\,.$$

The proof is by induction on j. Write

$$\partial_t^{j+1} w \ =\ \partial_t^j\Big(-\sum A_j \partial_j w + \mathcal{G}(u^\epsilon, w^\epsilon)\, w^\epsilon - r^\epsilon \Big)\,. \tag{8.3.16}$$

The inductive hypothesis shows that for $k \le j$, and s arbitrary

$$\sup_{0\le t\le T} \ \|\partial_t^k \mathcal{G}(u^\epsilon, w^\epsilon)\|_{H^s_\epsilon(\Omega(t))} \ =\ O(1)\,, \qquad \text{and} \qquad \sup_{0\le t\le T} \ \|\partial_t^k w^\epsilon\|_{H^s_\epsilon(\Omega(t))} \ =\ O(\epsilon^\infty)\,.$$

Therefore the $H^s_\epsilon(\Omega(t))$ norm of the right hand side of (8.3.16) is $O(\epsilon^\infty)$ uniformly on $[0, T]$ completing the induction. ∎

§8.4. Rays and nonlinear transport.

In the linear case, the equations for the leading amplitudes simplify to transport equations when the smooth variety hypothesis is satisfied. With suitable hypotheses on F and initial data one has a similar simplification in the nonlinear case.

The leading profile U_0 is determined from its initial data as the solution of (7.4.23) and (7.4.24) which we repeat here suppressing the y dependence of F,

$$\mathbf{E}\big(L\,U_0 + F(U_0)\big) \;=\; 0\,, \qquad \mathbf{E}U_0 \;=\; U_0. \tag{8.4.1}$$

Equations for periodic functions in θ are split into their oscillating and nonoscillating parts. Denote with an underline, the average value of a periodic function of θ

$$\underline{g}(\theta) := \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\; d\theta\,.$$

The oscillatory part is denoted with an asterisk,

$$g^*(\theta) := g - \underline{g}\,.$$

Splitting the equations for $U_0 = \underline{U} + U^*$ into their oscillating and nonoscillating parts yields the equivalent pair of equations

$$L(y,\partial_y)\,\underline{U} + \underline{F(y,\underline{U}+U^*)} = 0\,, \tag{8.4.2}$$

$$\pi(y)\Big(L\,U^* \;+\; F(\underline{U}+U^*)^*\Big) \;=\; 0\,, \qquad \pi(y)\,U^* \;=\; U^*\,. \tag{8.4.3}$$

Neither the mean $\underline{U}$ nor the oscillatory part U^* can be found by itself. They interact.

The equations for the principal profile are an integro-differential system which is basically a hyperbolic problem with one more space variable, namely θ. The equation does not have θ derivatives. To find the principal profile is a little harder than to solve a single hyperbolic Cauchy problem. The payoff is not the solution of a single initial value problem, but the solution (asymptotically) of a one parameter family of such problems which have short wavelength oscillations. As pointed out in the introduction, these small structures make such a family particularly difficult to solve by numerical methods. If $\operatorname{rank}\pi(y) = k$ then the unknown function U_0 takes values in a k dimensional space. The number of unknown functions is reduced from N to k. The equation for the profile is usually simpler that solving a single initial value problem for the original problem.

The pair of equations becomes significantly simpler when one can guarantee that $\underline{U} = 0$. The next result gives two such situations.

Proposition 8.4.1 i. *If the nonlinear map $U \mapsto F(U)$ is odd, that is $F(-U) = -F(U)$ and the initial value $U\big|_{t=0}$ is odd in θ, then the solution U is odd in θ.*

ii. *If $F(U)$ is a linear combination of polynomials of odd degree in U and its complex conjugate $\overline{U}$, and, $U\big|_{t=0}$ has spectrum contained in the odd integers, then the solution U has spectrum contained in the odd integers.*

In both cases $\underline{U} = 0$.

Proof. i. The assumptions imply that the function $-U(y,-\theta)$ is a solution with the same initial data. By uniqueness, $U(y,\theta) = -U(y,-\theta)$.

ii. Denote the initial data $g(x,\theta) = U\big|_{t=0}$. The Picard iterates converging to the solution are defined by $U^1(t,x,\theta) = g(x,\theta)$ and

$$\mathbf{E}\big(L\,U^{\nu+1} + F(U^\nu)\big) \;=\; 0\,, \qquad \mathbf{E}U^{\nu+1} \;=\; U^{\nu+1}, \qquad U^{\nu+1}\big|_{t=0} \;=\; g\,.$$

By induction, the U^ν have spectrum contained in the odd integers. ∎

For profiles which satisfy $\underline{U} = 0$, the profile equation becomes

$$\pi\, L\, \pi\, U + \pi(y)\, F(U) = 0\,, \qquad \pi(y)\, U = U\,.$$

Next *suppose that the smooth characteristic variety hypothesis is satisfied at* $(y, d\phi(y))$ *with* $y \in \Omega_T$. In this case using (5.4.4), the profile equation simplifies to the nonlinear transport equation,

$$\big(\partial_t + \mathbf{v}.\partial_x + \gamma\big)\, U + \pi(y)\, F(U) = 0\,, \qquad \pi(y)\, U = U\,.$$

For each fixed θ this is a semilinear ordinary differential equation for U_0 along the integral curves of $\partial_t + \mathbf{v}.\partial_x$. Solving such a family of equations is radically simpler than solving a multidimensional hyperbolic system. When the smooth variety hypothesis is satisfied as well as $\underline{U} = 0$ (e.g. as in Proposition 8.4.1), *the construction of the approximate solutions reduces to solving nonlinear ordinary differential equations along the rays.*

In special cases there are explicit solutions which give insight into the underlying dynamics defined by the nonlinear hyperbolic equation. It is in this way that the subject is often used in the applied scientific community. The reader is encouraged to browse the references given in the bibliography to find interesting applications both mathematical and physical. In the applied literature the method often goes under the name *slowly varying envelope approximation.*

Example. A striking example is the analysis of *self phase modulation* when a laser beam passes through glass. We will not introduce the appropriate nonlinear Maxwell equations but content ourselves with a cartoon which shares the key features. Consider the semilinear system

$$\frac{\partial u}{\partial t} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial u}{\partial x_1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial u}{\partial x_2} + F(u) \;=\; 0\,, \qquad F \;=\; (F_1, F_2)\,.$$

The characteristic equation is $\tau^2 - |\xi|^2 = 0$. Consider the phase $\phi(t, x) = t - x_1$ for which the group velocity is equal to $(1, 0)$. The associated spectral projection and polarization are given by

$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad U_0(t, x, \theta) = (a_0(t, x, \theta), 0)\,.$$

When the hypotheses of Propostion 8.4.1 are satisfied the principal profile equation is

$$\frac{\partial a_0}{\partial t} + \frac{\partial a_0}{\partial x_1} + F_1(a_0, 0) \;=\; 0\,. \qquad a_0(0, x_1, x_2, \theta) = g(x_1, x_2, \theta) \tag{8.4.4}$$

For the special case where $F = i|u|^2 u$ and a_0 has spectrum in the odd integers, Proposition 8.4.1**ii** applies and the profile equation is

$$\frac{\partial a_0}{\partial t} + \frac{\partial a_0}{\partial x_1} + i|a_0|^2 a_0 \;=\; 0\,. \qquad a_0(0, x_1, x_2, \theta) = g(x_1, x_2, \theta) \tag{8.4.5}$$

Exercise 8.4.1. *Prove that if* $a_0 \in C^1(\mathbb{R}\,;\, H^1(\mathbb{R}^d \times S^1))$ *satisfies (8.4.5), then* $\int |a_0|^2\, dx\, d\theta$ *is independent of* t. **Hint.** Differentiate the quantity with respect to time. **Discussion.** Conclude that one arrives at the conclusion by multiplying by $\overline{a_0}$, taking real part, and integrating.

Corresponding to this conservation one has the ray by ray conservation law proved by considering small tubes of rays (see §5.4.3). Precisely, solutions of (8.4.2) satisfy

$$\big(\partial_t + \mathbf{v}.\partial_x\big)|a_0|^2 \;=\; 0\,.$$

Exercise 8.4.2 *Prove this by taking the real part of the product of* (8.4.2) *with* $\overline{a_0}$ *as suggested in the discussion of Exercise 8.4.1.*

Thus $|a_0|$ is constant on rays and equation (8.4.5) becomes a linear equation exactly solved by,

$$a_0(t,x_1,x_2,\theta) \;=\; e^{-i|g(x_1-t,x_2,\theta)|^2 t}\, g(x_1-t,x_2,\theta)\,. \tag{8.4.6}$$

The leading term in the approximation soution is

$$u_{\rm approx} \;=\; e^{-i|g(x_1-t,x_2,(t-x_1)/\epsilon)|^2 t}\; g(x_1-t,x_2,(t-x_1)/\epsilon)\,.$$

A particularly simple case is when g is monochromatic, $g = g(x)\,e^{i\theta}$ in which case the solution simplies to

$$u_{\rm approx} \;=\; e^{i(t-x_1)/\epsilon}\, e^{-i|g(x_1-t,x_2,(t-x_1)/\epsilon)|^2 t}\, b(x_1-t,x_2)\,. \tag{8.4.7}$$

An interesting special case is when $g(0,x,\theta) = b(x)\,e^{i\theta}$. The approximate solution in the linear case, $F=0$, would be

$$u_{\rm approx} \;=\; e^{i(t-x_1)/\epsilon}\; b(x_1-t,x_2)\,.$$

In the nonlinear case one finds

$$u_{\rm approx} \;=\; e^{i(t-x_1)/\epsilon}\; e^{-i|b(x_1-t,x_2)|^2 t}\; b(x_1-t,x_2)\,.$$

Compared to linear case, what has happened is that the phase has been modified. Along rays there is a phase lag which grows linearly in time and is proportional to the square of the amplitude. In optics, this is called *self phase modulation*.

If the nonlinearity were mulitplied by -1, the phase lag would be converted to a phase advance. The linear solution is moving with speed exactly equal to one. Such phase advance should not be confused with movement faster than light. Such confusion is common in the science literature. For this problem, no information moves faster than one, as we proved in §2.3.

Example. We describe a second nonlinear optical phenomenon revealed by the nonlinear transport equation. For the Maxwell equations, the projectors π have rank two and the smooth variety hypothesis is satisfied, so the nonlinear transport equation governs the dynamics of a two dimensional vector. In the important case of the commonly occuring cubic Kerr nonlinearity, the equations are explicitly solvable almost as in the above example.

If the electric field of the initial value of the profile is parallel a fixed direction, for example $(0,1,0)$, the light is polarized. The solution of the transport equation preserves this polarization. For propagation in the x_1 direction this polarization is possible as is polarization parallel to any vector orthogonal to $(1,0,0)$. A linear combination, for example,

$$E(0,x) \;=\; e^{i(t-x_1)/\epsilon}\; a(x)\; (0,2,1)\,,$$

is called elliptically polarized and the axis of polarization is $(0,1,0)$ corresponding to the stronger direction.

For the linear Maxwell equations the elliptical polarization is preserved simply by superposition. *For nonlinear optical models with the most common Kerr nonlinearity, the solutions of the transport with elliptically polarized data, the axis of polarization rotates at a constant speed in the plane perpendicular to* $(1,0,0)$. This explanation of an observed physical phenomenon is a second striking succeses of the nonlinear geometric optics approximations. The successes were made about thirty years before the approximations were proved to be accurate in the 1990's.

Chapter 9. Resonant Interaction and Quasilinear Systems

This chapter describes two extensions. First, we describe the resonant interaction of wave trains with distinct phases. This is *multiphase nonlinear geometric optics*. Second, the semilinear analysis is extended to the quasilinear case with the goal of discussing compressible inviscid fluid dynamics.

§9.1. Introduction to resonance

Even at the level of formal asymptotic expansions, resonance poses a challenge. It was [Majda and Rosales, 1986] who got it right. The approach presented in this chapter is that of [Joly, Métivier, and Rauch, Duke, 1994]. The essence of the phenomenon is illustrated by a simple example.

Example. Consider the oscillatory semilinear initial value problem

$$\begin{aligned}
(\partial_t + \partial_x) u_1 &= 0 & \qquad u_1\big|_{t=0} &= a_1(x)\, e^{ix/\epsilon} \\
\partial_t u_2 &= u_1 u_3 & \qquad u_2\big|_{t=0} &= 0 \\
(\partial_t - \partial_x) u_3 &= 0 & \qquad u_3\big|_{t=0} &= a_3(x)\, e^{ix/\epsilon}
\end{aligned} \tag{9.1.1}$$

with initial amplitudes $a_j \in C_0^\infty(\mathbb{R})$. The exact solution is given by

$$u_1(t,x) = a_1(x-t)\, e^{i(x-t)/\epsilon}\,, \quad u_3(t,x) = a_3(x+t)\, e^{i(x+t)/\epsilon}\,, \quad u_2 = \int_0^t u_1(t,x)\, u_3(t,x)\ dt\,.$$

The phases, $(x \pm t)/\epsilon$, that appear in the integrand for u_2 sum to $2ix/\epsilon$ which is independent of t. The formula for u_2 is,

$$u_2 = e^{i2x/\epsilon} \int_0^t a_1(x-t)\, a_3(x+t)\, dt\,. \tag{9.1.2}$$

u_1 and u_3 are wave trains with phases

$$\phi_1 := (x-t)/\epsilon, \qquad \text{and}, \qquad \phi_2 := (x+t)/\epsilon\,.$$

They interact to generate a the wave train u_2 with phase

$$\phi_3 := 2x/\epsilon.$$

The phases satisfy the resonance relation

$$\phi_1 \;+\; \phi_2 \;=\; \phi_3\,.$$

The amplitude of the new wave is of the same order, ϵ^0, as the waves from which it is formed. The linear operator

$$L(\partial_t, \partial_x) \;:=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \partial_t \;+\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \partial_x,$$

in the background has principal symbol,

$$L(i\tau, i\xi) \;:=\; i \begin{pmatrix} \tau+\xi & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau-\xi \end{pmatrix}.$$

The characteristic variety of L has equation $0 = \det L(\tau,\xi) = \tau(\tau+\xi)(\tau-\xi)$. The three phases satisfy the eikonal equation

$$\phi_t(\phi_t+\phi_x)(\phi_t-\phi_x) = 0\,.$$

For the solutions with $\nabla_{t,x}\phi \neq 0$, this is equivalent to exactly one of the equations

$$\phi_t = 0, \qquad \phi_t+\phi_x = 0, \qquad \text{or} \qquad \phi_t-\phi_x = 0,$$

at all points, assuming the domain of definition is connected.

Variants of this example illustrate two properties of resonance.

Example. Suppose that the initial condition for u_3 is changed to $u_3\big|_{t=0} = a_3(x)e^{i\psi(x)}$ with $d\psi(x)/dx$ nowhere equal to 1. Then $u_3 = a_3(x+t)\,e^{i\psi(x+t)/\epsilon}$ and the integral defining u_2 is an oscillatory integral in time with phase $(x-t+\psi(x+t))/\epsilon$. The time derivative of the phase is $O(1/\epsilon)$ so the method of nonstationary phase shows that $u_2 = O(\epsilon)$. The resonant interaction is destroyed.

Exercise 9.1.1. *Prove that more generally, if $\{x : \psi'(x) = 1\}$ has measure zero, then $u_2 = o(1)$ as $\epsilon \to 0$. Again, the offspring wave is smaller than the parents.*

For those who know about Young measures, it is interesting to note that the Young measures of the intitial data are independent of the function ψ so long as $\psi' \neq 0$. Thus *data with the same Young measures yield solutions with different Young measures.*

Introduce the symmetric form $\sum \phi_j = 0$ for resonance relations. If ψ_k satisfy $\sum n_k\,\psi_k = 0$, then the phases $\phi_k := n_k\psi_k$ satisfy the symmetric form. The symmetric form is often easier to manipulate.

Example. We find all triples of resonant linear eikonal phases with pairwise independent differentials for $L = \partial_t + \operatorname{diag}(\lambda_1,\lambda_2,\lambda_3)\partial_x$ with λ_j distinct real numbers. Seek such ϕ_j satisfying the resonance relation $\sum \phi_j = 0$. The independent differentials together with the eikonal relation force (up to permutation),

$$\phi_j(t,x) \;=\; \alpha_j(x-\lambda_j\,t), \qquad (\alpha_1,\alpha_2,\alpha_3)\in\mathbb{R}^3\setminus 0.$$

Therefore, α is determined up to scalar mulitplication by the resonance relation which is equivalent to the pair of equations,

$$\sum_j \alpha_j \;=\; 0, \qquad \text{and,} \qquad \sum_j \alpha_j\,\lambda_j \;=\; 0\,.$$

Exercise 9.1.2. *For $(\lambda_1,\lambda_2,\lambda_3) = (1,0,-1)$ show that if $f,g,h \in C^\infty(\mathbb{R})$ each has nonvanishing derivative at the origin, and at least one of them has nonvanishing second derivate, then the three phases*

$$f(t), \qquad g(t-x), \qquad \text{and} \qquad h(t+x)$$

cannot be resonant on a neighborhood of the origin. **Discussion.** This constant coefficient stictly hyperpbolic operators on $\mathbb{R}^{1=1}$, linear phases are the only possibilities for resonant triples with pairwise independent differentials.

The example and exercises show that *the phenomenon of resonance is both rare and sensitive when viewed from the perspective of perturbing the phases.* On the other hand, *wave trains with resonant phases interact much more strongly, amplifying their importance.*

§9.2. The three wave interaction PDE

One can understand a rich variety of resonance phenomena by studying the following special example. Like the examples of Chapter 1, it illustrates many important principals which are part of a general theory discussed later.

Consider the system

$$\begin{aligned}(\partial_t + \partial_x) u_1 &= c_1\, u_3\, u_2\,,\\ \partial_t u_2 &= c_2\, u_1 u_3\,,\\ (\partial_t - \partial_x) u_3 &= c_3\, u_1\, u_2\,.\end{aligned} \tag{9.2.1}$$

with real $c_j \in \mathbb{R} \setminus 0$. This equation maximizes the intermode interaction. The absence of a term in u_j^2 in the j^{th} equation has a consequence that harmonics are not generated by self interaction.

Multiplying the first equation by $a_1\, u_1$, the second by $a_2\, u_2$, and the third by $a_3\, u_3$, shows that if a_1, a_2, and a_3 are real numbers so that $\sum a_j c_j = 0$ then for solutions one has the differential conservation laws

$$\frac{\partial}{\partial t}\big(a_1\, u_1^2 + a_2\, u_2^2 + a_3\, u_3^2\big) \;+\; \frac{\partial}{\partial x}\big(a_1\, u_1^2 \;-\; a_3\, u_3^2\big) \;=\; \big(2 \sum_j \; a_j\, c_j\big) u_1 u_2 u_3 \;=\; 0\,,$$

Integrating dx yields the integral conservation laws for solutions sufficiently smooth and sufficiently small at ∞,

$$\frac{d}{dt} \int a_1\, u_1^2 + a_2\, u_2^2 + a_3\, u_3^2 \;\; dx \;=\; 0\,.$$

This is a two dimensional space of conservation laws parameterized by the a.

In order to take advantage of the complex exponential function, we are interested in complex solutions. For complex solutions, conservation laws involving $|u_j|^2$ are more interesting than those involving u_j^2 as they yield L^2 bounds. The complex analogue of (9.2.1) with such conservation laws is,

$$\begin{aligned}(\partial_t + \partial_x) u_1 &= c_1\, u_2\, u_3^*\\ \partial_t u_2 &= c_2\, u_1 u_3\\ (\partial_t - \partial_x) u_3 &= c_3\, u_1^*\, u_2\,.\end{aligned} \tag{9.2.2}$$

with $c_j \in \mathbb{R} \setminus 0$.

Mutiplying the j^{th} equation by $a_j u_j^*$ and taking the real part shows that if $a_j \in \mathbb{R}$ satisfy $\sum a_j\, c_j = 0$, then solutions satisfy

$$\partial_t\Big(\sum_j a_j |u_j|^2\Big) \;+\; \partial_x\Big(a_1\, |u_1|^2 \;-\; a_3\, |u_3|^2\Big) \;=\; 2\, \Big(\sum_j a_j\, c_j\Big)\, \mathrm{Re}\Big(u_1^*\, u_2\, u_3^*\Big) \;=\; 0\,.$$

If the c_j do not all have the same sign, then there are conservation laws of this type with all the $a_j > 0$. This yields an L^2 bound on the solution. On the other hand, if the c_j are all positive then initial data with $u_j(0,x)$ real and positive yield real solutions such that for all j, u_j is nondecreasing along j characteristics. For sufficiently positive data there is finite time blow up.

Proposition 9.2.1. *Suppose that $c_j \ge c > 0$ and the real valued initial data satisfy*

$$\forall j, \quad \forall |x| \le R, \quad u_j(0,x) \;\ge A > 0.$$

i. *If $u(t,x) \in C^\infty([0,t_*[\times\mathbb{R})$ is a solution, then $u_j(t,x) \ge y(t)$ for $t_* > t \ge 0$ and $|x| \le R-t$, where $y = A/(1-cAt)$ is the solution of $y' = c\,y^2,\ y(0) = A$.*

ii. *If $T_* := (cA)^{-1}$ is the blow up time for y, and $R > T_*$, then u blows up on or before time T_* in the sense that one must have $t_* \le T_*$.*

Proof. The second assertion follows from the first.

Since the speed of propagation is no larger than 1, the values of u in $|x| \le R-t$ are unaffected by the values of the Cauchy data for $|x| > R$. Therefore, it suffices to prove that $u_j \ge y(t)$ when the data satisfy $u_j(0,x) \ge A$ for all x.

Define

$$m(t) \ := \ \min_{x\in\mathbb{R}\,,\,j} \ u_j(t,x)\,.$$

Since the u_j are nondecreasing on j-characteritics, it follows that $m(t)$ is nondecreasing. And, $m(0) \ge A > 0$. In addition one has the lower bound obtained by integration along j characteristics,

$$u_j(t,x) \ \ge \ m(0) + c\int_0^t m(t)^2 \ dt\,.$$

Taking the infinum on x yields

$$m(t) \ \ge \ m(0) + c\int_0^t m(t)^2 \ d \ \ge \ A + c\int_0^t m(t)^2\,dt\,.$$

The function y is characterized as the solution of

$$y(t) \ = \ A + c\int_0^t y(t)^2\,dt\,.$$

For $\epsilon > 0$ small, let y^ϵ be the solution of $(y^\epsilon)' = c(y^\epsilon)^2$ with $y^\epsilon(0) = A - \epsilon$ so

$$y^\epsilon(t) \ = \ A - \epsilon + c\int_0^t y^\epsilon(t)^2\,dt\,.$$

It follows that $m(t) > y^\epsilon(t)$ for all $0 \le t < t_*$. For, if this were not so there would be a smallest $\underline{t} \in]0,t[_*$ where $m(\underline{t}) = y^\epsilon(\underline{t})$. Then

$$y^\epsilon(\underline{t}) \ = \ m(\underline{t}) \ \ge \ A + c\int_0^{\underline{t}} m(t)^2\,dt \ > \ A - \epsilon + c\int_0^t (y^\epsilon(t))^2\,dt \ = \ y^\epsilon(\underline{t})\,.$$

This contradiction establishes $m > y^\epsilon$. Passing to the limit $\epsilon \to 0$ proves $m \ge y$ which is the desired conclusion. ∎

Only the signs of the c_j play a roll in the qualitative behavior of the equation (9.2.2).

Proposition 9.2.2. *There is exactly one positive diagonal linear transformation*

$$u \ := \ (d_1\,v_1\,,\, d_2\,v_2\,,\, d_3\,v_3)\,, \qquad d_j > 0\,,$$

which transform the the system to the analogous system with interaction coefficients $\{c_1, c_2, c_3\}$ replaced by

$$\left\{\frac{c_1}{|c_1|}\,,\,\frac{c_2}{|c_2|}\,,\,\frac{c_3}{|c_3|}\right\}.$$

Proof. The change of variables yields an anlogous system for v with the interaction coefficients replaced by

$$\Big\{\frac{d_2\,d_3}{d_1}\,c_1\,,\ \frac{d_3\,d_1}{d_2}\,c_2\,,\ \frac{d_1\,d_2}{d_3}\,c_3\Big\}\,.$$

Need d_j so that

$$\frac{d_2\,d_3}{d_1}\,c_1 = \frac{c_1}{|c_1|}\,,\qquad \frac{d_3\,d_1}{d_2}\,c_2 = \frac{c_2}{|c_2|}\,,\qquad \frac{d_1\,d_2}{d_3}\,c_3 = \frac{c_3}{|c_3|}\,.$$

Multiplying the $j^{\rm the}$ equation by d_j^2 yields the equivalent system,

$$\frac{d_1^2}{|c_1|} \;=\; \frac{d_2^2}{|c_2|} \;=\; \frac{d_3^2}{|c_3|} \;=\; d_1 d_2 d_3\,.$$

The first two equalities hold if and only if,

$$(d_1^2, d_2^2, d_3^2) = a(|c_1|, |c_2|, |c_3|) \qquad \text{with} \quad a>0\,.$$

Then, the last equation holds if and only if

$$a \;=\; a^3\,|c_1 c_2 c_3|\,.$$

This uniquely determines a, and therefore d. ∎

Remark. For general $d_j \neq 0$, the three quantities $d_1 d_2/d_3$, $d_2 d_3/d_1$, $d_3 d_1/d_2$ have the same sign. Using $d_j \neq 0$ allows us to multiply the three interactiion coefficients by -1 if desired. Thus every system is transformed to one with interaction coefficients all equal to $+1$ or two equal to $+1$. There are four equivalence classes, the last three depending on the location of the coefficient -1.

Proposition 9.2.3. i. *If the real interaction coefficients $c_j \neq 0$ do not all have the same sign, then the Cauchy problem for (9.4) has a unique global solution $u \in \cap_s C^s([0,\infty[\,;\, H^s(\mathbb{R}))$ for arbitrary Caucy data in $\cap_s H^s(\mathbb{R})$.*

ii. *If the c_j have the same sign there are smooth compactly supported data so that the solution of the Cauchy problem blows up in finite time.*

Proof. For real data, this equation reduces to the previous one and the explosive behavior has already been treated.

To prove **i.**, the results of section 6.4 show that it suffices to prove for every $T>0$, an *a priori* bound for the $L^\infty([0,T]\times\mathbb{R})$ norm.

From the conservation law, one has

$$\sup_{t\in[0,T]}\ \int \sum_j |u_j|^2\ dx \ \le\ K \ <\ \infty\,.$$

The equation for u_2 yields,

$$|u_2(t,\underline{x})| \ \le\ |u_2(0,\underline{x})\ | + \int_0^t |u_1\,u_3(t,\underline{x})|\ dt\,. \tag{9.2.3}$$

The key idea is to estimate the integral on the right hand side using energy estimates for u_1 and u_3.

For any $\underline{x} \in \mathbb{R}$ integrate the identity

$$\big(\partial_t + \partial_x\big)|u_1|^2 \;=\; 2\,\mathrm{Re}\Big(u_1^*\big(\partial_t + \partial_x\big)u_1\Big) \;=\; 2\,\mathrm{Re}\,c_1 u_1^* u_2 u_3\,,$$

over the strip $[0,t]\times\,]-\infty,\underline{x}]$ to find that

$$\int_0^t |u_1(t,\underline{x})|^2\,dt \;\le\; 2K + 2|c_1|\int_{[0,t]\times\mathbb{R}} |u_1u_2u_3|\;dt\,dx\,.$$

Estimate the integral dx on the right using the $L^\infty\times L^2\times L^2$ Hölder inequality to find

$$\int_0^t |u_1(t,\underline{x})|^2\,dt \;\le\; 2K \;+\; 2|c_1|\int_0^t K\,\|u_2(t)\|_{L^\infty(\mathbb{R})}\;dt\,.$$

By symmetry,

$$\int_0^t |u_3(t,\underline{x})|^2\,dt \;\le\; 2K \;+\; 2|c_3|\int_0^t K\,\|u_2(t)\|_{L^\infty(\mathbb{R})}\;dt\,.$$

The Cauchy-Schwarz inequality implies that

$$\int_0^t |u_1(t,\underline{x}))\,u_2(t,\underline{x})|\;dt \;\le\; 2K \;+\; 2\,\max\{|c_1|,|c_3|\}\int_0^t \|u_2(t)\|_{L^\infty(\mathbb{R})}\,dt\,. \tag{9.2.4}$$

Estimate the integral on the right in (9.2.3) using (9.2.4) to find

$$|u_2(t,\underline{x})| \;\le\; C \;+\; \int_0^t C\,\|u_2(t)\|_{L^\infty(\mathbb{R})}\,dt\,,$$

with C independent of $(t,\underline{x})\in[0,T]\times\mathbb{R}$. Taking the supremum of the left hand side over $\underline{x}$ yields

$$\|u_2(t)\|_{L^\infty(\mathbb{R})} \;\le\; C \;+\; \int_0^t C\,\|u_2(t)\|_{L^\infty(\mathbb{R})}\,dt\,, \qquad 0\le t\le T\,.$$

Gronwall's inequality bounds the sup norm of u_2 over bounded time intervals.

To estimate u_1 one needs L^2 estimates for u_2 and u_3 on the speed one characteristics $x=\underline{x}+t$. These are obtained by integrating $\partial_t|u_2|^2$ and $(\partial_t-\partial_x)|u_3|^3$ over $\{(s,x)\,:\,0\le s\le t,\text{ and } x\ge \underline{x}+s\}$. A similar argument works for u_3.

Exercise 9.2.1. *State and prove the $L^\infty([0,T]\times\mathbb{R})$ estimate for u_3.* ∎

§9.3. The three wave interaction ODE

For the three wave PDE, (9.2.2), and phases equal to the resonant triplet, waves of each pair of families influence, by resonant interaction, the wave of the third. The simplest examples showing this are solutions of the special form

$$u_1 = A_1(t)\,e^{i(t-x)/\epsilon}\,, \qquad u_2 = A_2(t)\,e^{-i2x/\epsilon} \qquad u_3 := A_3(t)e^{-i(t+x)/\epsilon}\,, \tag{9.3.1}$$

with amplitudes A_j independent of x. The oscillatory structure evolves in time, but is uniform in space. Equation (9.2.2) is satisfied if and only if the amplitudes A_j satisfy the *three wave interaction equations*

$$A_1' = c_1\, A_2\, A_3^*\,, \qquad A_2' = c_2\, A_1\, A_3\,, \qquad A_3' = c_3\, A_1^*\, A_2\,. \tag{9.3.2}$$

This is a nonlinear system of ordinary differential equation for three complex quantities A_j. The phase space is $\mathbb{C}^3$, hence six dimensional as a real vector space. It is the same equation that one would find if one sought solutions of the three wave interaction pde whch were independent of x.

The equilibria are the points where (at least) two of the three $\{A_j\}$ vanish. There are three linear subspaces of equilibria, each with real dimension equal to 2,

$$\big\{A_2 = A_3 = 0\big\}\,, \qquad \big\{A_3 = A_1 = 0\big\}\,, \qquad \text{and,} \qquad \big\{A_1 = A_2 = 0\big\}\,.$$

Each pair of planes intersect at the origin. The system (9.3.2) is highly symmetric.

Proposition 9.3.1. i. *The quantity* $\mathrm{Im}\big(A_1(t)\, A_2^*(t)\, A_3(t)\big)$ *, is constant on solutions of (9.3.2).*
ii. *If a_j are real numbers so that $\sum a_j\, c_j \;=\; 0$ then the quantity $\sum_j a_j\, |A_j(t)|^2$ is constant on solutions of (9.3.2).*
iii. *If A is a solution and $\theta \in \mathbb{R}$, then $\tilde A$ obtained by each of the three gauge transformations*

$$\tilde A := \big(e^{i\theta}\, A_1\,,\, A_2\,,\, e^{-i\theta}\, A_3\big), \quad \tilde A := \big(A_1\,,\, e^{i\theta}\, A_2\,,\, e^{i\theta}\, A_3\big), \quad \tilde A := \big(e^{i\theta}\, A_1\,,\; e^{i\theta}\, A_2\,,\, A_3\big),$$

is also a solution. The conserved quantities in **i,ii** *are invariant under the gauge transformations.*
iv. *If A is a solution and $\sigma \in \mathbb{R}\setminus 0$, then $\tilde A$ obtained by the scaling*

$$\tilde A_j(t) \;=\; \sigma A_j(\sigma t)\,,$$

is also a solution.

Proof. i. Compute

$$\begin{aligned}
(A_1 A_2^* A_3)_t &= (A_1)_t A_2^* A_3 + A_1 (A_2^*)_t A_3 + A_1 A_2^* (A_3)_t \\
&= c_1 A_2 A_3^* A_2^* A_3 + c_2 A_1 A_1^* A_3^* A_3 + c_3 A_1 A_2^* A_1^* A_2 \\
&= c_1 |A_2 A_3|^2 \;+\; c_1 |A_1 A_3|^2 \;+\; c_1 |A_2 A_1|^2 \;\in\; \mathbb{R}\,.
\end{aligned}$$

ii. Compute

$$\begin{aligned}
\frac{d}{dt}|A_1|^2 &= 2\mathrm{Re}\, A_1^* \frac{d}{dt} A_1 &= 2c_1 \mathrm{Re}(A_1^* A_2 A_3^*), \\
\frac{d}{dt}|A_2|^2 &= 2\mathrm{Re}\, A_1^* \frac{d}{dt} A_1 &= 2c_2 \mathrm{Re}(A_2^* A_1 A_3), \\
\frac{d}{dt}|A_3|^2 &= 2\mathrm{Re}\, A_1^* \frac{d}{dt} A_1 &= 2c_3 \mathrm{Re}(A_1^* A_2 A_3^*).
\end{aligned}$$

The real parts are of $A_1 A_2^* A_3$ or its complex conjugate, so are equal. Therefore one has

$$\frac{d}{dt}\Big(\sum a_j\, |A_j(t)|^2\Big) \;=\; \Big(2 \sum_j a_j\, c_j\Big) \mathrm{Re}(A_2^* A_1 A_3) \;=\; 0\,.$$

The assertions **iii,iv** are immediate. ∎

Remarks. 1. When the $c_j \neq 0$ do not all have the same sign one can choose the $a_j > 0$. In this case, the three wave interaction system is globally solvable.

2. When the three c_j have the same sign, there exist solutions which blow up in finite time. This is proved by comparison with an explosive Ricatti equation as for the three wave interaction PDE.

Exercise 9.3.1. *Suppose that the c_j have the same sign and that $A(t)$ is solution defined for $0 \le t < T_*$ so that $\limsup_{t \to T_*} \|A(t)\| = \infty$. Prove that for all j, $\limsup_{t\to T_*} \|A_j(t)\| = \infty$.* **Hint.** Use quadratic conservation laws.

3. The gauge transformations commute. The third gauge transformation is the product of the preceding two. The abelian group of gauge transformations is a two dimensional torus of mappings

$$A \quad \mapsto \quad \big(e^{i\theta_1}\, A_1\,,\, e^{i\theta_2} A_2\,,\, e^{i\theta_2}\, e^{-i\theta_1}\, A_3\big)\,.$$

Theorem 9.3.2. i. *The equilibrium $(0,0,0)$ is unstable if and only if the three c_j have the same sign.*

ii. For i, j, k distinct, the equilibrium $A_i = A_j = 0\,,\ \underline{A}_k \neq 0$ of (9.3.2), is unstable if the interaction coefficients c_i and c_j have the same sign. The stable and unstable manifolds have real dimension equal to 2.

iii. For the same equilibrium, if c_i and c_j have opposite signs, orbits of the linearized equation are bounded. For initial data starting close to the equilibrium, the solutions of the nonlinear system exist for all time and $A_i(t), A_j(t)$ and $|A_k(t)|$ stay close to their initial values uniformly in time. The equilibrium is unstable. If $\underline{A}_k$ is real then the equilibrium is stable for the restriction of the dynamics to $A \in \mathbb{R}^3$.

Proof. i. The stability of the origin when the c_j do not have the same sign follows from the conservation of $\sum a_j |A_j|^2$ with positive a_j. The instability is proved using explosive positive (resp. negative) solutions when the c_j are positive (resp. negative).

ii. For ease of reading consider the equilibrium $(0, 0, \underline{A}_3) \neq 0$. The linearized equation at this equilibrium is

$$B' \;=\; \begin{pmatrix} 0 & c_1\, \underline{A}_3^* & 0 \\ c_2\, \underline{A}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} B\,.$$

The eigenvalues of the coefficient matrix are the solutions λ of

$$\lambda(\lambda^2 - c_1 c_2 |\underline{A}_3|^2) \;=\; 0\,.$$

If c_1 and c_2 have the same sign, then the roots are $0, \pm|c_1 c_2|^{1/2} |\underline{A}_3|$. The positive eigenvalue implies that the equilibrium is unstable. The stable and unstable manifolds have complex dimension equal to 1 and real dimension equal to 2.

If c_1, c_2 have opposite signs then $|c_2||B_1|^2 + |c_1||B_2|^2$ is constant on orbits of the linearized equation. It follows that each orbit of the linearized equation is uniformly bounded in time.

iii. In the case of opposite signs, the functional $|c_2|^2|A_1|^2 + |c_1||A_2|^2$ is constant on orbits of (9.3.2). For initial data which start near $(0, 0, \underline{A}_3)$ the components $A_1(t), A_2(t)$ stay close to zero for all time.

A conserved quantity $a_1|A_1|^2 + a_2|A_2|^2 + |A_3|^2$ together with the control of $A_1(t), A_2(t)$ implies that $|A_3(t)|$ stays close to $|A_3(0)|$. In particular, the orbit exists for all time. For real solutions this implies stability since in that case the sign of $A_3(t)$ does not change and

$$|A_3(t) - A_3(0)| = \Big||A_3(t)| - |A_3(0)|\Big|.$$

For complex solutions write $A_3 = \rho\, e^{i\theta}$, $\rho := |A_3|$ and compute *

$$A_3' = \rho' e^{i\theta} + i\rho e^{i\theta}\theta' = \rho e^{i\theta}\Big(\rho'/\rho + i\theta'\Big), \qquad \theta' = \mathrm{Im}\left(\frac{A_3'}{A_3}\right) = \mathrm{Im}\left(\frac{c_1 A_1^* A_2}{A_3}\right) = \frac{\mathrm{Im}\, c_1 A_1^* A_2 A_3^*}{|A_3|^2}.$$

To prove instability choose $\delta = |\underline{A}_3| > 0$. For $\epsilon > 0$ choose complex initial data $(A_1(0), A_2(0), \underline{A}_3)$ with $|A_1(0), A_2(0)| < \epsilon$ and $\mathrm{Im}\ A_1 A_2^* A_3 \neq 0$. In the expression for θ', the numerator is a nonzero constant, and the denominator is always $\sim |\underline{A}_3)|$, so the angle θ has derivative bounded below. There is a $t > 0$ so that $\theta(t) - \theta(0) = \pi$ so $|A_3(t) - A_3(0)| = 2|\underline{A}_3| > \delta$ proving instability. ∎

For any triple of interaction coefficients, there exists $i \neq j$ so that c_i and c_j have the same sign. Then the equilibria defined by $A_i = A_j = 0$ is unstable. The unstable equilibrium exists even in the globally solvable case where the c_j do not all have the same sign. For example, if c_1 and c_2 have the same sign and c_3 the opposite, then their is a conserved Euclidean norm $\sum a_j |A_j|^2$. On the other hand, most orbits starting near $A_1 = 0, A_2 = 0, A_3 = 1$ stray far from this state. This situation is described as saying oscillations on the third mode generate *frequency conversion* to modes one and two. The solution cannot grow, but it can wander far from its initial state. The energy originally localized nearly entirely on mode 3, moves substantially away. An appreciable portion of the energy passes to modes one and two.

The analysis of the interactions in the highly oscillatory family (9.2.2) reduces to the analysis of a system of nonlinear ordinary differential equations This is a special case of a general phenomenon for *homogeneous oscillations*, that is oscillations which in a sense are the same at all positiions of space. Proving general results of this sort is one of our goals. Another is to extend our semilinear analysis of Chapters 7 and 8 to the quasilinear case. We return to the construction of high frequency asymptotic solutions, this time with several phases and in the quasilinear case.

§9.4. Formal asymptotic solutions for resonant quasilinear geometric optics

We give a self contained, but rapid derivation of the equations of quasilinear geometric optics. Consider the quasilinear system of partial differential operators,

$$L(u, \partial)u \ := \ \sum_{\mu=0}^{d} A_\mu(u)\, \partial_\mu u\,.$$

Suppose that the system is symmetric in the sense of the first paragraph of §6.6. Consider solutions whose values are close to to a constant state $\underline{u}$. The change of independent variable $u \mapsto u - \underline{u}$ reduces to the case $\underline{u} = 0$. Without loss of generality we study solutions close to 0.

As in the last paragraph of §6.6, the change dependent variable, $u := A_0(0)^{1/2} v$, yields the equivalent system

$$\sum_{\mu=0}^{d} \tilde{A}_\mu(v)\ \partial_\mu v \ = \ 0, \qquad \tilde{A}_\mu(v) \ := \ A_0(0)^{-1/2}\ A_\mu(A_0(0)^{1/2} v)\ A_0(0)^{-1/2}\,,$$

* Thanks to G. Métivier for this short proof. See [Alber *et. al.*] for more information on this system.

with

$$\tilde{A}_\mu \;=\; \tilde{A}_\mu^*\,, \qquad \text{and,} \qquad \tilde{A}_0(0) \;=\; I\,.$$

We suppose that such a change has been performed and suppress the tildes.

For $u \approx 0$ use the approximation

$$A_\mu(u) \approx A_\mu(0) + A'_\mu(0)u.$$

to show that the nonlinear terms are equal to

$$\big(A'_\mu(0)u\big)\,\partial_\mu u \;+\; \text{higher order terms.}$$

We assess the time of nonlinear interaction for solutions built from an oscillatory wave trains $\epsilon^p\, e^{i\phi(y)/\epsilon}$. The power p will be chosen so that this time is ~ 1.

For Burgers' equation $u_t + uu_x = 0$, with compactly supported initial data with $\|\partial_x u(0,x)\|_{L^1} \sim 1$, solutions break down at times $t \sim 1$. Thus for initial data $\epsilon^p a(x)\, e^{i\phi(0,x)/\epsilon}$ the lifetime is $O(1)$ when $p = 1$ and is much longer (resp. shorter) when $p > 1$ (resp. $p < 1$). This shows that nonlinear effects are important for $t \sim 1$ for the critical power $p = 1$.

A second estimate proceeds as follows. Assume that $A'_\mu(0) \neq 0$ for some μ, so that the leading nonlinear terms are quadratic. The analysis when the leading Taylor polynomial is higher order can be carried out as in earlier sections. For the important examples from inviscid fluid dynamics, the hypothesis of quadratic nonlinearity is usually verified. Consider solutions built from wave trains $\epsilon^p a(y)\, e^{i\phi(y)/\epsilon}$ whose amplitudes are $O(\epsilon^p)$ and whose derivates are $O(\epsilon^{p-1})$. The nonlinear terms then have amplitude $O(\epsilon^{2p-1})$. For phases satisfying the eikonal equation terms of this size yield a response which is $O(\epsilon^{2p-1})$ for $t \sim O(1)$. We choose the amplitudes so that the time of nonlinear interaction is $O(1)$ so we want $\epsilon^{2p-1} \sim \epsilon^p$. This yields the critical power $p = 1$. When $p = 1$ our analysis will show that nonlinear effects usually affect the leading order asymptotics for times $t \sim 1$.

Exercise 9.4.1 i. *Perform a perturbation computation as in §6.5, to show that the solution of,*

$$\partial_t u + u\partial_x u \;=\; 0, \qquad u(0,x) \;=\; 1 + \delta g(x),$$

is given by

$$u \;\sim\; u_0 \;+\; \delta u_1 \;+\; \delta^2 u_2 \;+\; \cdots,$$

where $u_0(t,x) = 1$ is the unperturbed state, $u_1(t,x) = g(x-t)$ solves the linearized equation, and the leading nonlinear term u_2 is determined by

$$(\partial_t + \partial_x)u_2 + g(x-t)\,g'(x-t) \;=\; 0, \qquad u_2(0,x) = 0\,.$$

ii. *As in §7.1 take $g = \epsilon^p\, a(x)\, e^{ix/\epsilon}$ and $\delta = 1$ to see that for $t \sim 1$ the leading nonlinear term is small compared to the leading when $p > 1$ and they become comparable as $p \to 1$.* **Discussion.** This is a third motivation for the critical exponent $p = 1$.

Consider the interaction of waves with linear phases $\phi_j(y)$ which by nonlinear interaction yield possible phases $\sum n_j \phi_j$ with $n_j \in \mathbb{Z}$. Each of these candidate phases is a linear function $\alpha.y$ with $\alpha \in \mathbb{R}^{1+d}$. Thus the expression

$$\epsilon\, U(y, y/\epsilon) \qquad \text{with} \qquad U(y,Y) \sim \sum_{\alpha\in\mathbb{R}^d} U_\alpha(y)\, e^{i\alpha.Y}$$

is of the critical amplitude and includes all the anticipated terms. In a formal trigonometric sum over α it is understood that there are at most a countable number of nonvanishing coefficients U_α. The exact structure of the function of Y given by $U(y,Y)$ is left unspecified for the moment. Equivalently, consider $U(y,Y)$ as a formal trigonometric series in Y with coefficients which are smooth functions of y. To solve the profile equations and prove accuracy will require supplementary hypotheses. These hypotheses do not play a role in the derivation of the profile equations.

Pose the *ansatz*

$$u^\epsilon = \epsilon\, U(\epsilon, y, y/\epsilon)\,, \tag{9.4.1}$$

$$U(\epsilon, y, Y) \;\sim\; \sum_{j=0}^{\infty} \epsilon^j\, U_j(y,Y) \;\sim\; U_0(y,Y) + \epsilon\, U_1(y,Y) + \cdots\,, \tag{9.4.2}$$

$$U_j(y,Y) \;\sim\; \sum_{\alpha\in\mathbb{R}^d} U_{j,\alpha}(y)\, e^{i\alpha.Y}\,. \tag{9.4.3}$$

Write

$$L(u^\epsilon, \partial) u^\epsilon \;=\; L(\epsilon U,\, \partial)\Big(\epsilon\, U(\epsilon, y, y/\epsilon)\Big)\,. \tag{9.4.4}$$

Expand in a Taylor series at $\epsilon = 0$,

$$A_\mu\big(\epsilon U(\epsilon, y, Y)\big) \;\sim\; A(0) \;+\; \epsilon A'_\mu(0) U_0 \;+\; \cdots\,, \tag{9.4.5}$$

to find

$$L(\epsilon U(\epsilon, y, Y)\,,\, \partial_y) \;\sim\; L_0 \;+\; \epsilon L_1 \;+\; \cdots \;=\; \sum_{j=0}^{\infty} \epsilon^j\, L_j(y, Y, \partial_y)\,.$$

The L_j are operators whose coefficients are functions of y, Y involving the derivatives $\partial_u^\beta A_\mu(0)$ and the profiles U_k with $k \le j-1$. The most important come from the leading terms in (9.4.5),

$$L_0 = L(0, \partial_y)\,, \qquad \text{and,} \qquad L_1 = \sum_\mu A'_\mu(0) U_0(y,Y)\, \partial_\mu\,.$$

The chain rule shows that

$$\frac{\partial}{\partial y_\mu} U(\epsilon, y, y/\epsilon) \;=\; \Big(\frac{\partial}{\partial y_\mu} + \frac{1}{\epsilon}\,\frac{\partial}{\partial Y_\mu}\Big) U(\epsilon, y, Y)^\epsilon\Big|_{Y=y/\epsilon}\,. \tag{9.4.6}$$

So

$$L(u^\epsilon, \partial) u^\epsilon \;=\; W(\epsilon, y, Y)\Big|_{Y=y/\epsilon}\,, \tag{9.4.7}$$

where

$$W(\epsilon, y, Y) \;=\; L\Big(\epsilon U(\epsilon, y, Y)\,,\, \frac{\partial}{\partial y_\mu} + \frac{1}{\epsilon}\,\frac{\partial}{\partial Y_\mu}\Big)\, \epsilon\, U(\epsilon, y, Y)$$

Expand to find

$$\begin{aligned}
W(\epsilon, y, Y) \;&\sim\; \Big(\sum_{j=0}^{\infty} \epsilon^j L_j\Big(y, Y, \frac{\partial}{\partial y_\mu} + \frac{1}{\epsilon}\,\frac{\partial}{\partial Y_\mu}\Big)\Big)\Big(\epsilon \sum_{k=0}^{\infty} \epsilon^k U_k(y,Y)\Big) \\
&\sim\; \Big(\sum_{j=0}^{\infty} \Big[\epsilon^j L_j\Big(y, Y, \frac{\partial}{\partial y_\mu}\Big) + \epsilon^{j-1} L_j\Big(y, Y, \frac{\partial}{\partial Y_\mu}\Big)\Big]\Big)\Big(\epsilon \sum_{k=0}^{\infty} \epsilon^k U_k(y,Y)\Big) \\
&\sim\; \sum_{j=0}^{\infty} \epsilon^j\, W_j(y,Y)\,.
\end{aligned}$$

In particular,

$$W_0(y,Y) = L(0,\partial_Y)\, U_0(y,Y)\,, \tag{9.4.8}$$

and

$$\begin{aligned} W_1(y,Y) &= L(0,\partial_y)\, U_0 + L_1(y,Y,\partial_Y)U_0 + L(0,\partial_Y)U_1 \\ &= L(0,\partial_y)\, U_0 + \sum_\mu A'_\mu(0)U_0\,\partial_{Y_\mu}U_0 + L(0,\partial_Y)\, U_1\,. \end{aligned} \tag{9.4.9}$$

This expression involves both U_0 and U_1 which is typical of multiscale methods. The quadratically quasilinear terms $A'_\mu(0)U_0\,\partial_Y U_0$ involve derivatives in the fast variables Y. For $j \geq 2$ the formula for W_j is

$$\begin{aligned} W_j \;&=\; \sum_{k+\ell=j} \Big(L_k(y,Y,\partial_y) + L_{k+1}(y,Y,\partial_Y)\Big)U_\ell \\ &=\; L(0,\partial_y)\, U_j + L_1(y,Y,\partial_Y)U_j + L(0,\partial_Y)\, U_{j+1} + \text{terms in } U_0, U_1, \ldots U_{j-1}\,. \end{aligned}$$

The strategy is to choose profiles U_j so that $W_j(y,Y) = 0$ for all $y, Y,$ not just on the $d+2$ dimensional subset $\{Y = y/\epsilon\}$ parameterized by $(\epsilon, y) = (\epsilon, t, x)$.

Setting $W_0 = 0$ in (9.4.8) shows that W_0 must lie in the kernel of $L(0,\partial_Y)$. For (9.4.9), the profile U_1 is as yet undetermined. However, in order that it is possible to choose a U_1 so that (9.4.9) holds, requires the second of the equations,

$$U_0 \in \operatorname{Kernel} L(0,\partial_Y)\,, \quad \text{and,} \quad L(0,\partial_y)\, U_0 + \sum_\mu A'_\mu(0)U_0\,\partial_{Y_\mu}U_0 \in \operatorname{Range} L(0,\partial_Y)\,. \tag{9.4.10}$$

To understand (9.4.10) requires a study of the operator $L(0,\partial_Y)$. This is straight forward using the Fourier representation,

$$L(0,\partial_Y)\, U \;=\; L(0,\partial_Y) \sum_\alpha U_\alpha(y)\, e^{i\alpha.Y} \;=\; i \sum_\alpha L(0,\alpha)\, U_\alpha(y)\, e^{i\alpha.Y}\,. \tag{9.4.11}$$

As an operator acting on formal trigonometric series, $L(0,\partial_Y)$ has kernel consisting of those series whose α^{th} coefficient belongs to the kernel of $L(0,\alpha)$. Recall the definition of $\pi(\alpha)$ as the projection onto the kernel of $L(0,\alpha)$ along its range. The kernel of $L(0,\partial_Y)$ is then the set of trigonometric series such that $\pi(\alpha)\, U_\alpha = U_\alpha$. The image is the set of series with U_α belonging to the image of $L(0,\alpha)$. Equivalently, $\pi(\alpha)\, U_\alpha = 0$.

Define an operator $\mathbf{E}$ from formal trigonometric series to themselves by

$$\mathbf{E} \sum_\alpha U_\alpha(y)\, e^{i\alpha.Y} := \sum_\alpha \pi(\alpha)\, U_\alpha(y)\, e^{i\alpha.Y}\,. \tag{9.4.12}$$

The previous remarks show that on formal trigonometic series the operator $\mathbf{E}$ projects along the image of $L(0,\partial_Y)$ onto its kernel. Therefore, the two conditions in (9.4.10) are equivalent to the pair of equations

$$\mathbf{E}\, U_0 = U_0\,, \tag{9.4.13}$$

and

$$\mathbf{E}\Big(\, L(0,\partial_y)\, U_0 + \sum_\mu A'_\mu(0)U_0\,\partial_{Y_\mu}U_0\,\Big) = 0\,. \tag{9.4.14}$$

These are the fundamental equations of resonant quasilinear geometric optics. They are analogues of (7.4.23) and (7.4.24).

Since $A(0) = I$ equation (9.4.14)) is equivalent to

$$\partial_t U_0 + \mathbf{E}\Big(\sum_j A_j(0)\,\partial_j U_0 + \sum_\mu A'_\mu(0) U_0\,\partial_{Y_\mu} U_0\Big) = 0\,. \tag{9.4.15}$$

Written this way, the equation looks like an evolution equation for U_0. Since the operator $\mathbf{E}$ does not depend on t one has, at least formally,

$$\partial_t\,\big(I - \mathbf{E}\big)U_0 = 0$$

so the constraint (9.4.13) is satisfied as soon as it is satisfied at $t = 0$. It is reasonable to expect that U_0 can be determined from its initial data required to satisfy $\mathbf{E}U_0(0, x, Y) = U_0(0, x, Y)$.

The equation $W_1 = 0$ is equivalent to the pair of equations $\mathbf{E}\,W_1 = 0$ and $(I - \mathbf{E})\,W_1 = 0$. The first, $\mathbf{E}\,W_1 = 0$, is the second equations in (9.4.14).

Define $Q(\alpha)$ to be the partial inverse of $L(0, \alpha)$ that is,

$$Q(\alpha)\,\pi(\alpha) \;=\; 0, \qquad Q(\alpha)\,L(0, \alpha) \;=\; I - \pi(\alpha)\,.$$

Introduce the operator $\mathbf{Q}$ on trigonometric series by

$$\mathbf{Q}\sum_\alpha U_\alpha(y)\,e^{i\alpha.Y} \;:=\; \sum_\alpha Q(\alpha)\,U_\alpha(y)\,e^{i\alpha.Y}\,, \tag{9.4.16}$$

$\mathbf{Q}$ is a partial inverse to $L(0, \partial_Y)$. It is determined by

$$\mathbf{Q}\,\mathbf{E} \;=\; 0, \qquad \mathbf{Q}\,L(0, \partial_Y) \;=\; I - \mathbf{E}\,. \tag{9.4.17}$$

Since $Q(\alpha)$ commutes with $L(0, \alpha)$, it follows that $\mathbf{Q}$ commutes with $L(0, \partial_Y)$.

The second part, $(I - \mathbf{E})W_1 = 0$, of the equation $W_1 = 0$ is equivalent to $\mathbf{Q}W_1 = 0$. Multiplying (9.4.9) by $\mathbf{Q}$ shows this is equivalent to

$$\big(I - \mathbf{E}\big)U_1 \;=\; -\mathbf{Q}\Big(L(0, \partial_y)\,U_0 + \sum_\mu A'_\mu(0) U_0\,\partial_{Y_\mu} U_0\Big)\,. \tag{9.4.18}$$

Once U_0 is determined, this determines $(I - \mathbf{E})U_1$.

Multiplying $W_{j-1} = 0$ by $\mathbf{Q}$ and $W_j = 0$ by $\mathbf{E}$ shows that the equations $(I - \mathbf{E})W_{j-1} = 0$ together with $\mathbf{E}W_j = 0$ are equivalent to a pair of equations

$$\big(I - \mathbf{E}\big)\,U_j = \mathbf{Q}\,\big(\text{terms in } U_0, U_1, \dots U_{j-1}\big)\,, \tag{9.4.19}$$

and

$$\mathbf{E}\Big(L(0, \partial_y)\,U_j + \sum_\mu A'_\mu(0) U_0\,\partial_{Y_\mu} U_j\Big) = \mathbf{E}\,\big(\text{terms in } U_0, U_1, \dots U_{j-1}\big)\,. \tag{9.4.20}$$

Note that once $U_0, \dots, U_{j-1}$ are determined, equations (9.4.19) and (9.4.20) will serve to determine U_j from initial values $\mathbf{E}U_j(0, x, Y)$.

§9.5. Existence for quaisiperiodic prinicipal profiles.

An essential step is to pass from formal trigonometric series in Y to a more manageable class. One class which will serve us well is to consider profiles $U_0(y,Y)$ which are periodic in Y. Though this suffices for the most interesting examples we construct, the general theory should and does go further. Consider, the one dimensional problem with leading part $\partial_t + \mathrm{diag}\,(\lambda_1,\lambda_2,\lambda_3)\partial_x$ from §9.1. One wants to treat functions oscillating with resonant trio of phases $\alpha_j(x-\lambda_j t)$. For the phases $n.y/\epsilon$ which appear for periodic profiles the ratio of the coefficients of t and x are rational. Thus one could only treat the case of λ parallel to an element of $\mathbb{Q}^3$. Quasiperiodic functions as in the next definition are sufficient to treat a wide variety of problems including general λ.

Notation. *Suppose that the real linear functions $\{\phi_j(Y)\}_{j=1}^m$ are linearly independent over the rationals. To a function $\mathcal{U}(y,\theta_1,\dots,\theta_m)$ smooth and 2π multiply periodic in θ, associate the quasiperiodic profile $U(y,Y) := \mathcal{U}\big(y,\phi_1(Y),\dots,\phi_m(Y)\big)$. An induced operator $\mathcal{E}$ mapping periodic functions to themselves is defined by*

$$\mathcal{E}\Big(\sum_{n\in\mathbb{Z}^m}\mathcal{U}_n(y)e^{in.\theta}\Big) \;:=\; \sum_{n\in\mathbb{Z}^m}\pi\big(\sum_k n_k d\phi_k\big)\,\mathcal{U}_n(y)e^{in.\theta}\,,$$

so that $(\mathcal{E}\mathcal{U})(y,\phi(Y)) := \mathbf{E}\,U(y,Y)$. Similarly define the partial inverse,

$$\mathcal{Q}\mathcal{U} \;:=\; \sum_{n\in\mathbb{Z}^m} Q\big(\sum_k n_k d\phi_k\big)\,\mathcal{U}_n(y)e^{in.\theta}\,.$$

Introduce the shorthand, $n.d\phi := \sum_k n_k d\phi_k$.

To write (9.4.13)-(9.4.14) as an equation for $\mathcal{U}$ note that

$$\frac{\partial}{\partial Y_\mu}\mathcal{U}(y,\phi_1(Y)\dots,\phi_m(Y)) = \sum_{k=1}^m \frac{\partial\phi_k}{\partial Y_\mu}\,\frac{\partial\mathcal{U}_0}{\partial\theta_k}\,.$$

The profile equation for U_0 are equivalent to,

$$\mathcal{E}\,\mathcal{U}_0 = \mathcal{U}_0\,,\qquad \mathcal{E}\,\Big(L(0,\partial_y)\,\mathcal{U}_0 + \sum_\mu A'_\mu(0)\mathcal{U}_0\sum_k\frac{\partial\phi_k}{\partial Y_\mu}\,\frac{\partial\mathcal{U}_0}{\partial\theta_k}\Big) = 0\,, \tag{9.5.1}$$

for $\mathcal{U}_0$. This equation has the form

$$\partial_t\mathcal{U}_0 \;+\; G(\mathcal{U}_0,\partial_{y,\theta})\,\mathcal{U}_0 \;=\; 0\,,$$

where

$$G(\mathcal{U},\partial_{y,\theta}) \;:=\; \mathcal{E}\Big(\sum_{j=0}^d A_j(0)\,\partial_{y_j} \;+\; \sum_\mu A'_\mu(0)\mathcal{U}\sum_k\frac{\partial\phi_k}{\partial Y_\mu}\,\frac{\partial}{\partial\theta_k}\;\Big)\mathcal{U} \;:=\; \mathcal{E}\,K(\mathcal{U},\partial_{y,\theta})\mathcal{U}\,.$$

The notation is chosen to suggest a quasilinear hyperbolic system. But, the operator $\mathcal{E}$ is nonlocal in θ. However, $\mathcal{E}$ is an orthogonal projection operator in $L^2(\mathbb{R}^d_x\times\mathbb{T}^m_\theta)$ which commutes with $\partial_{y,\theta}$.

Theorem 9.5.1. (Joly, Métivier and Rauch, Duke '94) *Suppose that $H_0(y,\theta)\in\cap_s(H^s(\mathbb{R}^d\times\mathbb{T}^m)$ satisfies the constraint $\mathcal{E}H_0 = H_0$. Then there is $T_*>0$ and a unique maximal solution*

$$\mathcal{U}_0\in\cap_s\,C^s\big([0,T_*[\,;\;H^s(\mathbb{R}^d\times\mathbb{T}^m)\big)$$

satisfying (9.29) together with the intitial condition $\mathcal{U}\big|_{t=0} = H_0$. *If* $T_* < \infty$ *then*

$$\limsup_{t\nearrow T_*} \ \big\|\,\mathcal{U}_0(t)\,\big\|_{\mathrm{Lip}(\mathbb{R}^d\times\mathbb{T}^m)} = \infty\,.$$

Sketch of Proof. The key idea is to derive *a priori* estimates as in the case of quasilinear hyperbolic systems. One differentiates the equation applying $\partial^\beta_{x,\theta}$, and takes the real part of the $L^2(\mathbb{R}^d\times\mathbb{T}^m)$ scalar product with $\partial^\beta_{x,\theta}\mathcal{U}_0$ (suppressing the subscript 0) to find

$$\frac{d}{dt}\Big(\frac12\|\partial^\beta_{x,\theta}\mathcal{U}\|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^m)}\Big) \ = \ \mathrm{Re}\,\Big(\partial^\beta_{x,\theta}\mathcal{U}\,,\,\partial^\beta_{x,\theta}\mathcal{E}K(\mathcal{U},\partial_{y,\theta})\mathcal{U}\Big)_{L^2(\mathbb{R}^d\times\mathbb{T}^m)}\,.$$

Using the commutation and symmetry properties of $\mathbf{E}$ yields

$$\begin{aligned}\big(\partial^\beta_{x,\theta}\mathcal{U}\,,\,\partial^\beta_{x,\theta}\mathcal{E}K(\mathcal{U},\partial_{y,\theta})\mathcal{U}\big)_{L^2(\mathbb{R}^d\times\mathbb{T}^m)} \ &= \ \big(\mathcal{E}\,\partial^\beta_{x,\theta}\mathcal{U}\,,\,\partial^\beta_{x,\theta}K(\mathcal{U},\partial_{y,\theta})\mathcal{U}\big)_{L^2(\mathbb{R}^d\times\mathbb{T}^m)}\\ &= \ \big(\partial^\beta_{x,\theta}\mathcal{U}\,,\,\partial^\beta_{x,\theta}K(\mathcal{U},\partial_{y,\theta})\mathcal{U}\big)_{L^2(\mathbb{R}^d\times\mathbb{T}^m)}\,,\end{aligned}$$

the last equality using $\mathcal{E}\,\mathcal{U} = \mathcal{U}$. The last *is* a quasilinear hyperbolic expression. Using Gagliardo-Nirenberg estimates as in the treatment of the quasilinear Cauchy problem, one has

$$\mathrm{Re}\,\big(\partial^\beta_{x,\theta}\mathcal{U}\,,\,\partial^\beta_{x,\theta}K(\mathcal{U},\partial_{y,\theta})\mathcal{U}\big)_{L^2(\mathbb{R}^d\times\mathbb{T}^m)} \ \le\ C\big(\|\mathcal{U}\|_{\mathrm{Lip}(\mathbb{R}^d\times\mathbb{T}^m)}\big)\|\mathcal{U}\|^2_{H^{|\beta|}(\mathbb{R}^d\times\mathbb{T}^m)}\,.$$

Summing on $|\beta|\le s\in\mathbb{N}$ yields

$$\frac{d}{dt}\big\|\mathcal{U}(t)\big\|^2_{H^s(\mathbb{R}^d\times\mathbb{T}^m)} \ \le\ C(\|\mathcal{U}\|_{\mathrm{Lip}(\mathbb{R}^d\times\mathbb{T}^m)})\|\mathcal{U}\|^2_{H^s(\mathbb{R}^d\times\mathbb{T}^m)}\,.$$

Local well-posedness in H^s for $\mathbb{N}\ni s > 1+(d+m)/2$ so that $H^s\subset \mathrm{Lip}$ is then proved as for quasilinear hyperbolic systems. ∎

The principal profile is constructed as the limit of solutions $\mathcal{U}^h(t,x,\theta)$ of difference approximations,

$$\partial_t\mathcal{U}^h \ + \ \mathcal{E}\Big(L(y,(0,\delta^h_x)\mathcal{U}^h \ + \ \sum_\mu\ \sum_k\ A'_\mu(0)\mathcal{U}^{\nu-1}\,\frac{\partial\phi_k}{\partial Y_\mu}\,\delta^h_{\theta_k}\mathcal{U}^h\Big) \ = \ 0, \qquad \mathcal{U}^h\big|_{t=0} = g\,.$$

For either method one has the following upper bound on the spectrum of $\mathcal{U}$ as a function of θ

Definition. *If* $V(\theta_1,\ldots,\theta_m)$ *is a periodic distribution, then the* **spectrum** *of* V *is the set of* $n\in\mathbb{Z}^m$ *so that the* n^{th} *Fourier coefficient of* V *is not equal to zero. The spectrum is denoted* spec V.

Theorem 9.5.2. *Suppose that* $\mathcal{U}$ *is as in Theorem 9.5.2. Denote by* $\mathbb{S}$ *the smallest* $\mathbb{Z}$*-module containing the spectrum of* $\mathcal{U}(0,x,\theta)$ *for all* $x\in\mathbb{R}^d$. *Then,*

$$\forall\ (t,x)\in[0,T_*[\times\mathbb{R}^{d+m}, \qquad \mathrm{spec}\,\mathcal{U}(t,x,\theta)\ \ \subset\ \ \mathbb{S}\cap\mathrm{Char}\,L(0,\partial)\,.$$

Proof. Since $\mathcal{E}\mathcal{U} = \mathcal{U}$ it follows that the Fourier coefficient $\mathcal{U}_\alpha$ is in $\ker L(0,\alpha)$ so is nonvanishing only if $\alpha \in \mathrm{Char}\, L$.

The set

$$\Big\{ t \in [0, T_*[\ : \ \forall\ (t,x) \in [0,t] \times \mathbb{R}^{d+m}, \ \ \mathrm{spec}\, \mathcal{U}(t,x,\theta) \ \ \subset \ \ \mathbb{S} \Big\} \tag{9.5.2}$$

is closed by definition and contains $\{t = 0\}$ by hypothesis. It suffices to show that it is a relatively open subset of $[0, T_*[$.

If $\underline{t}$ is in the set then we first prove that there is a $T > \underline{t}$ so that for all $h < h_0$, the difference approximations $\mathcal{U}^h$ exists for $\underline{t} \le t \le T$ and have spectrum contained in $\mathbb{S}$ on $[\underline{t}, T] \times \mathbb{R}^{d+m}$.

To prove this, write $\mathcal{U}^h$ as the limit of Picard iterates V^ν, defined by $V^1(t,x,\theta) = \mathcal{U}^h(\underline{t},x,\theta)$, and,

$$\partial_t V^\nu \ + \ \mathcal{E}\Big(L(y, (0, \delta_x^h) V^{\nu-1} \ + \ \sum_\mu \sum_k \ A'_\mu(0) V^{\nu-1} \frac{\partial \phi_k}{\partial Y_\mu} \, \delta^h_{\theta_k} V^{\nu-1} \Big) \ = \ 0, \qquad V^\nu\big|_{t=0} = g\,.$$

This will converge on a small interval $\underline{t} \le t \le \underline{t} + \delta(h)$. Since the set of functions with spectrum in $\mathbb{S}$ is an algebra it follows by induction on ν that $\mathrm{spec} V^\nu \subset \mathbb{S}$ on this interval. Passing to the limit $\nu \to \infty$ shows that $\mathrm{spec} V^\nu \subset \mathbb{S}$ for $\underline{t} \le t \le t + \delta(h)$.

The length $\delta(h)$ depends on the size of the initial data. The existence proof yields an *a priori* estimate for the difference equation for $\underline{t} \le t \le \underline{t} + \delta$ with delta independent of h.

Conclude that a finite number of applications of the $\delta(h)$ result shows that for $\underline{t} \le t \le \underline{t} + \delta$ $\mathrm{spec} V^\nu \subset \mathbb{S}$. Passing to the limit $\nu \to \infty$ shows that for $\underline{t} \le t \le \underline{t} + \delta$ $\mathrm{spec} \mathcal{U}^h \subset \mathbb{S}$.

The proof of existence shows that there is a $T \in]\underline{t}, \underline{t} + \delta]$ so that $\mathcal{U}^h \to \mathcal{U}$ on $[\underline{t}, T] \times \mathbb{R}^{d+m}$. Passing to the limit $h \to 0$, shows that the spectrum of $\mathcal{U}$ is contained in $\mathbb{S}$ on $[\underline{t}, T'] \times \mathbb{R}^{d+m}$. This prove that the set (9.5.2) is relatively open. ∎

Examples. i. If the initial data has spectrum contained in $\mathbb{Z}\alpha$ with $\alpha \in \mathbb{Z}^m$, then $\mathrm{spec}\mathcal{U} \subset \mathbb{Z}\alpha$ and one finds an expansion as in one phase geometric optics.

ii. In the extreme opposite case, is the $\mathbb{Z}$-span of the spectrum of the initial data meets the characteristic variety of L in a set much larger than the initial spectrum this suggests the possibility of the creation of many new oscillatory modes. In Chapter 11, we show that such new oscillations can be generated for the compressible inviscid Euler equations in dimensions $d \ge 2$.

§9.6. Small divisors and correctors

The equations for the correctors $\mathcal{U}_j$ with $j \ge 1$, involve the operator $\mathcal{Q}$, for example (9.4.18). *Without further hypotheses, $\mathcal{Q}$ may be very ill behaved.* The matrices $Q(\alpha)$ may grow very rapidly as α grows. This has two consequence. First, **Q** may not even map smooth profiles in θ to distributions in θ. In that case the equations for the higher profiles do not make sense. Second, there are known examples where the error of approximation by the leading term is $o(\epsilon^p)$ but is not $O(\epsilon^{p+\delta})$ for any $\delta > 0$ ([Joly, Métivier and Rauch, 1992]).

What is needed in order to get a reasonably well behaved operator $\mathcal{Q}$ is that the matrices $Q(n.d\phi)$ grow no faster than polynomially in $|n|$. The trouble spots for $\mathcal{Q}$ are eigenvalues of $L(0, n.d\phi)$ which though not equal to zero, are very close to zero.

The proof in §8.3 is short and relatively simple in part because the approximate solution constructed is infinitely accurate. For example, the loss of $d/2$ powers of epsilon in passing from (8.3.13) to (8.3.14) is absorbed by the small size of the residual. While the equations for the prinicipal profile derived in §9.4 is robust, there are serious problems with the equations for the correctors. One either must make additional hypotheses or continue the analysis without or with weaker correctors. This section presents a simple example illustrating the nature of the problem.

Begin by considering what appears to be a very special case, but which captures the essence of the difficulty posed by small divisors. Suppose that $\phi = \alpha . y$ is a linear phase that satisfies the eikonal equation for $L(0, \partial)$. Consider what happens for a nonlinear problem with initial data which oscillate with the same phase but with different and incomensurate frequencies. For example one can take the initial data corresponding to the Lax solution

$$a(\epsilon, y)\, e^{i\phi(y)/\epsilon} \; + \; a(\epsilon, y)\, e^{i\rho\,\phi(y)/\epsilon}\,,$$

where ρ is irrational. Then the two phases $\phi_1 = \phi$ and $\phi_2 = \rho\,\phi$ are linearly independent over the rationals. By resonant interaction one expects the solution to involve at least the phases $n_1\phi_1 + n_2\phi_2$ with $n_j \in \mathbb{Z}$. The leading profile is expected to be at least as complicated as

$$U(y, \theta) \;=\; \sum_{n \in \mathbb{Z}^2} a_n(y)\; e^{i(n_1 + \rho\, n_2)\theta}\,.$$

The set of numbers $n_1 + \rho\, n_2$ is countable but dense in $\mathbb{R}$. Even if the a_n are smooth and rapidly decreasing we have an almost periodic function with dense set of frequencies.

The operator $\mathbf{Q}$ is given by

$$\mathbf{Q}\, U \;:=\; \sum_n Q((n_1 + \rho\, n_2)\phi)\, a_n(y)\; e^{i(n_1 + \rho\, n_2)\theta}\,.$$

This is perfectly well defined on formal Trigonometric series. However since Q is the partial inverse of $L(0\,,\,(n_1 + \rho\, n_2)\phi) = (n_1 + \rho\, n_2) L(0\,,\,\phi)$,

$$Q((n_1 + \rho n_2)\phi) \;=\; \frac{1}{n_1 + \rho\, n_2}\, Q(\phi)\,.$$

Where $n_1 + \rho n_2$ is small, these matrices are large. There are divisors $n_1 + \rho n_2$ arbitrarily close to 0. The operator Q is not bounded on L^2. The divisor is small when $\rho \approx -n_1/n_2$. The mapping properties depend on how well the irrational number ρ is approximated by rational numbers.

If ρ is algebraic, and the solution of an irreducible integer polynomial of degree $d \geq 2$, then Liouville's theorem asserts that there is a constant $c > 0$ depending on ρ so that for all integers p, q

$$\Big|\rho \;-\; \frac{p}{q}\Big| \;\geq\; \frac{c}{q^d}\,.$$

In this case,

$$\big|n_1 + \rho\, n_2\big| \;=\; \Big|n_2\Big(\frac{n_1}{n_2} + \rho\Big)\Big| \;\geq\; \frac{c\,|n_2|}{|n_2|^d}\,.$$

Therefore $Q\big((n_1 + \rho\, n_2)\phi\big)\;|n|$ can grow no faster than polynomially in n. This can be used to show that $\mathbf{Q}$ is bounded $H^s \to L^2$ for s sufficiently large.

If ρ were exceptionally well approximable by rationals, (for example for the Liouville number $\rho = \sum_{j=1}^{\infty} 10^{-(j!)}$) then $\mathbf{Q}$ would not have this desirable property and the construction of correctors hits a serious snag.

The next hypothesis describes small divisors that can be tolerated.

Small divisor hypothesis. *There is a $C>0$ and an integer N so that for all $n\in\mathbb{Z}^m\setminus 0$, if $\lambda\neq 0$ is an eigenvalue of $L(0,\sum_k n_k d\phi_k)$, then*

$$|\lambda|\ \geq\ \frac{C}{|n|^N}\,. \tag{9.6.1}$$

If $\phi_j=\alpha_j.y$ then the hypothesis is satisifed for Lebesgue almost all choices of α_j. It is often not difficult to verify this hypothesis.

Examples. In the example of §9.5 the small divisor hypothesis is satisfied for any irrational number ρ that is not exceptionally well approximated by rationals, that is when

$$\exists c>0,\quad \exists N,\quad \forall p,q\in\mathbb{Z},\qquad \Big|\rho\ -\ \frac{p}{q}\Big|\ \geq\ \frac{c}{|q|^N}\,.$$

For example, if ρ is the an algebraic number of degree $d\geq 2$.

On the other hand if ρ is too well approximable by rationals, for example the Liouville number, then the hypothesis is violated.

Proposition 9.6.1. *If the small divisor hypothesis is satisfied then there is a constant $C>0$ and an integer M so that for all $n\in\mathbb{Z}^m$,*

$$\|Q(n.d\phi)\|\ \leq\ C\,\langle n\rangle^M\,.$$

Proof. From the small divisor hypothesis, one knows that the nonzero eigenvalues of $Q(n.d\phi)$ lie in an annulus $2c/\langle n\rangle^N\leq|z|\leq\langle n\rangle^N/2$. Define a larger annulus containing the eigenvalues strictly in its interior by

$$D(n)\ :=\ \big\{z\,:\,c/\langle n\rangle^N\leq|z|\leq\langle n\rangle^N\big\}\,.$$

Then

$$Q(n.d\phi)\ =\ \frac{1}{2\pi i}\oint_{\partial D(n)}\ \frac{1}{z}\ \big(zI-L(0,n.d\phi)\big)^{-1}\ dz\,.$$

For $z\in\partial D(n)$, $\|zI-L(0,n.d\phi)\|\leq C\langle n\rangle^N$. The nearest eigenvalue is no closer than $C\langle n\rangle^{-N}$. Therefore $\|(zI-L(0,n.d\phi))^{-1}\|\leq C\langle n\rangle^{N'}$, and the Proposition follows. ∎

The Proposition implies that when the small divisor hypothesis is satisfied, $\mathbf{Q}$ maps $\cap_s H^s(\mathbb{R}^d\times\mathbb{T}^m)$ continuously to itself. The next Theorem is linear and easier than the previous one.

Theorem 9.6.2. *Suppose that the small divisor hypothesis is satisfied and that $\mathcal{U}_0$ is as in Theorem 9.1 and for $j\geq 1$ initial profiles $H_j(y,\theta)\in\cap_s(H^s(\mathbb{R}^d\times\mathbb{T}^m)$ satisfy $\mathcal{E}H_j=H_j$. Then higher order profiles $\mathcal{U}_j\in\cap_s C^s([0,T_*[\,;\,H^s(\mathbb{R}^d\times\mathbb{T}^m))$ for $j\geq 1$ are uniquely determined by the initial conditions $\mathcal{E}\,\mathcal{U}_j=H_j$ and the transcriptions of (9.4.19) and (9.4.20) to the reduced profiles.*

Suppose that the profiles $\mathcal{U}_j$ of all orders are determined as in Theorems 9.2 and 9.3. Borel's Theorem constructs

$$C^\infty\big([0,1]\times[0,T_*[\,:\,\cap_s H^s(\mathbb{R}^d)\big)\ \ni\ \mathcal{U}(\epsilon,y,\theta)\ \sim\ \sum_j\ \epsilon^j\,\mathcal{U}_j(y,\theta)\,. \tag{9.6.2}$$

Define approximate solutions

$$u^\epsilon(t,x) := \epsilon\,\mathcal{U}(\epsilon, t, x, \phi_1(t,x)/\epsilon, \dots, \phi_m(t,x)/\epsilon) \in \cap_s C^s([0,T_*[\,;\, H^s(\mathbb{R}^d))\,. \tag{9.6.3}$$

Theorem 9.6.3. *With the above definitions, the residual*

$$r^\epsilon := L(u^\epsilon, \partial)\, u^\epsilon \tag{9.6.4}$$

is infinitely small in the sense that

$$\forall T \in [0, T_*[\,,\ \gamma \in \mathbb{N}^{d+1}\,,\ N \in \mathbb{N}\,,\ \ \exists c > 0\,,\ \ \forall \epsilon \in]0,1], \qquad \|\partial_y^\gamma r^\epsilon\|_{L^2([0,T]\times\mathbb{R}^d)} \le c\,\epsilon^N \tag{9.6.5}$$

The small divisor hypothesis is needed to construct correctors. Without it, the leading profile U_0 still exists. And it can sometimes be proved to provide an approixmation with relative error $o(1)$ as $\epsilon \to 0$, [Joly, Métiver, Rauch, Ann.Inst.Fourier, 1994].

§9.7. Stability and accuracy of the approximate solutions

The approximate solutions are of size $O(\epsilon)$ but taking a derivative costs a power of ϵ. Thus $(\epsilon\partial_y)^\gamma$ applied to the approximate solutions which is $O(\epsilon)$. The next theorem implies that the approximate solutions are infinitely close to the exact solutions with the same initial values.

The result differs from Theorem 8.6 in two ways. First it is on $\mathbb{R}^d$ rather than local in Ω_T. Much more important it is quasilinear instead of semilinear and that requires some changes in the proof. The reader is referred to the original papers, for example [Joly, Métiver, Rauch Duke J. 1994] for details. We present statements only. Examples are discussed in the next two chapters.

Stability Theorem 9.7.1 *Suppose that $T > 0$ and that u^ϵ is a family of smooth approximate solutions to $L(u, \partial)\, u = 0$ which are $O(\epsilon)$ in the sense that for all $\gamma \in \mathbb{N}^{1+d}$, $\exists c(\gamma) > 0$, $\forall \epsilon \in]0,1]$*

$$\|\, (\epsilon\partial_y)^\gamma u^\epsilon\, \|_{L^\infty([0,T]\times\mathbb{R}^d)} \ \le\ c(\gamma)\,\epsilon\,. \tag{9.7.1}$$

Suppose that the residuals $r^\epsilon := L(u^\epsilon, \partial)\, u^\epsilon$ are infinitely small in the sense that

$$\forall \gamma \in \mathbb{N}^{d+1}\,,\ N \in \mathbb{N}\,,\ \ \exists c > 0\,,\ \ \forall \epsilon \in]0,1], \qquad \|\partial_y^\gamma r^\epsilon\|_{L^2([0,T]\times\mathbb{R}^d)} \le c\,\epsilon^N\,. \tag{9.7.2}$$

Define $v^\epsilon \in C^\infty([0, T_*(\epsilon)[\times\mathbb{R}^d)$ to be the maximal solution of the initial value problem

$$L(v^\epsilon, \partial)\, v^\epsilon \ =\ 0\,, \qquad v^\epsilon(0,x) \ =\ u^\epsilon(0,x)\,. \tag{9.7.3}$$

Then there is an $\epsilon_0 > 0$ so that for $\epsilon < \epsilon_0$, the time of existence satisfies $T_*(\epsilon) > T$, and the approximate solution u^ϵ is infinitely close to the exact solution v^ϵ in the sense that for all integers s and N

$$\|u^\epsilon - v^\epsilon\|_{H^s([0,T]\times\mathbb{R}^d)} \ \le\ c(s,N)\,\epsilon^N\,.$$

§9.7. Semilinear resonant nonlinear geometric optics

The simplest examples, like those in §9.2, are semilinear. The first examples in §10 are semilinear. In this section we simply state the form of the *ansatz* and profile equations in the semilinear case.

The precise theorem statements and proofs closely resemble the quasilinear case and can be found in the references.

For a semilinear system

$$L(\partial)\, u + f(u) = 0\,, \qquad L(\partial) := \sum_{\mu=0}^{d} A_\mu\, \partial_\mu\,,$$

recall that $\pi(\alpha)$ is orthogonal projection on the kernel of $L(\alpha)$ and $\mathbf{E}$ is the operator on formal trigonometric series $\mathbf{E} \sum a_\alpha(y)\, e^{i\alpha.\theta} := \sum \pi(\alpha)\, a_\alpha(y)\, e^{i\alpha.\theta}$. The critical size for semilinear problems is amplitudes $O(\epsilon^p)$ with $p = 0$. The approximate solutions have the form

$$u^\epsilon \;\sim\; U_0^\epsilon(y, y/\epsilon)\,, \tag{9.7.1}$$

$$U_0(y,Y) \;\sim\; \sum_{\alpha \in \mathbb{R}^d} U_{0,\alpha}(y)\, e^{i\alpha.Y}\,. \tag{9.7.2}$$

The amplitudes are $O(1)$ as $\epsilon \to 0$ in contrast to the quasilinear case where the amplitudes where $O(\epsilon)$ but in agreement with the one phase semilinear theory.

The profile equations for U_0 are

$$\mathbf{E}\, U_0 = U_0\,, \tag{9.7.3}$$

$$\mathbf{E}\Big(L(\partial_y)\, U_0(y,\theta) + f\big(U_0(y,\theta)\big)\Big) = 0\,. \tag{9.7.4}$$

Solutions of the profile equation of the quasiperiodic form

$$U(\epsilon, y, Y) = \mathcal{U}(\epsilon, y, \phi_1(Y), \dots, \phi_m(Y)) \;\in\; C^\infty([0,\epsilon_0]\,;\, \cap_s H^s([0,T] \times \mathbb{R}^d \times \mathbb{T}^m)\,,$$

with

$$\mathcal{U}(\epsilon, y, \theta_1, \dots, \theta_m) \;\sim\; \sum_{j=0}^{\infty} \epsilon^j\, \mathcal{U}_j(y,\theta)\,,$$

in the sense of Taylor series exist provided the small divisor hypothesis of the preceding section holds with $L(n.d\phi)$ in place of $L(0, n.d\phi)$. They yields approximate solutions with infinitely small residual. The accuracy of these solutions follows from the stability Theorem 8.3.2.

The equations for $\mathcal{U}$ involve the nonlinear term $f(\mathcal{U})$. To prove the analogue of Theorem 9.5.2 in the semilinear context requires the following lemma.

Lemma 9.7.1 *If $V \in L^\infty(\mathbb{T}^m\,;\,\mathbb{C}^N)$ and $F \in C(\mathbb{C}^N\,;\,\mathbb{C}^N)$ then,*

$$\operatorname{spec} F(V) \;\subset\; \mathbb{Z} - \operatorname{span}\big(\operatorname{spec} V\big)\,. \tag{9.7.5}$$

Proof. The Weierstrass approximation theorem allows us to choose polynomials P^ν in $U, \overline{U}$ so that $P^\nu(W) \to F(W)$ uniformly on $\{|W\| \le \|V\|_{L^\infty}\}$. Then $P^\nu(V) \to F(V)$ uniformly.

Since $\operatorname{spec}(UV) \subset \operatorname{spec} U \,+\, \operatorname{spec} V$ it follows that,

$$\operatorname{spec} P^\nu(V) \;\subset\; \mathbb{Z} - \operatorname{span}\big(\operatorname{spec} V\big)\,.$$

Passing to the limit $\nu \to \infty$ proves (9.7.5). ∎

Chapter 10. Examples of Resonance in One Dimensional Space

§10.1. Resonance relations.

The examples in this chapter share a common spectral structure. The semilinear examples have

$$A_0 = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_1 = \operatorname{diag}\big\{1, 0, -1\big\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The quasilinear examples have $A_0 = I$ and $A_1'(0) = \operatorname{diag}\big\{1, 0, -1\big\}$. The operator

$$L \ := \ \partial_t \ + \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \partial_x \tag{10.1.1}$$

is equal to $L(\partial)$ in the first case and to $L(0, \partial)$ in the second.

In this chapter we consider exclusively profiles that are 2π periodic in Y,

$$U(y, Y) \ = \ \sum_{n \in \mathbb{Z}^2} a_n \, e^{in.Y} \,, \tag{10.1.2}$$

so the formal trigonometric series from §9.4 are Fourier series. In the language of quasiperiodic profiles with reduced profile $\mathcal{U}$ from §9.5, this corresponds to taking $m = 2$ and phases $\phi_\mu(y) := y_\mu \,, \mu = 0, 1$. For the more general operator, $\partial_t + \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)\partial_x$, the quasiperiodic setting is required in order to capture the triad of resonant phases.

Proposition 10.1.1. *For the operator (10.1.1) and phases $\phi_\mu = y_\mu$, the small divisor hypothesis is satisfied.*

Proof. The matrix

$$L(n.d\phi) = L(n_0, n_1) = \begin{pmatrix} n_0 + n_1 & 0 & 0 \\ 0 & n_0 & 0 \\ 0 & 0 & n_0 - n_1 \end{pmatrix}$$

has eigenvalues $n_0 + n_1 \,, \ n_0 \,, \ n_0 - n_1$. For $n \in \mathbb{Z}^2$ the eigenvalues are integers.

When an eigenvalue is nonzero, it is bounded below by 1 in modulus. This proves small divisor hypothesis (9.6.1) with $N = 0$ and $C = 1$. ∎

Denote the standard basis elements of $\mathbb{C}^3$ by

$$r_1 := (1, 0, 0) \,, \qquad r_2 := (0, 1, 0) \,, \qquad r_3 := (0, 0, 1) \,. \tag{10.1.3}$$

The matrix $L(0, n)$ is diagonal in this basis and when $n_1 \neq 0$ the eigenvalues are distinct and the corresponding eigenprojectors are,

$$\pi_1 \ := \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \pi_2 \ := \ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \pi_3 \ := \ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $n_1 \neq 0$, $\pi(n_0, n_1)$ is nonzero in exactly three circumstances

$$\begin{aligned} n_0 + n_1 &= 0\,, \qquad \text{in which case} \qquad \pi(n) = \pi_1\,, \\ n_0 &= 0\,, \qquad \text{in which case} \qquad \pi(n) = \pi_2\,, \\ n_0 - n_1 &= 0\,, \qquad \text{in which case} \qquad \pi(n) = \pi_3\,. \end{aligned} \tag{10.1.4}$$

When $n_1 = 0$, $\pi(n) = I$.
Let

$$\lambda_1 := +1\,, \qquad \lambda_2 := 0\,, \qquad \lambda_3 := -1\,.$$

The characteristic variety of L is the union of the three lines

$$\ell_j \ := \ \Big\{\eta = (\eta_0, \eta_1) \ : \ \eta_0 + \lambda_j \eta_1 = 0\Big\}\,, \qquad j = 1, 2, 3\,.$$

In the figure the characteristic points of $\mathbb{Z}^2$ are indicated by dots. The circled dots yield two resonance relations,

$$(1,1) + (1,-1) + (-2,0) = 0, \qquad \text{and} \qquad (-1,1) + (-1,-1) + (2,0) = 0\,.$$

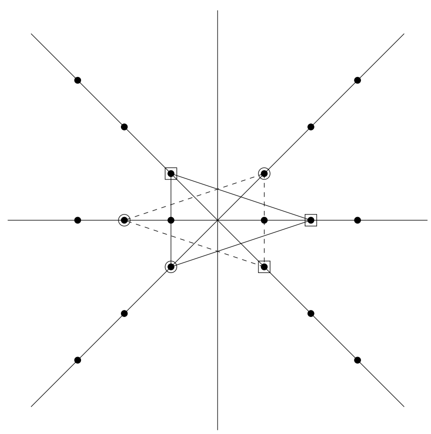


Figure 10.1 Char L and two resonant triads.

Equation (9.4.13) shows that the profile satisfies $\mathbf{E}U_0 = U_0$. In particular, the Fourier coefficients $\widehat{U_0}(y, n_0, n_1)$ vanish unless $n \in \cup_j \ell_j$. The coefficients are polarized,

$$n \in \ell_j \setminus 0 \qquad \Longrightarrow \qquad \pi(n) = \pi_j \quad \text{and} \quad \pi_j\, \widehat{U_0}(y, n) = \widehat{U_0}(y, n)\,. \tag{10.1.5}$$

Since the π_j sum to I, one has

$$\mathbf{E} = \sum_1^3 \mathbf{E}_j\,, \qquad \text{where,} \qquad \mathbf{E}_j := \pi_j\, \mathbf{E}\,.$$

The definition of $\mathbf{E}$ yields,

$$\mathbf{E}_j \sum_{\alpha \in \mathbb{Z}^2} a_\alpha(y)\, e^{i\alpha.Y} = \sum_{\alpha \in \ell_j \cap \mathbb{Z}^2} \pi_j\, a_\alpha(y)\ e^{i\alpha.Y}.$$

For $k \in \mathbb{Z}$, define the scalar Fourier coefficients $\widehat{\sigma_j}$ encoding the spectra of $\widehat{U}_0$ from ℓ_j

$$\begin{aligned}
\widehat{\sigma_1}(y,k) &:= \left\langle \hat{U}_0(y,(k,-k),r_1\right\rangle, \\
\widehat{\sigma_2}(y,k) &:= \left\langle \hat{U}_0(y,(0,k),r_2\right\rangle, \\
\widehat{\sigma_3}(y,k) &:= \left\langle \hat{U}_0(y,(k,k),r_3\right\rangle.
\end{aligned}$$

The corresponding 2π periodic functions are

$$\sigma_j(y,\phi) := \sum_{k\in\mathbb{Z}} \widehat{\sigma_j}(y,k)\, e^{ik\phi}, \qquad j=1,2,3. \tag{10.1.6}$$

Then,

$$U_0(y,Y_0,Y_1) = \Big(\sigma_1(y,Y_0-Y_1)\,,\, \sigma_2(y,Y_1)\,,\, \sigma_3(y,Y_0+Y_1)\Big), \tag{10.1.7}$$

and,

$$\begin{aligned}
\mathbf{E}_1\, U_0 &= r_1 \sum_{k\in\mathbb{Z}} \widehat{\sigma_1}(y,k)\, e^{ik(Y_0-Y_1)}, \\
\mathbf{E}_2\, U_0 &= r_2 \sum_{k\in\mathbb{Z}} \widehat{\sigma_2}(y,k)\, e^{ikY_1}, \\
\mathbf{E}_3\, U_0 &= r_3 \sum_{k\in\mathbb{Z}} \widehat{\sigma_1}(y,k)\, e^{ik(Y_0+Y_1)},
\end{aligned}$$

The next proposition shows that the projection operators $\mathbf{E}$ are simple integral operators. This is a special case of a general phenomenon.

Proposition 10.1.2. *For $g(Y) \in \cap_s H^s(\mathbb{T}^2)$, the operators $\mathbf{E}_j$ are given by the formulas*

$$\begin{aligned}
(\mathbf{E}_1 g)(Y)\big) &= \int_0^{2\pi} \pi_1 g(\psi + (Y_0 - Y_1), \psi)\ \frac{d\psi}{2\pi} \\
(\mathbf{E}_2 g)(Y) &= \int \pi_2 g(Y_0, Y_1)\ \frac{dY_0}{2\pi} \\
(\mathbf{E}_3 g)(Y)\big) &= \int_0^{2\pi} \pi_3 g(-\psi + (Y_0 - Y_1), \psi)\ \frac{d\psi}{2\pi}.
\end{aligned} \tag{10.1.8}$$

The expressions show that the integrals depend only on $Y_0 - Y_1$, Y_1 and $Y_0 + Y_1$ respectively.

Proof. The case $\mathbf{E}_2$ is the easiest. One has

$$\mathbf{E}_2(a\, e^{in.Y}) = \begin{cases} \pi_2 a\, e^{in.Y} & \text{when} \quad n_0 = 0 \\ 0 & \text{when} \quad n_0 \neq 0. \end{cases}$$

On monomials $\mathbf{E_2}$ agrees with $\int \ldots dY_0/2\pi$. By linearity and density

$$\mathbf{E}_2 g(Y_0,Y_1) = \int \pi_2 g(Y_0,Y_1)\ \frac{dY_0}{2\pi},$$

proving the middle formula.

Consider next $\mathbf{E}_1$ for which the preserved monomials are of the form $e^{ik(Y_0-Y_1)}$ with integer k. These monomials are constant on the lines $Y_0 - Y_1 = c$. The general monomial is of the form $e^{imY_0}e^{ik(Y_0-Y_1)}$. To kill those with $m \neq 0$ it is sufficient to integrate over $Y_0 - Y_1 = c$. Parameterize $\{Y_0 - Y_1 = c\}$ by Y_1 to obtain,

$$\mathbf{E}_1 g \;=\; \int_0^{2\pi} \pi_1 g(Y_1 + c, Y_1)\; \frac{dY_1}{2\pi}\,.$$

On the domain of integration, $c = Y_0 - Y_1$ and Y_1 is a dummy variable yielding

$$\mathbf{E}_1 g \;=\; \int_0^{2\pi} g(\psi + (Y_0 - Y_1), \psi)\; \frac{d\psi}{2\pi}\,.$$

For $\mathbf{E}_3$ the monomials $e^{imY_0}e^{ik(Y_0+Y_1)}$ with $m = 0$ are the ones preserved. One singles them out by integrating over $Y_0 + Y_1 = c$ which can be parameterized by Y_1 to yield

$$\mathbf{E}_3 g \;=\; \int_0^{2\pi} \pi_3 g(-Y_1 + c, Y_1)\; \frac{dY_1}{2\pi}\,,$$

which is the third formula. ∎

§10.2. Semilinear examples.

For initial data $U_0(0,x,Y) = \mathbf{E}\,U_0 \in \cap_s H^s(\mathbb{R}^d \times \mathbb{T}^{1+1})$ the leading profile equation has a unique smooth solution locally in time. Since the small divisor hypothesis is satisfied, the corrector profiles U_j exist and are uniquely determined from the initial values of $\mathbf{E}U_j\big|_{t=0}$. The semilinear analogues of Theorems 9.5.3-9.5.4 described in §9.7 imply that they yield infinitely accurate approximate solutions.

The profile equation (9.4.14) is a vector equation with three components. The j^{th} component asserts that

$$\pi_j\, \mathbf{E}\Big(L(\partial_y)U_0 + f(U_0)\Big) \;=\; 0\,.$$

The next computation shows that this equation yields an evolution equation for σ_j coupled by lower order terms to the other components.

Use the diagonal structure of L and $[\mathbf{E}, \partial_y] = [\mathbf{E}, \pi_j] = 0$ to find,

$$\begin{aligned}\pi_1\, \mathbf{E}\, L(\partial_y)\, U_0 \;&=\; \mathbf{E}_1\, \pi_1 L(\partial_y)\, U_0 \;=\; \mathbf{E}_1\, (\partial_t + \partial_x)\, \pi_1\, U_0 \\ &=\; (\partial_t + \partial_x)\, \mathbf{E}_1\, \big(\sigma_1(y, Y_0 - Y_1)\, r_1\big) \;=\; (\partial_t + \partial_x)\, \sigma_j(y, Y_0 - Y_1)\, r_1\,.\end{aligned}$$

Thus,

$$(\partial_t + \partial_x)\, \sigma_j(y, Y_0 - Y_1) \;+\; \Big\langle \mathbf{E}_1\big(f(U_0)\big), r_1 \Big\rangle \;=\; 0\,.$$

Equivalently,

$$\big(\partial_t + \partial_x\big)\, \sigma_1(y, Y_0 - Y_1) + \Big\langle \mathbf{E}_1\Big(f_1\big(\sigma_1(y, Y_0 - Y_1), \sigma_2(y, Y_1), \sigma_3(y, Y_0 + Y_1)\big) r_1\Big), r_1 \Big\rangle = 0. \tag{10.2.1}$$

Similarly, the second and third equations are equivalent to

$$\partial_t \sigma_2(y, Y_1) + \Big\langle \mathbf{E}_2\Big(f_2\big(\sigma_1(y, Y_0 - Y_1)\,,\, \sigma_2(y, Y_1)\,,\, \sigma_3(y, Y_0 + Y_1)\big) r_2\Big)\,,\, r_2 \Big\rangle = 0, \tag{10.2.2}$$

and,

$$\big(\partial_t - \partial_x\big)\,\sigma_3(y, Y_0 + Y_1) + \Big\langle \mathbf{E}_3\Big(f_3\big(\sigma_1(y, Y_0 - Y_1), \sigma_2(y, Y_1), \sigma_3(y, Y_0 + Y_1)\big) r_3\Big), r_3\Big\rangle = 0. \quad (10.2.3)$$

Equations (10.2.1)-(10.2.3) form a coupled system of three integrodifferential equations. They are differential in the variables t, x and integral in the variables $(Y_0, Y_1) \in \mathbb{T}^2$. The system is easy to approximate numerically. For these problems with $d = 1$, the highly oscillatory initial value problem is at the borderline of directcomputer simulation for times $t \sim 1$ and $\epsilon \sim 10^{-3}$.

Example 10.2.1. Consider the three wave interaction system (9.2.2). The transport equation for σ_2 is

$$\partial_t\, \sigma_2(y, \theta_1) \;=\; c_2 \Big\langle \mathbf{E}_2 \Big(\sigma_1(y, Y_0 - Y_1)\, \sigma_3(y, Y_0 + Y_1)\, r_2 \Big)\,,\, r_2 \Big\rangle\,. \qquad (10.2.4)$$

The profile equations are best understood in Fourier. Exand to find

$$\sigma_1(y, Y_0 - Y_1)\, \sigma_3(y, Y_0 + Y_1) \;=\; \sum_{\mu,\nu} \widehat{\sigma_1}(y, \nu)\, e^{i\nu(Y_0 - Y_1)}\; \widehat{\sigma_3}(y, \mu)\, e^{i\mu(Y_0 + Y_1)}\,.$$

The operator $\mathbf{E}_2$ selects the phases $n.Y$ with $n_0 = 0$. As the phase is equal to $\nu(Y_0 - Y_1) + \mu(Y_0 + Y_1)$, this yields $\mu = -\nu$, so

$$\mathbf{E}_2\Big(\big(\sigma_1(y, Y_0 - Y_1)\, \sigma_3(y, Y_0 + Y_1)\, r_2\Big) \;=\; \sum_{\nu} \widehat{\sigma_1}(y, \nu)\; \widehat{\sigma_3}(y, -\nu)\; e^{-2i\nu Y_1}\, r_2\,.$$

The profile equation (10.2.4) for $\widehat{\sigma_2}(y, \nu)$ splits according to the parity of ν,

$$\partial_t\, \widehat{\sigma_2}(y, -2\nu) = c_2\, \widehat{\sigma_1}(y, \nu)\; \widehat{\sigma_3}(y, -\nu)\,, \qquad \partial_t\, \widehat{\sigma_2}(y, -2\nu + 1) = 0\,, \quad \nu \in \mathbb{Z}\,. \qquad (10.2.5)$$

The dynamics for σ_1 is given by

$$\big(\partial_t + \partial_x\big)\, \sigma_1(y, Y_0 - Y_1) \;=\; c_1 \Big\langle \mathbf{E}_1 \Big(\sigma_2(y, Y_1)\, \overline{\sigma_3}(y, Y_0 + Y_1)\, r_1 \Big)\,,\, r_1 \Big\rangle\,. \qquad (10.2.6)$$

The third profile equation is,

$$\big(\partial_t + \partial_x\big)\, \sigma_3(y, Y_0 + Y_1) \;=\; c_3 \Big\langle \mathbf{E}_3 \Big(\sigma_2(y, Y_1)\, \overline{\sigma_1}(y, Y_0 - Y_1) r_3 \Big)\,,\, r_3 \Big\rangle\,. \qquad (10.2.7)$$

For (10.2.6) use,

$$\overline{\sigma_3}(y, \phi) \;=\; \Big(\sum_{\nu} \widehat{\sigma_3}(y, \nu)\; e^{i\nu\phi}\Big)^* \;=\; \sum_{\nu} \widehat{\sigma_3}(y, \nu)^*\; e^{-i\nu\phi}\,,$$

$$\sigma_2(y, Y_1)\, \overline{\sigma_3}(y, Y_0 + Y_1) \;=\; \sum_{\mu,\nu} \widehat{\sigma_2}(y, \mu)\; e^{i\mu Y_1}\; \widehat{\sigma_3}(y, \nu)^*\; e^{-i\nu(Y_0 + Y_1)}\,.$$

The phase of the product of the exponentals is $-\nu Y_0 + (\mu - \nu) Y_1$. The operator $\mathbf{E}_1$ selects only those phases $n.Y$ with $n_0 + n_1 = 0$. In this case,

$$-\nu \;+\; (\mu - \nu) \;=\; 0, \qquad \Longleftrightarrow \qquad \mu \;=\; 2\nu.$$

Therefore,

$$\big(\partial_t + \partial_x\big)\,\sigma_3(y, Y_0 + Y_1) \;=\; c_3\,\sum_\nu \widehat{\sigma_2}(y,2\nu)\;\widehat{\sigma_3}(y,\nu)\;e^{-i\nu(Y_0 - Y_1)}\,.$$

In terms of the Fourier coefficients this is equivalent to,

$$(\partial_t + \partial_x)\widehat{\sigma_1}(y,\nu) \;=\; c_3\,\widehat{\sigma_2}(y,-2\nu)\;\widehat{\sigma_3}(y,-\nu)^*\,. \tag{10.2.8}$$

An analgous computation shows that the third profile equation is equivalent to

$$(\partial_t - \partial_x)\widehat{\sigma_3}(y,-\nu) \;=\; c_3\,\widehat{\sigma_1}(y,\nu)^*\;\widehat{\sigma_2}(y,-2\nu)\,. \tag{10.2.9}$$

Exercise 10.2.1. *Verify (10.2.9).*

The equations (10.2.5), (10.2.8), (10.2.9) show that the nonlinear interactions are localized in the triads

$$\Big\{\widehat{\sigma_1}(y,k)\,,\,\widehat{\sigma_2}(y,-2k)\,,\,\widehat{\sigma_3}(y,-k)\Big\}\,. \tag{10.2.10}$$

The corresponding Fourier coefficients of U_0 are

$$\widehat{U_0}(y,k,-k), \qquad \widehat{U_0}(y,0,-2k) \qquad \widehat{U_0}(y,-k,-k)\,.$$

Two such triads are indicated in Figure 10.1. The interaction comes about through the resonance relation

$$-2x \;=\; (t-x) \;-\; (t+x)\,, \qquad (0,-2k) = (k,-k) + (-k,-k)\,.$$

For each k, the triple (10.2.10) satisfies the three wave interaction pde decoupled from the other Fourier coefficients. The initial data for the triad of Fourier coefficients are indpendent of ϵ and not rapidly oscillating. The fact that the triads are isolated shows that there is no possibility of interactions moving far in the scale of wave numbers.

Consider three special cases. For the initial value problem (9.1.1), $c_1 = c_3 = 0$, and the intial data are

$$\sigma_1(0,x,\phi) = a_1(x)\,e^{i\phi}\,, \qquad \sigma_2(0,x,\phi) = 0\,, \qquad \sigma_3(0,x,\phi) = a_3(x)\,e^{-i\phi}\,.$$

The initial data ignite the single resonant triad. The function $\sigma(y,\phi)$ is given by

$$\sigma_1 = a_1(t-x)e^{i\phi}\,, \quad \sigma_2 = e^{-2i\phi}\int_0^t a_1(t-x)\,a_3(t+x)\,dt\,, \quad \sigma_3 = a_1(t+x)e^{-i\phi}\,.$$

In this particular case, the approximation of nonlinear geometric optics gives the exact solution.

Modify the third initial datum to

$$u_3(0,x) = a_3(x)e^{inx/\epsilon}\,, \qquad n \in \mathbb{Z}\setminus\{-1\}\,, \tag{10.2.11}$$

to find $\sigma_3(t,x,\phi) = a_3(t+x)\,e^{in\phi}$ and,

$$\mathbf{E}_2(\sigma_1(y,Y_0 - Y_1)\,\sigma_3(y,Y_0 + Y_0)r_2 = \mathbf{E}_2\Big(a_1(t-x)\,a_3(t+x)\,e^{i\{(t+x)+n(t-x)\}}r_2\Big) = 0\,.$$

The product inside $\mathbf{E}_2$ always oscillates in time so is annihilated by $\mathbf{E}_2$ to give $\partial_t\sigma_2 = 0$. The oscillations in the second component of U_0 do not change in time and there is no interaction with the oscillations in the other components. This agrees with the nonstationary phase analysis in §9.1.

Consider the real initial data

$$\sigma_1(0,x,\phi) = a_1(x)\,\sin\phi\,, \qquad \sigma_2(0,x,\phi) = 0\,, \qquad \sigma_3(0,x,\phi) = a_3(x)\,\sin(-\phi)\,.$$

In this case the initial data ignite two resonant triads

$$\Big\{(0,-2n),(n,n),(-n,-n)\Big\}, \qquad \text{and,} \qquad \Big\{(0,2n),(-n,-n),(n,n)\Big\}.$$

Each triad of coefficients,

$$\widehat{\sigma_1}(t,x,1)\,,\,\widehat{\sigma_1}(t,x,-2)\,,\,\widehat{\sigma_1}(t,x,-1)\,, \qquad \text{and} \qquad \widehat{\sigma_1}(t,x,-1)\,,\,\widehat{\sigma_1}(t,x,2)\,,\,\widehat{\sigma_1}(t,x,1)\,,$$

solves the three wave interaction pde. All other coefficients vanish identically.

In the last two cases, the approximate solution is not an exact solution.

Proposition 10.2.11. *Consider the system of profile equations for the three wave interaction system with $c_j \in \mathbb{R}\setminus 0$. The following are equivalent.*

i. *For arbitrary initial data $\sigma(0,x,\phi) \in \cap_s H^s(\mathbb{R}\times\mathbb{T})$ there is a unique global solution $\sigma(t,x,\phi) \in \cap_s C^s\big(\mathbb{R}\,;\,H^s(\mathbb{R}\times\mathbb{T})\big)$.*

ii. *The coefficients c_j do not have the same sign.*

Proof. The explosive behavior is proved by considering a single resonant triad which blows up in finite time T_*.

For existence it suffices to observe that the $L^\infty([0,T]\times\mathbb{R})$ bound for solutions of the three wave system with c_j not all of the same sign proves an estimate

$$\|\widehat{\sigma}(t,x,n)\|_{L^\infty([0,T]\times\mathbb{R})} \;\le\; C\Big(\|\widehat{\sigma}(0,x,n)\|_{L^2(\mathbb{R})}\,,\,T\Big)\;\big\|\widehat{\sigma}(0,x,n)\big\|_{L^\infty(\mathbb{R})}\,,$$

with the function $C(\cdot,\cdot)$ independent of n. Summing on n, this suffices to establish an *apriori* estimate

$$\|\sigma(t,x,\phi)\|_{L^\infty([0,T]\times\mathbb{R}\times\mathbb{T})} \;\le\; C\Big(\|\sigma(0,x,\phi)\|_{H^s(\mathbb{R}\times\mathbb{T})}\;,\,s\,,\,T\Big)\,, \qquad s>1\,.$$

This implies global solvability using Moser's inequality as in §6.4. ∎

When the profiles exist globally in time, Theorem 9.4 shows that the approximation of resonant nonlinear geometric optics is accurate on arbitrary long time intervals $0\le t\le T$. In particular the interval of existence of the exact solution grows infinitely long in the limit $\epsilon\to 0$. In the present case we know more, namely that the solutions exist globally. Note that the approximation is not justified on the infinite time interval $0<t<\infty$. One must exercise care in drawing conclusions about the large time behavior of exact solutions from the large time behavior of the profiles.

There is similar caution for the case of explosive profiles. It is tempting to conclude from profile blowup that there is a parallel blowup of exact solutions. This is not justified. Thoerem 9.4 justifies the approximation on arbitrary intervals of smoothness, $0<T<T_*$. One can draw some

conclusions which have the flavor of explosion. Denote by v^ϵ the exact solution with the same intial data as the approximate solution u^ϵ. Choosing T very close to T_* on shows that

$$\lim_{T\to T_*}\ \liminf_{\epsilon\to 0}\ \big\|v^\epsilon(T,x)\big\|_{L^\infty(\mathbb{R}_x)}\ =\ \infty$$

This asserts that the family of exact solutions v^ϵ is unbounded, but it does not assert that any given member of the family explodes.

Example 10.2.2. Consider the modification of equation (9.1.1) where the equation for u_2 is changed to a general real quadratic interaction

$$\partial_t u_2 = \sum_{1\le i\le j\le 3} A_{i,j}\, u_i\, u_j \tag{10.2.12}$$

The profile equation for σ_2 is

$$\partial_t\sigma_2(y,Y_1) = \Big\langle \mathbf{E}_2\Big(\sum_{1\le i\le j\le 3} A_{i,j}\ \sigma_i(y,h_i(Y))\ \sigma_j(y,h_j(Y))r_2\Big)\,,\, r_2\Big\rangle\,, \tag{10.2.13}$$

with

$$h_1(Y) := Y_0 - Y_1, \qquad h_2(Y) := Y_1, \qquad h_3(Y) := Y_0 + Y_1\,.$$

Write U_0 as in (10.1.7). The contribution of the term $A_{1,3}\,\sigma_1\,\sigma_3$ to the profile equation is computed exactly as before and yields,

$$\begin{aligned}\mathbf{E}_2\big(A_{1,3}\,\sigma_1(y,Y_0-Y_1)\,\sigma_3(y,Y_0+Y_1)\,r_2\,\big)\ &=\ =\ A_{1,3}\sum_n \widehat{\sigma_1}(y,n)\ \widehat{\sigma_3}(y,-n)\ e^{-2inY_1}\,r_2\\ &=\ A_{1,3}\big(\sigma_1 * \check{\sigma}_3\big)(y,-2Y_1)\,r_2\,.\end{aligned}$$

Denoting with an underline the mean value of a 2π periodic function one then computes the formulas

$$\begin{aligned}\mathbf{E}_2\big(A_{1,2}\,\sigma_1(y,Y_0-Y_1)\,\sigma_2(y,Y_1)\,r_2\,\big)\ &=\ A_{1,2}\,\underline{\sigma_1}\,\sigma_2(y,Y_1)\,r_2\,,\\ \mathbf{E}_2\big(A_{2,3}\,\sigma_2(y,Y_1)\,\sigma_3(y,Y_0+Y_1)\,r_2\,\big)\ &=\ A_{2,3}\,\sigma_2(y,Y_1)\,\underline{\sigma_3}\,r_2\,,\\ \mathbf{E}_2\big(A_{2,2}\,\sigma_2(y,Y_1)\,\sigma_2(y,Y_1)\,r_2\,\big)\ &=\ A_{2,2}\,\sigma_2(y,Y_1)^2\,r_2\,.\end{aligned}$$

Combining yields the profile equation

$$\partial_t\sigma_2 = A_{2,2}\,\sigma_2^2(y,\phi) + A_{1,2}\,\sigma_2(y,\phi)\,\underline{\sigma_1} + A_{2,3}\,\sigma_2(y,\phi)\,\underline{\sigma_3} + A_{1,3}\,(\sigma_1 * \check{\sigma}_3)(y,-2\phi)\,. \tag{10.2.14}$$

Notice that the first three terms are local in y,ϕ while the quadratic convolution interaction term which comes from the resonance is local in Fourier and not in ϕ. For the initial data from (9.1.1), $\underline{\sigma_1} = \underline{\sigma_3} = 0$ and the profile equation simplifies to

$$\partial_t\sigma_2(y,\phi) = A_{2,2}\,\sigma_2^2 + A_{1,3}\,a_1(t-x)\,a_3(t+x)\,e^{-2i\phi}\,. \tag{10.2.15}$$

Only one Fourier component of σ_2 is affected by the resonant term. The $A_{2,2}\,\sigma_2^2$ broadens the spectrum of σ_2, one of whose Fourier components is influenced by the waves from modes one and three.

With general quadratic interactions in all the equations, one finds coupled integrodifferential equations with quadratic self interaction terms for all j. The resulting three by three systems are analogous to

$$\partial_t \sigma = a\,\sigma^2 + b\,\sigma * \sigma\,, \qquad \sigma = \sigma(t,\phi)\,. \tag{10.2.16}$$

It would be interesting to understand well the competition between the two quadratic terms on the right of (10.2.16). The term that is local in ϕ is a convolution in n while the convolution in ϕ is local in n.

§10.3. Quasilinear examples.

The next examples resemble the semilinear examples. An important difference is that the amplitudes of the approximate solutions are smaller. The approximate solution defined by the leading profile is given by

$$u^\epsilon(t,x) = \epsilon\, U_0(t,x,t/\epsilon,x/\epsilon)\,.$$

The prefactor of ϵ was absent in the semilinear case. For profiles periodic in Y, equation (9.5.1) simplifies to,

$$\mathbf{E}\, U_0 = \mathcal{U}_0\,, \qquad \mathbf{E}\Big(L(0,\partial_y)\, U_0 + \sum_{\mu=0}^{1} A_\mu'(0)\, U_0\, \frac{\partial U_0}{\partial Y_\mu} \Big) = 0\,. \tag{10.3.1}$$

Example 10.3.1 A quasilinear analogue of the system (9.1.1) is the system of conservation laws

$$\begin{aligned} (\partial_t + \partial_x) u_1 &= 0 \\ \partial_t u_2 + \partial_x (u_1 u_3) &= 0 \\ (\partial_t - \partial_x) u_3 &= 0 \end{aligned} \tag{10.3.2}$$

Proposition 10.1.1 shows that with $\phi_\mu := y_\mu$, the small divisor hypothesis is satisfied and equations (10.1.1) through (10.1.7) are unchanged. And, $A_0(u) = I$. The second component of the profile reads,

$$\partial_t\, \sigma_2 + \Big\langle \mathbf{E}_2\Big(A_1'(0) U_0\, \frac{\partial U_0}{\partial Y_1}\Big)\,,\, r_2 \Big\rangle = 0\,. \tag{10.3.3}$$

Equation (10.1.7) yields,

$$\frac{\partial U_0}{\partial Y_1} = \Big(-\sigma_1'(y, Y_0 - Y_1)\,,\, \sigma_2'(y, Y_1)\,,\, \sigma_3'(y, Y_0 + Y_1) \Big)\,. \tag{10.3.4}$$

For (10.3.2),

$$A_1(U) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ U_3 & 0 & U_1 \\ 0 & 0 & 0 \end{pmatrix} \;=\; A_1(0) + A'(0) U$$

so

$$A_1'(0) U_0 = \begin{pmatrix} 0 & 0 & 0 \\ \sigma_3(y, Y_0 + Y_1) & 0 & \sigma_1(y, Y_0 - Y_1) \\ 0 & 0 & 0 \end{pmatrix}\,.$$

Suppressing the y dependence,

$$\begin{aligned} A_1'(0)\, U_0\, \frac{\partial U_0}{\partial Y_1} &= \Big(-\sigma_3(Y_0 + Y_1)\, \sigma_1'(Y_0 - Y_1) + \sigma_1(Y_0 - Y_1) \sigma_3'(Y_0 + Y_1) \Big)\, r_2 \\ &= r_2\, \frac{\partial}{\partial Y_1} \Big(\sigma_1(Y_0 - Y_1) \sigma_3(Y_0 + Y_1) \Big)\,. \end{aligned}$$

$\mathbf{E}_2$ commutes with $\partial/\partial_{Y_\mu}$ and $\mathbf{E}_2$ applied to the product is computed as earlier to find,

$$\partial_t \sigma_2(t,x,Y_1) \;=\; \frac{\partial}{\partial Y_1}\Big(\sum_\nu e^{-2i\nu Y_1}\ \widehat{\sigma_1}(t,x,\nu)\ \widehat{\sigma_3}(t,x,-\nu)\Big) \tag{10.3.5}$$

The odd Fourier coefficients of σ_2 are stationary and the even ones evolve according to

$$\partial_t \hat\sigma_2(y,-2\nu) = -2i\nu\, \hat\sigma_1(y,\nu)\, \hat\sigma_3(y,-\nu)\,. \tag{10.3.6}$$

The profile equations read

$$\begin{aligned} \big(\partial_t + \partial_x\big)\sigma_1 \;&=\; 0\,,\\ \partial_t \sigma_2 \;&=\; \partial_\phi\Big(\big(\sigma_1 * \check\sigma_3\big)(t,x,-2\phi)\Big)\,,\\ \big(\partial_t - \partial_x\big)\sigma_3 \;&=\; 0\,. \end{aligned} \tag{10.3.7}$$

The system (10.3.7) is in conservation form. This is a general phenomenon. If the original system is in conservation form,

$$\sum_{\mu=0}^{d} \partial_\mu A_\mu(u) \;=\; 0\,, \tag{10.3.8}$$

then the terms of the equation are $A'_\mu(u)\partial_\mu u$. The coefficients, $A'_\mu(u)$, have the special structure of being derivatives.

Exercise 10.3.1. *If the original system is real and in conservation form (10.3.7) and $A_0(0) = I$, then the profile equation (9.4.14) can be written in the conservation form*

$$\partial_t U_0 + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\Big(\mathbf{E}\big(A_j(0)U_0\big)\Big) + \sum_{\mu=0}^{d}\ \sum_{j,k=1}^{N} \frac{\partial}{\partial\theta_k}\left(\mathbf{E}\,\frac{\partial^2 A_\mu(0)}{\partial u_j \partial u_k}\, U_j U_k\right) = 0\,. \tag{10.3.9}$$

For complex equations there are more terms because of the derivatives with respect to the conjugate variables but the conservation form persists.

Equation (10.3.9) implies that in the case of conservation laws a profile as in Theorem 9.1 that has mean zero with respect to θ at $\{t = 0\}$ remains mean zero throughout its maximal interval of existence. As in Example 10.2.2, the profile equations in the mean zero case are simpler.

Proposition 10.3.1. *Consider a real 3×3 system of conservation laws with $d = 1$,*

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} A(u) \;=\; 0\,, \qquad A(u) = \big(A_1(u), A_2(u), A_3(u)\big)\,, \tag{10.3.10}$$

satisfying $A'(0) = \operatorname{diag}\big\{1, 0, -1\big\}$. Introduce σ_j as in (10.1.7) and six interaction constants

$$b_j := \frac{\partial^2 A_j(0)}{\partial u_j^2}\,, \quad j = 1,2,3\,, \qquad c_1 := \frac{\partial^2 A_1(0)}{\partial u_2 \partial u_3}\,, \quad c_2 := \frac{\partial^2 A_2(0)}{\partial u_1 \partial u_3}\,, \quad c_3 := \frac{\partial^2 A_3(0)}{\partial u_1 \partial u_2}\,. \tag{10.3.11}$$

The profile equation (9.5.1) for periodic profiles (10.1.1) of mean zero is equivalent to the system of equations for the Fourier coefficients,

$$
\begin{aligned}
\big(\partial_t + \partial_x\big)\, \hat\sigma_1(t,x,m) &= b_1\, im\, \widehat{(\sigma_1^2)}(t,x,m) + c_1\, im\, \hat\sigma_2(t,x,2m)\, \hat\sigma_3(t,x,-m)\,,\\
\partial_t\, \sigma_2(t,x,2m) &= b_2\, 2im\, \widehat{(\sigma_2^2)}(t,x,2m) + c_2\, 2im\, \hat\sigma_1(t,x,m)\, \hat\sigma_3(t,x,m)\,,\\
\big(\partial_t - \partial_x\big)\, \hat\sigma_3(t,x,m) &= b_3\, im\, \widehat{(\sigma_3^2)}(t,x,m) + c_3\, im\, \hat\sigma_2(t,x,2m)\, \hat\sigma_1(t,x,-m)\,,\\
\partial_t \hat\sigma_2(t,x,2m+1) &= b_3\, 2im\, \widehat{(\sigma_2^2)}(t,x,2m+1)\,.
\end{aligned}
\tag{10.3.12}
$$

Exercise 10.3.2. *Prove Proposition 10.3.1.*

The next goal is to analyse more closely the resonance terms. First consider the case where all the b_j vanish so the self interaction terms are absent.

Example 10.3.2. Consider the case where $b_1 = b_2 = b_3 = 0$. Then for each $m \in \mathbb{Z}$, the three Fourier components $\{\hat\sigma_1(y,m), \hat\sigma_2(y,-2m), \hat\sigma_3(y,-m)\}$ evolve independent of the other Fourier components according to the laws

$$
\begin{aligned}
\big(\partial_t + \partial_x\big)\, \hat\sigma_1(t,x,m) &= -c_1\, im\, \hat\sigma_2(t,x,-2m)\, \hat\sigma_3(t,x,\mp m)\,,\\
\partial_t\, \hat\sigma_2(t,x,2m) &= -c_2\, 2im\, \hat\sigma_1(t,x,m)\, \hat\sigma_3(t,x,-m)\,,\\
\big(\partial_t - \partial_x\big)\, \hat\sigma_3(t,x,-m) &= -c_3\, im\, \hat\sigma_2(t,x,-2m)\, \hat\sigma_1(t,x,m)\,.
\end{aligned}
\tag{10.3.13}
$$

The odd components of σ_2 belong to no such triad and are stationary,

$$
\partial_t \widehat{\sigma_2}(t,x,2m+1) = 0\,. \tag{10.3.14}
$$

For fixed $m \neq 0$, the triple $(i\sigma_1, i\sigma_2, i\sigma_3)$ satisfies the three wave interaction pde which we understand well. In addition to the information already gleaned, one has the following invariance properties.

Proposition 10.3.2. *The profile equations (10.3.13) have the following properties.*

1. *The set of σ so that for a fixed $m \in \mathbb{Z}$, and $\forall x, \phi$,*

$$
\begin{aligned}
&\sigma_1(t,x,m) = -\sigma_1(t,x,-m)\,, \qquad \sigma_2(t,x,2m) = -\sigma_2(t,x,-2m)\,,\\
&\sigma_3(t,x,m) = -\sigma_3(t,x,-m)\,,
\end{aligned}
$$

is invariant. Imposing this condition for all m shows that the set of σ so that $\hat\sigma(y,m)$ is odd in m is invariant under the dynamics. These are exactly the functions $\sigma(y,\phi)$ which are odd in ϕ.

2. *The set of σ so that for a fixed $m \in \mathbb{Z}$, $\{\hat\sigma_1(y,m), \hat\sigma_2(y,-2m), \hat\sigma_3(y,-m)\}$ are purely imaginary for all x, ϕ is invariant. Therefore the set of σ so that $\hat\sigma(y,m)$ is purely imaginary for all $m, x \in \mathbb{Z}\times\mathbb{R}$ is invariant.*

3. *The set of σ so that for a fixed $m \in \mathbb{Z}$, $\{\hat\sigma_1(t,x,m), \hat\sigma_2(t,x,-2m), \hat\sigma_3(t,x,-m)\}$ do not depend on x is invariant. Imposing this for all m shows that the set of σ which do not depend on x is invariant.*

Global solvability of the profile equations when $b = 0$ is completely resolved by our analysis of the three wave interaction pde.

Proposition 10.3.3. *If $b_1 = b_2 = b_3 = 0$ and the three constants c_1, c_2, and c_3 do not have the same sign, then the profile equations (10.3.13) are globally solvable in the sense that for arbitrary initial data $\sigma(0,x,\phi) \in \cap_s \mathrm{Re}\, H^s(\mathbb{R}\times\mathbb{T})$ there is a unique global solution $\sigma(t,x,\phi) \in \cap_s C^s\big(\mathbb{R}\,;\, H^s(\mathbb{R}\times\mathbb{T})\big)$. The norms $\|\sigma(t)\|_{H^s(\mathbb{R}\times\mathbb{T})}$ are bounded independent of $t\in\mathbb{R}$. In contrast, if the b_j vanish, and c_1, c_2, and c_3 have the same sign, then the profile equations (10.3.13) have solutions with finite blowup time $0 < T_* < \infty$.*

This blowup is quite striking. Consider for example the case of a profile whose Fourier series is supported on a single pair of resonant triads as in Figure 10.1,

$$U_0(t,x,Y) \;=\; -\Big(\,\zeta_1(t)\,\sin m(Y_0-Y_1)\,,\,\zeta_2(t)\,\sin(-2mY_1)\,,\,\zeta_3(t)\,\sin(-m(Y_0+Y_1))\,\Big)\,. \tag{10.3.15}$$

The exact solution is described by

$$u^\epsilon(t,x) \;\sim\; -\epsilon\,\Big(\,\zeta_1(t)\,\sin\frac{m(t-x)}{\epsilon}\,,\,\zeta_2(t)\,\sin\frac{-2mt}{\epsilon}\,,\,\zeta_3(t)\,\sin\frac{-m(t+x)}{\epsilon}\,\Big)\,. \tag{10.3.16}$$

Suppose that $\zeta(t)$ is a solution of the three wave interaction ode. whose components have the same sign and blow up at time $T_* < \infty$ so that

$$\lim_{t\to T*_-}\;\big|\zeta(t)\big| \;=\; \infty\,. \tag{10.3.17}$$

The initial data and solutions are periodic in x. The data are bounded in $BV(I)$ for any bounded interval, and are $O(\epsilon)$ in $L^\infty(\mathbb{R})$. For the exact solutions, Theorem 9.4 together with finite speed of propagation yields the following result of unbounded amplification.

Theorem 10.3.4 *Suppose that the system (10.3.12) satifies $b=0$ and that c_1, c_2, and c_3 have the same sign. Choose $\zeta(t)$ a real solution of the profile equation which explodes at time $0 < T_* < \infty$ and define the profile U_0 by (10.3.15). Let u^ϵ be the exact solution with the initial data $U_0(0,0,x/\epsilon) = U_0(t,t/\epsilon,x/\epsilon)|_{\{t=0\}}$. Then for any $T\in]0,T_*]$, u^ϵ smooth on $[0,T]\times\mathbb{R}$ for ϵ small. The data is bounded in the sense that*

$$\|u^\epsilon(0)\|_{L^\infty(\{|x|\le T^*+1\})} \;\le\; C\,\epsilon\,, \qquad \|u^\epsilon(0)\|_{BV(\{|x|\le T^*+1\})} \;\le\; C\,. \tag{10.3.18}$$

The family of solutions explodes in BV in the sense that

$$\lim_{T\to T^*_-}\;\lim_{\epsilon\to 0+}\;\Big|\int_{\{|x|\le T^*+1-T\}} u^\epsilon(T,x)\,\sin\frac{m(x+t)}{\epsilon}\,dx\Big| \;=\; \infty\,. \tag{10.3.19}$$

The solutions are small in L^∞ with data bounded in BV. The BV norm is amplified by as large a constant as one likes in the following sense. For any large $M>0$ and small $\delta>0$ one can chose $T\in[0,T^*[$ and $\epsilon_0>0$ so that for $0<\epsilon<\epsilon_0$, u^ϵ is smooth on $[0,T]\times\mathbb{R}$,

$$\big\|u^\epsilon\big\|_{L^\infty([0,T]\times\mathbb{R})} < \delta\,, \tag{10.3.20}$$

and

$$\big\|u^\epsilon(T)\big\|_{BV\{|x|\le T^*+1-T\}} \;\ge\; M\,\big\|u^\epsilon(0)\big\|_{BV\{|x|\le T^*+1-T\}}\big\|\,. \tag{10.3.21}$$

Proof. Theorem 9.6.2 together with Theorem 9.7.1 imply that

$$\lim_{\epsilon\to 0+}\int_{\{|x|\le T^*+1\}} u_3^\epsilon(T,x)\,\sin\frac{m(x+t)}{\epsilon}\;dx \;=\; \frac{T^*+1}{\pi}\,\zeta_3(T,m)\,.$$

Exercise 9.3.1 shows that each component of ζ must explode as $T\to T^*$, and (10.41) follows. To prove the last assertion of the proposition, choose $T<T^*$ and then ϵ_0 so that

$$|\zeta(T)|>M\,,\qquad \text{and}\qquad \sup_{t\in[0,T]}\ \epsilon_0\,\big|\zeta(t)\big|<\delta\,.$$

Theorems 9.6.2 and 9.7.1 complete the proof. ∎

A weakness of this result demonstrating unbounded amplification of the BV norm of a family of solutions with *sup* norm tending to zero and initial BV norms bounded is that the hypothesis $b=0$ implies that the system is *not* genuinely nonlinear. In [Joly, Métiver, Rauch, Comm. Math. Phys.1994], it is verified that for b sufficiently small, the profile equations have explosive solutions near those just constructed. In this way one has examples of families of solutions of a fixed genuinely nonlinear system which are uniformly small in L^∞, uniformly bounded in BV and for which the BV norm at time $t=1$ is as large a multiple of the BV norm at $t=0$ as one likes. This that for some genuinely nonlinear 3×3 systems, the desirable estimates of the form

$$\big\|u(1)\big\|_{BV}\ \le\ C\,\big\|u(0)\big\|_{BV}$$

are not true for L^∞ small solutions. Such estimates for the scalar case were proved by Conway-Smoller and Kruzskov while Glimm and Lax proved such estimates for 2×2 systems when $d=1$. The above examples show that the Glimm-Lax result cannot be extended to general genuinely nonlinear 3×3 systems. After its discovery using nonlinear geometric optics, alternate constructions of such amplification were found by [Young].

Chapter 11. Dense Oscillations for the Compressible Euler Equations

In this chapter it is proved that the compressible Euler equations have a cascade of resonant nonlinear interactions that can create waves moving in a dense set of directions from three incoming waves.

§11.1. The $2-d$ isentropic Euler equations

This system describes compressible, inviscid, fluid flow with negligible heat flow. The velocity and density are denoted $v = (v(t,x), v(t,x))$ and $\rho(t,x)$. The pressure is assumed to be a function, $p(\rho)$, of the density. The governing equations (away from shocks) are

$$\begin{aligned}
&(\partial_t + v_1\partial_1 + v_2\partial_2)v_1 + (p'(\rho)/\rho)\partial_1\rho = 0\,,\\
&(\partial_t + v_1\partial_1 + v_2\partial_2)v_2 + (p'(\rho)/\rho)\partial_2\rho = 0\,,\\
&(\partial_t + v_1\partial_1 + v_2\partial_2)\rho + \rho(\partial_1 v + \partial_2 v_2) = 0\,.
\end{aligned} \tag{11.1.1}$$

Here

$$x = (x_1, x_2), \qquad \text{and}, \qquad \partial_j := \partial/\partial x_j,$$

Denote by

$$u := (v_1, v_2, \rho), \qquad \text{and}, \qquad f(\rho) := p'(\rho)/\rho\,.$$

The system (11.1.1) is then of the form $L(u,\partial)u = 0$, with coefficient matrices,

$$A_0 = I\,, \qquad A_1(u) = \begin{pmatrix} v_1 & 0 & f(\rho)\\ 0 & v_1 & 0\\ \rho & 0 & v_1\end{pmatrix}, \qquad A_2(u) = \begin{pmatrix} v_2 & 0 & 0\\ 0 & v_2 & f(\rho)\\ 0 & \rho & v_2\end{pmatrix}. \tag{11.1.2}$$

The system is symmetrized by multiplying by

$$D(\rho) \ := \ \operatorname{diag}\big(\rho, \rho, f(\rho)\big) \ := \ \begin{pmatrix} \rho & 0 & 0\\ 0 & \rho & 0\\ 0 & 0 & f(\rho)\end{pmatrix}.$$

At a constant background state, $\underline{u} := (\underline{v}, \underline{\rho})$, the linearized operator has symbol

$$L(\underline{u},\,\tau,\xi) \ = \ \begin{pmatrix} \tau + \underline{v}.\xi & 0 & \underline{f}\,\xi_1\\ 0 & \tau + \underline{v}.\xi & \underline{f}\,\xi_2\\ \underline{\rho}\,\xi_1 & \underline{\rho}\,\xi_2 & \tau + \underline{v}.\xi\end{pmatrix}, \qquad \text{where} \qquad \underline{f} := f(\underline{\rho}).$$

The operator is symmetrized by multiplying by the constant matrix $D(\underline{\rho}) = \operatorname{diag}\big(\underline{\rho}, \underline{\rho}, \underline{f}\big)$.
Compute,

$$\det\, L(\underline{u}\,,\,\tau,\xi) = (\tau + \underline{v}.\xi)\big[(\tau + \underline{v}.\xi)^2 - c^2|\xi|^2\big] \tag{11.1.3}$$

where the sound speed c is defined by

$$c^2 \ := \ p'(\rho)\,, \qquad c > 0\,. \tag{11.1.4}$$

Convention. *By a linear change of time variable $t' = c\,t$ we may assume without loss of generality that $c = 1$.*

The asymptotic relations

$$L(u^\epsilon,\partial)u^\epsilon \ \sim\ 0, \qquad \text{and,} \qquad D(\rho)\, L(u^\epsilon,\partial)u^\epsilon \ \sim\ 0,$$

are equivalent. The latter is symmetric. Solutions are constructed in §9.4 and §9.6. The construction was carried out after the change of variable $\tilde u = D^{1/2}u$ with new coefficients $D^{-1/2}\, D\, A_\mu\, D^{-1/2}$. As the formulae are somewhat simpler, we compute with the equations (11.1.1). The background state $\underline{u}$ is constant.

The method of §9.4 is to expand the symmetric hyperbolic expression

$$(D\,L)\Big(\underline{u} + \epsilon U^\epsilon(y,Y)\,,\ \frac{\partial}{\partial y} + \frac{1}{\epsilon}\,\frac{\partial}{\partial Y}\Big)\ \Big(\underline{u} + \epsilon\, U^\epsilon(y,Y)\Big)$$

in powers of ϵ and determine the profiles in the expansion of U^ϵ so that coefficients of each power of ϵ vanishes. Multiplying by D^{-1} shows that

$$L\Big(\underline{u} + \epsilon U^\epsilon(y,Y)\,,\ \frac{\partial}{\partial y} + \frac{1}{\epsilon}\,\frac{\partial}{\partial Y}\Big)\ \Big(\underline{u} + \epsilon\, U^\epsilon(y,Y)\Big)\ \sim\ \epsilon^\infty\,.$$

The operator on the left hand side is not symmetric. Expanding as in §9.4, the leading two terms yield

$$L(\underline{u},\partial_Y)\,U_0(y,Y)\ =\ 0\,, \tag{11.1.5}$$

and,

$$L(\underline{u},\partial_y)\,U_0 + \sum_\mu A'_\mu(\underline{u})U_0\,\partial_{Y_\mu}U_0 + L(\underline{u},\partial_Y)\,U_1\ =\ 0 \tag{11.1.6}$$

On Fourier series,

$$L(\underline{u},\partial_Y)\,\sum_\alpha a_\alpha(y)\,e^{i\alpha.Y}\ =\ i\,\sum_\alpha L(\underline{u},\alpha)\,a_\alpha(y)\,e^{i\alpha.Y}\,.$$

The result of the next lemma is trivial in the symmetric case.

Lemma 11.1.1. *For $(\tau,\xi)\in\mathbb{R}^{1+d}$, $\mathbb{C}^N$ is the direct sum of the image and kernel of $L(\underline{u},\tau,\xi)$. The the norms of the spectral projections $\pi(\tau,\xi)$ along the image onto the kernel are bounded independent of τ,ξ.*

Proof. Write $L(\tau,\xi) = D^{-1}\big(DL(\tau,\xi)\big)$. Since DL is hermitian and $D = D^* > 0$ it follows that L is hermitian in the scalar product

$$(u,v)_D\ :=\ (Du,v)\,.$$

In fact,

$$(Lu,v)_D := (DLu,v) = (u,DLv) = (Du,Lv) := (u,Lv)_D\,.$$

The image and kernel are orthogonal in this scalar product and therefore complementary.

The spectral projections are orthogonal with respect to the scalar product $(\ ,\ )_D$. They have norm equal to 1 in the corresponding matrix norm.

Since this norm is equivalent to the euclidean matrix norm, the $\pi(\tau,\xi)$ are uniformly bounded. ∎

The profile U is constructed as a periodic function of $Y = (T,X_1,X_2)$. In the language of §9.4, that means choosing

$$\phi_1(T)\ :=\ T, \qquad \phi_2(Y)\ :=\ X_1\,, \qquad \text{and} \qquad \phi_3(Y)\ :=\ X_2\,.$$

With this choice $\mathbf{e} = \mathcal{E}$,

$$\mathbf{E}\Big(\sum a_\alpha(y)\, e^{i\alpha.Y}\Big) \;:=\; \sum \pi(\alpha)\, a_\alpha(y)\, e^{i\alpha.Y}\,.$$

The lemma shows that $\mathbf{E}$ is bounded on all $H^s([0,T]_t \times \mathbb{R}^d_x \times \mathbb{T}^{1+d}_Y)$. It projects onto the kernel of $L(\underline{u}, \partial_Y)$, and

$$\mathbf{E}\, L(\underline{u}, \partial_Y) \;=\; 0\,.$$

Thus (11.1.5) is equivalent to

$$\mathbf{E}\, U_0 \;=\; 0\,. \tag{11.1.7}$$

Multiplying (11.1.6) by $\mathbf{E}$ yields

$$\mathbf{E}\Big(L(\underline{u}, \partial_y) \;+\; \sum_\mu A'_\mu(\underline{u}) U_0 \partial_{Y_\mu}\Big) U_0 \;=\; 0\,. \tag{11.1.8}$$

We use equations (11.1.7)-(11.1.8) for the principal profile. The advantage is that the coefficients of the non symmetric system L are simple.

The equations (1.17)-(1.18) have the exact same form as the equations derived for the symmetric operator DL, only the operator $\mathbf{E}$ has changed. Mulitplying the equations (1.17)-(1.18) by any operator whose restriction to $\ker L(\underline{u}, \partial_Y)$ is equal to the identity, does not affect the equations. So, there are many equivalent versions.

§11.2. Homogeneous oscillations and many wave interaction systems

For homogeneous oscillations, the equations for the leading profile, are equivalent to a system of coupled ordinary differential equations for its Fourier coefficients.

Consider profiles $U_0(t,x,Y)$ that are independent of x and $2\pi \times 2\pi \times 2\pi$ periodic in (Y_0, Y_1, Y_2). The approximate solution has the form

$$u^\epsilon(t,x) \;=\; u^\epsilon(t, x_1, x_2) \;\sim\; \epsilon\, U_0(t, t/\epsilon, x/\epsilon)\,.$$

The profile equations are

$$\mathbf{E}\, U_0 \;=\; U_0\,, \qquad \mathbf{E}\Big(\partial_t U_0 + \sum_{\mu=0}^{2} \big(A'_\mu(\underline{u})\, U_0\big)\, \frac{\partial U_0}{\partial Y_\mu}\Big)\,. \tag{11.2.1}$$

Define

$$B_\mu(V) \;:=\; \big(A'_\mu(\underline{u})\big)(V) \tag{11.2.2}$$

so that B_μ is a linear matrix valued function of the vector V. Since the leading coefficient of L is equal to the identity, $B_0 = 0$.

For convenience in denoting Fourier coefficients, suppress the subscript 0 and expand the leading profile,

$$U(t,Y) = U(t, Y_0, Y_1, Y_2) = \sum_{\alpha \in \mathbb{Z} \times \mathbb{Z}^2} U_\alpha(t)\, e^{i\alpha.Y}\,. \tag{11.2.3}$$

Inserting this in (11.2.1), the nonlinear term is equal to

$$\sum_\mu \Big(\sum_\alpha B_\mu(U_\alpha)\; e^{i\alpha.Y}\Big)\Big(\sum_\beta i\beta_\mu\; U_\beta\; e^{i\beta.Y}\Big)\,.$$

Setting the coefficient of $e^{i\gamma.Y}$ equal to zero yields

$$\frac{dU_\gamma}{dt} \ + \ \pi(\gamma) \sum_{\alpha+\beta=\gamma} \Big[\sum_{\mu=0}^{2} B_\mu(U_\alpha)(i\beta_\mu)U_\beta\Big] \ = \ 0\,. \tag{11.2.4}$$

The factor $\pi(\gamma)$ is from the operator $\mathbf{E}$.
When $\alpha+\beta=\gamma$, define

$$U_\alpha, U_\beta \quad \longmapsto \quad -\pi(\gamma)\Big[\sum_{\mu=0}^{2} B_\mu(U_\alpha)\,\beta_\mu\,U_\beta\Big]\,, \tag{11.2.5}$$

defines a bilinear map

$$\ker(\pi(\alpha)) \times \ker(\pi(\beta)) \quad \longmapsto \quad \ker(\pi(\gamma))\,. \tag{11.2.6}$$

In the important special case when the kernels are one dimensional, choose bases r_α homogeneous of degree zero in α. Define scalar valued $a_\alpha(t)$ and $\Gamma(\alpha\beta)$ by

$$U_\alpha(t) \ = \ a_\alpha(t)\,r_\alpha\,, \qquad \Gamma(\alpha,\beta)\,r_\gamma \ := \ -\pi(\gamma)\Big[\sum_{\mu=0}^{2} B_\mu(r_\alpha)\,\beta_\mu\,r_\beta\Big]\,. \tag{11.2.7}$$

The coefficient $\Gamma(\alpha,\beta)$ measures one of the two quadratic terms by which the α and β components can influence the $\gamma=\alpha+\beta$ component. The second such term is $\Gamma(\beta,\alpha)$.
In (11.2.4) the contribution of these two terms yields,

$$\frac{da_\gamma(t)}{dt} \ = \ \frac{i}{2} \sum_{\alpha+\beta=\gamma} \Big(\Gamma(\alpha,\beta)+\Gamma(\beta,\alpha)\Big)\,a_\alpha(t)\,a_\beta(t)\,. \tag{11.2.8}$$

where the *interaction coefficient*, $\Gamma(\alpha,\beta)+\Gamma(\beta,\alpha)$, is scalar valued, homogeneous of degree 1 in α,β,γ, and symmetric in α,β. Equation (11.2.8) is a three wave equation with possibly infinitely many oscillations with quadratic interaction.
Theorem 9.5.2 implies that

$$\forall\, y \in [0,T_*[\times\mathbb{R}^d \qquad \mathrm{spec}\,(U_0(y,\,\cdot\,)) \ \subset \ \mathrm{Char}\,L(\underline{u},\partial) \ \cap \ \mathbb{Z}-\mathrm{Span}\,\big(\mathrm{spec}(U_0(0,.,.)\big)\,. \tag{11.2.9}$$

The main result of this chapter is the following. It asserts that three initial waves at $t=0$ lead by nonlinear interaction to waves moving in a dense set of directions. It is an improvement of two articles of [Joly, Metivier, Rauch, 1994, 1998] devoted to the subject.

Theorem 11.2.1. *There is a real smooth local in time solution $U_0(t,Y)$ of the leading profile equation which is $2\pi\times2\pi\times2\pi$ periodic in (Y_0,Y_1,Y_2) and satisfies*

$$U_\alpha(0)=0 \ \text{ except for } \ \alpha\in\mathbb{Z}(1,1,0)\cup\mathbb{Z}(0,1,0)\cup\mathbb{Z}(0,3,4)\,, \tag{11.2.10}$$

and

$$\left\{\frac{(\alpha_1,\alpha_2)}{\|(\alpha_2,\alpha_3)\|} \ : \ \exists(\alpha_0,\alpha_1,\alpha_2)\in\mathbb{Z}^3, \quad \alpha_0>0, \qquad \text{s.t.} \quad \frac{d^2U_\alpha(0)}{dt^2}\neq 0\right\} \tag{11.2.11}$$

is dense in the unit circle. It contains all points on the unit circle with rational slope r so that $1 + r^2$ is the square of a rational with at most three exceptions.

The first assertion shows that at $\{t = 0\}$ there are only three oscillating wavetrains and the second asserts that in the future there are waves traveling in directions dense in the unit circle. In the second article† it is shown that the $U_\alpha(t)$ are real analytic in time. The U_α with $d^2U_\alpha/dt^2 \neq 0$, therefore vanish at most at a discrete set of points. Thus the dense set of wavetrains are simultaneously illuminated with at most a countable set of exceptional times.

§11.3. Linear oscillations for the Euler equations

Consider background states with $\underline{v} = 0$. Thanks to Gallilean invariance, this not an essential restriction. Equation (11.1.3) simplifies to $\tau(\tau - |\xi|^2)$, and

$$\mathrm{Char}\,(L(0, \partial) \;=\; \{\tau = 0\} \cup \{\tau^2 = |\xi|^2\}\,,$$

is the union of a horizontal plane and a light cone as for Maxwell's equations. In contrast to the case of electrodynamics, all the sheets of the characteristic variety are physical whereas for Maxwell, the divergence free constraints eliminate the plane.

Convention. *For the rest of the chapter the background state is $\underline{v} = 0$ and is suppressed when indicating the symbol $L(0, \partial)$ as $L(\partial)$.*

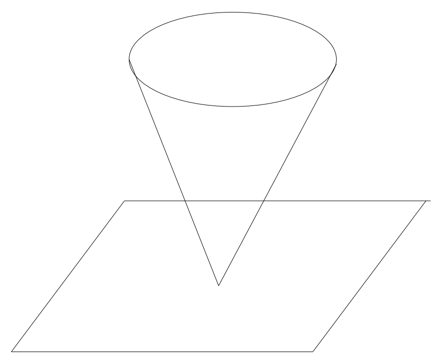


Figure 11.1. The characteristic variety of linearized Euler

Proposition 11.3.1. *The small divisor hypothesis of §9.6 is satisfied.*

Proof. Compute

$$\det\Big(L(\tau, \xi) - \sigma I\Big) \;=\; (\tau - \sigma)\,\big((\tau - \sigma)^2 - |\xi|^2\big)\,.$$

For,

$$(\tau, \xi) \;=\; (n_0, n_1, n_2) \quad \text{with} \quad n \in \mathbb{Z}^3,$$

one has

$$\det\Big(L(\tau, \xi) - \sigma I\Big) \;=\; (n_0 - \sigma)\,\big((n_0 - \sigma)^2 - n_1^2 - n_2^2\big)\,.$$

If σ is a nonzero eigenvalue, then

$$\sigma = n_0 \in \mathbb{Z} \setminus 0, \qquad \text{or,} \qquad (n_0 - \sigma)^2 = n_1^2 + n_2^2 \quad \text{with} \quad n_0 - \sigma \neq 0\,.$$

† The book containing the first article appeared six years after the talk that it summarizes. The second article has publication date 1994 which looks like it is four years earlier than the first!

In the first case one has the lower bound, $|\sigma| \geq 1$.

In the second case,
$$2n_0\sigma - \sigma^2 \;=\; n_0^2 - n_1^2 - n_2^2\,.$$
If the left hand side vanishes then $\sigma = 2n_0$ is a nonzero integer hence of modulus ≥ 1. When it is nonzero one has
$$|2n_0\sigma - \sigma^2| \;\geq\; 1.$$
If
$$|\sigma| \;<\; \min\left\{\frac{1}{\sqrt{2}}\,,\,\frac{1}{4|n_0|}\right\}$$
then each summand on the left is smaller than $1/2$ so the sum cannot be larger than 1. Therefore the eigenvalue satisfies
$$|\sigma| \;\geq\; \min\left\{\frac{1}{\sqrt{2}}\,,\,\frac{1}{4|n_0|}\right\},$$
verifying the small divisor hypothesis. ∎

Theorems 9.5.1, and 9.6.2 then yield profiles and Theorem 9.6.3 affirms that he residual is infinitely small. Theorem 9.7.1 shows that these solutions are infinitely close to the exact solutions with the same initial data. We analyse the resonance relations and the profile equations in detail in order to prove Theorem 11.2.1.

It is sometimes confusing that (t,x), (τ,ξ), (v,ρ), and dual vectors to the (v,ρ) space, are all objects with three components. To maintain some distinction we use round brackets (t,x), (τ,ξ), for the first two and Dirac brackets $|v,\rho\rangle$ for the third. Dual vectors to the $|v,\rho\rangle$ are denoted, with reversed brackets $\langle .\,,.\,|$. The pairings of $\mathbb{R}^2_x$ and $\mathbb{R}^2_\xi$ and of $\mathbb{R}^3_{t,x}$ and $\mathbb{R}^3_{\tau,\xi}$ are indicated with a period, e.g. $(t,x).(\tau,\xi)$. Dirac's notation $\langle\ |\ \rangle$ is used for the third pairing.

Begin by computing $\ker L(\tau,\xi)$, $\operatorname{range} L(\tau,\xi)$ and the spectral projection $\pi(\tau,\xi)$ for each $(\tau,\xi) \in \operatorname{Char}(L)$. Since $L(\tau,\xi)$ is homogeneous of degree one with $\det\, L(1,0,0) \neq 0$, it suffices to consider $|\xi| = 1$. For ξ fixed there are three points in the characteristic variety, $\tau = 0$ and $\tau = \pm|\xi|$.

For $\tau = 0$ one has
$$L(0,\xi) = \begin{pmatrix} 0 & 0 & \underline{f}\xi_1 \\ 0 & 0 & \underline{f}\xi_2 \\ \underline{\rho}\xi_1 & \underline{\rho}\xi_2 & 0 \end{pmatrix}$$
$$\ker\left(L(0,\xi)\right) = \mathbb{R}\,|-\xi_2, -\xi_1, 0\rangle \tag{11.3.1}$$
$$\operatorname{range}\left(L(0,\xi)\right) = \operatorname{Span}\big\{|\xi_1,\xi_2,0\rangle, |0,0,1\rangle\big\}. \tag{11.3.2}$$
Note that the range is orthogonal to the kernel so the spectral projection is the orthogonal projection,
$$\pi(0,\xi) \;:=\; \Big|\xi_2, -\xi_1, 0\Big\rangle\Big\langle \xi_2, -\xi_1, 0\Big|. \tag{11.3.3}$$
Similarly
$$L(\pm 1,\xi) = \begin{pmatrix} \pm 1 & 0 & \underline{f}\xi_1 \\ 0 & \pm 1 & \underline{f}\xi_2 \\ \underline{\rho}\xi_1 & \underline{\rho}\xi_2 & \pm 1 \end{pmatrix}$$
$$\ker\big(L(\pm 1,\xi)\big) \;=\; \mathbb{R}\,|\xi_1,\xi_2,\mp 1/\underline{f}\rangle$$
$$\operatorname{range}\big(L(\pm 1,\xi)\big) = \operatorname{Span}\big\{|\pm 1, 0, \underline{\rho}\xi_1\rangle, |0,\pm 1,\underline{\rho}\xi_2\rangle\big\}$$
$$\pi(\pm 1,\xi) \;=\; \frac{1}{2}\Big|\xi_1,\xi_2,\mp 1/\underline{f}\Big\rangle\Big\langle \xi_1,\xi_2,\mp 1/\underline{\rho}\Big|\,.$$

These computations yield the plane wave solutions of the linearized equation;

$$|\xi_2, -\xi_1, 0\rangle \, F(\xi.x), \qquad |\xi_1, \xi_2, \mp 1/\underline{f}\rangle \, F(\pm|\xi|t + \xi.x)$$

where F is an arbitrary real valued function of one variable.

The first family of waves satisfy $\operatorname{div} v = 0$, $|\operatorname{curl} v| \sim |\xi|^2$, and has no variation in density. They are standing waves. Given the background state with velocity zero this means that they are convected with the background fluid velocity. They are called *vorticity waves.*

Waves of the second family have $\operatorname{curl} v = 0$ and $|\operatorname{div} v| \sim |\xi|^2$. The group velocity associated to $\tau = \pm|\xi|$ is equal to $-\nabla_\xi(\pm|\xi|) = \mp\xi/|\xi|$. These solutions are called are called *acoustic waves* or *compression waves.* They can move in any direction with speed one.

This prediction of the speed of sound, $c = p'(\rho)^{1/2}$, from the static measurement of $p(\rho)$ is an early success of continuum mechanics. It is also a model of what is found in science text analyses of a nonlinear hyperbolic model. That is, linearization at constant states, and computation of plane waves solutions and group velocities for the resulting constant coefficient equation governing small perturbations.

The solution of the linear oscillatory initial value problem

$$L(\partial_t, \partial_x) w = 0\,, \qquad w(0,x) = g\, e^{i\zeta.x}\,, \qquad g \in \mathbb{C}^3\,, \quad \zeta \in \mathbb{R}^2\,.$$

is equal to,

$$w \;=\; \big(\pi(0, \zeta/|\zeta|)g\big)\, e^{i\zeta.x} \;+\; \sum_{\pm} \big(\pi(\pm c, \zeta/|\zeta|)g\big) e^{\pm|\zeta|t + \zeta.x}\,.$$

§11.4. Resonance Relations.

Quadratic nonlinear interaction of oscillations $r_\alpha e^{i\alpha.(t,x)}$ and $r_\beta e^{i\beta.(t,x)}$ with α and β belonging to $\operatorname{Char}(L)$ produce terms in $e^{i(\alpha+\beta).(t,x)}$ which will propagate as soon as $\alpha+\beta$ is characteristic.

Definition. A *(quadratic) resonance* is a linear relation $\alpha + \beta + \eta = 0$ between three nonzero elements of $\operatorname{Char}(L)$.

These are sometimes called resonances of order 3, as they involve three points of the characteristic variety. The corresponding interactions are called *three wave interactions.* The simplest are colinear resonances, always present for homogeneous systems L, when α, β, and η are multiples of a fixed vector.

For semilinear problems one must consider linear relations among any number of characteristic covectors. Treating small amplitude oscillations for quasilinear problems, yields only quadratic nonlinearities in the equations for the profiles and thereby permits us to consider only triples.

Theorem 11.4.1. *Quadratic resonances for the Euler equations fall into three families:*

i. Colinear vectors satisfying $\tau^2 = |\xi|^2$.

ii. Triples α, β, η which belong to $\{\tau = 0\}$.

iii. Relations equivalent by $\mathbb{R}$-dilation, x-rotation, x-reflection and permutation of the three covectors to,

$$(\pm 1, \alpha_1, \alpha_2) + (0, 0, -2\alpha_2) + (\mp 1, -\alpha_1, \alpha_2) \;=\; 0, \qquad \alpha_1^2 + \alpha_2^2 = 1, \quad \alpha_1 \geq 0\,.$$

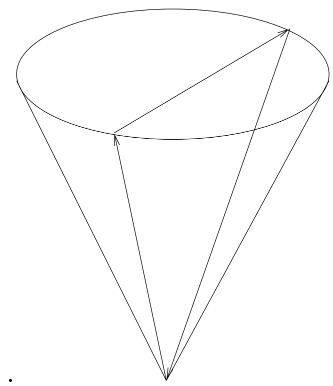


Figure 11.2 Resonance of type iii.

Proof. Seek α, β, η in $\{\tau(\tau^2 - |\xi|^2) = 0\}$ whose sum is zero. The classification above depends on counting how many of the α, β, η belong to $\{\tau = 0\}$.

If all three belong, it is case ii.

If exactly two belong, the relation $\alpha + \beta + \eta = 0$ is impossible since the τ component of the sum will equal the τ component of the covector which does not lie in $\tau = 0$.

If exactly one belongs, rotate axes so that the covector in $\tau = 0$ is parallel to $(0, 0, 1)$. The relation then is a multiple of case iii. except possibly for the sign of α_1 which can be adjusted by a reflection in x_1.

If none of the covectors belong to $\tau = 0$ we must show that the only possibility is colinear resonance.

A rotation followed by multiplication by a nonzero real reduces to the case $\alpha = (1, 1, 0)$.

Seek $\beta \in \mathrm{Char}\,(L)$ such that $\alpha + \beta \in \mathrm{Char}\,(L)$. If the τ component of β is positive then $\alpha + \beta$ belongs to the interior of the positive light cone unless α and β are colinear.

Thus it suffices to look for $\beta = (-|\xi|, \xi)$ with

$$(1, 1, 0) + (-|\xi|, \xi) \in \left\{\tau^2 = |\xi|^2\right\}.$$

This holds if and only if

$$(1 - |\xi|)^2 = [(1 + \xi_1)^2 + \xi_2^2].$$

Canceling common terms shows that this holds if and only if $-2|\xi| = 2\xi_1$, so we must have $\xi_2 = 0$ and $\xi_1 < 0$. Thus $(1, 1, 0)$ and $(-|\xi|, \xi)$ are colinear. ∎

§11.5. Interaction coefficients for Euler's equations

To define interaction coefficients, basis vectors $r_{(\tau,\xi)}$ for $\ker L(\tau, , \xi)$ with $(\tau, \xi) \in \mathrm{Char}\, L$ are needed. Choose vectors which are the extensions of formulas (11.3.1), (11.3.2) homogeneous of degree zero,

$$r_{(0,\xi)} \ := \ |\xi|^{-1} \left|\xi_2, -\xi_1, 0\right\rangle \tag{11.5.1}$$

$$r_{(\pm c|\xi|,\xi)} \ := \ |\xi|^{-1} \left|\xi_1, \xi_2, \mp|\xi|\underline{\rho}\right\rangle. \tag{11.5.2}$$

Theorem 11.5.1. *Suppose that α, β, and $\alpha + \beta := \gamma$ are nonzero elements of* $\mathrm{Char}\, L$.

i. *If α and β are colinear elements of $\{\tau^2 = |\xi|^2\}$ with $\beta = a\alpha$, $a \neq 0, -1$, then the interaction coefficient $\Gamma(\alpha, \beta) + \Gamma(\beta, \alpha)$ is given by*

$$\Gamma(\alpha, \beta) + \Gamma(\beta, \alpha) \ = \ \mathrm{sgn}(a)\, (\gamma_1^2 + \gamma_2^2)^{1/2}\, (3 + h\underline{\rho}^2)/2\,. \tag{11.5.3}$$

where h is defined in (11.5.6).

ii. *If α and β belong to $\{\tau = 0\}$, the interaction coefficient is given by*

$$\Gamma(\alpha,\beta) + \Gamma(\beta,\alpha) \;=\; -(1/2)|\beta-\alpha| \,\sin\big(\angle(\alpha,\beta)\big)\,\cos\big(\angle(\alpha+\beta,\alpha-\beta)\big) \tag{11.5.4}$$

where $\angle(\alpha,\beta) \in \mathbb{R}/2\pi\mathbb{Z}$ denotes the angle between α and β measured in the positive sense.

iii. *If $\alpha = (\pm 1, \alpha_1, \alpha_2)$, $\alpha_1 > 0$, and $\beta = (0, 0, -2\alpha_2)$ as in Theorem 11.2.***iii***, then the interaction coefficient is given by*

$$\Gamma(\alpha,\beta) + \Gamma(\beta,\alpha) \;=\; \cos(\phi/2)\,\cos(\phi)\,\mathrm{sgn}(-\alpha_2)\,(\gamma_1^2+\gamma_2^2)^{1/2}, \quad \phi := \angle\big((\gamma_1,\gamma_2),(\alpha_1,\alpha_2)\big). \tag{11.5.5}$$

iiibis. *If $\alpha = (\pm 1, \alpha_1, \alpha_2)$, $\beta = (\mp 1, -\alpha_1, \alpha_2)$, $\gamma = (0,0,2\alpha_2)$ then the interaction coefficient $\Gamma(\alpha,\beta)+\Gamma(\beta,\alpha)$ vanishes.*

The exceptional case, **iiibis**, asserts that for the creation of a vorticity wave by the interaction of two acoustic waves, the interaction coefficient vanishes.

Proof. First compute the matrices $B_j = D_u A_j(\underline{u})$. Define the constant h by

$$h := \frac{d}{d\rho}\left(\frac{p'(\rho)}{\rho}\right)\bigg|_{\rho=\underline{\rho}}. \tag{11.5.6}$$

From (11.1.4) one finds

$$B_1\big(|\delta v_1, \delta v_2, \delta\rho\rangle\big) = \begin{pmatrix} \delta v_1 & 0 & h\delta\rho \\ 0 & \delta v_1 & 0 \\ \delta\rho & 0 & \delta v_1 \end{pmatrix} \tag{11.5.7}$$

$$B_2\big(|\delta v_1, \delta v_2, \delta\rho\rangle\big) = \begin{pmatrix} \delta v_2 & 0 & 0 \\ 0 & \delta v_2 & h\delta\rho \\ 0 & \delta\rho & \delta v_2 \end{pmatrix} \tag{11.5.8}$$

Case ii. When $\alpha = (0,\alpha_1,\alpha_2)$ and $\beta = (0,\beta_1,\beta_2)$ belong to $\{\tau=0\}$, (11.5.1) shows that $\delta\rho = 0$ for r_α and r_β so

$$B_1(r_\alpha) = |\alpha|^{-1}\,\alpha_2\, I, \qquad B_2(r_\alpha) = -|\alpha|^{-1}\alpha_1 I\,,$$

and (11.3.3) yields $\pi(\gamma) = |r_\gamma\rangle\langle r_\gamma|$. Thus

$$\begin{aligned}\Gamma(\alpha,\beta) \;&= \langle r_\gamma|[\beta_1 B_1(r_\alpha) + \beta_2 B_2(r_\alpha)] r_\beta\rangle = |\alpha|^{-1}(\beta_1\alpha_2 - \beta_2\alpha_1)\langle r_{\alpha+\beta}|r_\beta\rangle \\ &= |\alpha|^{-1}(\beta_1\alpha_2 - \beta_2\alpha_1)|\alpha+\beta|^{-1}|\beta|^{-1} < \alpha_2+\beta_2, -\alpha_1-\beta_1, 0 \,\big|\, \beta_2, -\beta_1, 0\rangle\,.\end{aligned}$$

The last duality is equal to the scalar product $\langle \alpha+\beta|\beta\rangle$, so

$$\Gamma(\alpha,\beta) \;=\; |\alpha|^{-1}(\beta_1\alpha_2 - \beta_2\alpha_1)|\alpha+\beta|^{-1}|\beta|^{-1}\,\langle\alpha+\beta|\beta\rangle\,.$$

Interchanging the role of α and β and summing yields

$$\Gamma(\alpha,\beta) + \Gamma(\beta,\alpha) \;=\; |\alpha|^{-1}(\beta_1\alpha_2 - \beta_2\alpha_1)|\alpha+\beta|^{-1}|\beta|^{-1}\,\langle\alpha+\beta|\beta-\alpha\rangle\,. \tag{11.5.9}$$

Formula (11.5.4) follows since

$$
\begin{aligned}
|\alpha|^{-1}(\beta_1\alpha_2 - \beta_2\alpha_1)|\beta|^{-1} &= -\sin\big(\angle(\alpha,\beta)\big)\\
|\alpha+\beta|^{-1}|\beta-\alpha|^{-1}\,\langle \alpha+\beta|\beta-\alpha\rangle &= \cos\big(\angle(\alpha+\beta,\beta-\alpha)\big)\,.
\end{aligned}
$$

Case i. By homogeneity and Euclidean invariance it suffices to compute the case

$$\alpha = (\pm 1, 1, 0), \qquad \beta = a\alpha, \qquad a \in \mathbb{R}\setminus 0\,.$$

Then r_α, r_β, and $\pi(\gamma)$ are given by

$$r_\alpha = |1, 0, \mp\underline{\rho}\rangle\,, \qquad r_\beta = \mathrm{sgn}(a) r_\alpha\,, \qquad \pi(\gamma) = \frac{1}{2}\, r_\alpha \langle 1, 0, \mp 1/\underline{\rho}|\,.$$

Since $\beta_2 = 0$, one has

$$
\begin{aligned}
\Gamma(\alpha,\beta) r_\gamma = {}& \pi(\gamma)\Big[\beta_1 B_1(r_\alpha) r_\beta\Big] = a\,\mathrm{sgn}\,(a)\,\pi(\gamma)\begin{pmatrix} 1 & 0 & \mp h\underline{\rho}\\ 0 & 1 & 0\\ \mp\underline{\rho} & 0 & 1\end{pmatrix}\\
= {}& |a|\,\frac{r_\alpha}{2}\Big\langle 1\,,\,0\,,\,\mp 1/\underline{\rho}\,\Big|\, 1 + h\underline{\rho}^2\,,\,0\,,\,\mp 2\underline{\rho}\Big\rangle \;=\; |a|\,\big(3 + h\underline{\rho}^2\big)\,\frac{r_\alpha}{2}\,.
\end{aligned}
$$

Now $r_\alpha = \pm r_\gamma$ the sign depending on whether $\gamma = (1+a)\alpha$ is a positive or negative multiple of α, that is by $\mathrm{sgn}\,(1+a)$. Thus

$$\Gamma(\alpha,\beta) \;=\; \mathrm{sgn}(1+a)\,|a|\,(3 + h\underline{\rho}^2)/2.$$

By homogeniety the case of general α is given by

$$
\begin{aligned}
\Gamma(\alpha,\beta) &= (\alpha_1^2+\alpha_2^2)^{1/2}\,\mathrm{sgn}(1+a)\,|a|\,(3+h\underline{\rho}^2)/2\\
&= (\beta_1^2+\beta_2^2)^{1/2}\,\mathrm{sgn}(1+a)\,(3+h\underline{\rho}^2)/2\,,
\end{aligned}
$$

since $|a|\|\alpha_1,\alpha_2\| = \|\beta_1,\beta_2\|$. Noting that $\alpha = a^{-1}\beta$, the reversed coefficient is given by

$$\Gamma(\beta,\alpha) \;=\; (\alpha_1^2+\alpha_2^2)^{1/2}\,\mathrm{sgn}(1+a^{-1})\,(3+h\underline{\rho}^2)/2\,.$$

Adding yields

$$\Gamma(\alpha,\beta) + \Gamma(\beta,\alpha) \;=\; \Big[(\beta_1^2+\beta_2^2)^{1/2}\,\mathrm{sgn}(1+a) + (\alpha_1^2+\alpha_2^2)^{1/2}\,\mathrm{sgn}(1+a^{-1})\Big](3+h\underline{\rho}^2)/2\,.$$

In the three cases $a>0$, $-1<a<0$, and $a<-1$, the factor in square bracket is given by

$$\|\beta\| + \|\alpha\|\,, \qquad \|\beta\| - \|\alpha\|\,, \qquad \text{and,} \qquad -\|\beta\| + \|\alpha\|\,,$$

respectively. In all cases this is equal to $\mathrm{sgn}\,(a)\|\gamma\|$ which proves (11.5.3).

Case iii. It is sufficient to consider α with $\alpha_1^2+\alpha_2^2 = 1$ and $a>0$. Then

$$
\begin{aligned}
&\alpha = (\pm 1, \alpha_1, \alpha_2)\,, \quad \beta = (0, 0, -2\alpha_2)\,, \quad \gamma = (\pm 1, \alpha_1, -\alpha_2)\,, \quad r_\alpha = |\alpha_1, \alpha_2, \mp\underline{\rho}\rangle,\\
&r_\beta = |\mathrm{sgn}\,(-\alpha_2), 0, 0\rangle\,, \quad r_\gamma = |\alpha_1, -\alpha_2, \mp\underline{\rho}\rangle\,, \quad \pi(\gamma) = \frac{r_\gamma}{2}\,\langle \alpha_1, -\alpha_2, \pm 1/\underline{\rho}|\,.
\end{aligned}
$$

Since $\beta_1 = 0$,

$$\begin{aligned}\Gamma(\alpha,\beta) r_\gamma &= \pi(\gamma)\Big[\beta_2 B_2(r_\alpha) r_\beta\Big] = \pi(\gamma)\left[-2\alpha_2 \begin{pmatrix}\alpha_2 & 0 & 0\\ 0 & \alpha_2 & \mp h\underline{\rho}\\ 0 & \mp\underline{\rho} & \alpha_2\end{pmatrix}\begin{pmatrix}\operatorname{sgn}(-\alpha_2)\\ 0\\ 0\end{pmatrix}\right]\\ &= \frac{r_\gamma}{2}\Big\langle \alpha_1\,,\,-\alpha_2\,,\,\mp 1/\underline{\rho}\;\Big|\; -2\alpha_2^2\operatorname{sgn}(-\alpha_2)\,,\,0\,,\,0\Big\rangle\,.\end{aligned}$$

Therefore, $\Gamma(\alpha\beta) = -\alpha_1\alpha_2^2\operatorname{sgn}(-a)$.

To compute the coefficient $\Gamma(\beta,\alpha)$ note first that $B_2(r_\beta) = 0$ and $B_1(r_\beta) = \operatorname{sgn}(-\alpha_2) I$, since for r_β , $\delta v_2 = \delta\rho = 0$ and $\delta v_1 = \operatorname{sgn}(-\alpha_2)$. Therefore

$$\begin{aligned}\Gamma(\beta,\alpha) r_\gamma &= \pi(\gamma)\Big[\alpha_1 B_1(r_\beta) r_\alpha\Big] = \pi(\gamma)\Big[\alpha_1 \operatorname{sgn}(-\alpha_2) r_\alpha\Big]\\ &= \frac{\alpha_1}{2}\operatorname{sgn}(-\alpha_2)\, r_\gamma\,\Big\langle \alpha_1\,,\,-\alpha_2\,,\,\mp 1/\underline{\rho}\;\Big|\;\alpha_1\,,\,\alpha_2\,,\,\mp\underline{\rho}\Big\rangle.\end{aligned}$$

Therefore,

$$\Gamma(\beta,\alpha) = \frac{\alpha_1}{2}\operatorname{sgn}(-\alpha_2)\,(\alpha_1^2 - \alpha_2^2 + 1) = \alpha_1\operatorname{sgn}(-\alpha_2)\,\alpha_1^2\,.$$

Adding the previous results yields

$$\Gamma(\alpha,\beta) + \Gamma(\beta,\alpha) = \alpha_1\,(\alpha_1^2 - \alpha_2^2)\operatorname{sgn}(-\alpha_2) = \cos(\phi/2)\big(\cos^2(\phi/2) - \sin^2(\phi/2)\big)\operatorname{sgn}(-\alpha_2),$$

since $(\alpha_1,\alpha_2) = (\cos(\phi/2), \sin(\phi/2))$. Formula (11.5.5) for the case $\alpha_1^2 + \alpha_2^2 = 1$ follows. The general case follows by homogeneity.

Case iiibis. When $|\alpha_1,\alpha_2| = 1$, one has

$$r_\alpha = |\alpha_1,\alpha_2,\mp\underline{\rho}\rangle\,, \qquad r_\beta = |-\alpha_1,\alpha_2,\mp\underline{\rho}\rangle\,,$$

$$r_\gamma = \operatorname{sgn}(\alpha_2)\,|1,0,0\rangle\,, \qquad \pi(\gamma) = |1,0,0\rangle\langle 1,0,0|\,.$$

By definition, $\operatorname{sgn}(\alpha_2)\,\Gamma(\alpha,\beta)$ is equal to the first component of the vector

$$\left[-\alpha_1\begin{pmatrix}\alpha_1 & 0 & \mp h\underline{\rho}\\ 0 & \alpha_1 & 0\\ \mp\underline{\rho} & 0 & \alpha_1\end{pmatrix} + \alpha_2\begin{pmatrix}\alpha_2 & 0 & 0\\ 0 & \alpha_2 & \mp h\,\underline{\rho}\\ 0 & \mp\underline{\rho} & \alpha_2\end{pmatrix}\right]\begin{pmatrix}-\alpha_1\\ \alpha_2\\ \mp\underline{\rho}\end{pmatrix}.$$

Therefore

$$\Gamma(\alpha,\beta) = -\alpha_1\operatorname{sgn}(\alpha_2)\big(-\alpha_1^2 + \alpha_2^2 + h\underline{\rho}\big)\,.$$

To compute the coefficient $\Gamma(\beta,\alpha)$ it suffices to change the sign of α_1 in the above computation which simply changes the sign of the result. Thus $\Gamma(\beta,\alpha) = -\Gamma(\alpha,\beta)$ which proves **iiibis**. ∎

§11.6. Dense oscillations for the Euler equations

§11.6.1. The algebraic/geometric part.

The characteristic variety is given by the equation

$$\tau\big(\tau^2 - (\xi_1^2 + \xi_2^2)\big) = 0.$$

Definition. *In the* (τ, ξ) *space denote by* Λ *the lattice of integer linear combinations of the characteristic covectors*

$$(1,1,0)\,, \quad (0,1,0)\,, \quad \text{and} \quad (0,3,4)\,. \tag{11.6.1}$$

The next result improves substantially the corresonding result from the original paper. The proof is entirely different.

Theorem 11.6.1. *For every rational* r *there is a point*

$$(\tau, \xi) \;\in\; \Lambda \cap \{\tau = |\xi|\}\,,$$

with $\xi_1/\xi_2 = r$.

Remark. The group velocity associated to (τ, ξ) is equal to $-\xi$ so this asserts that there are wave numbers with group velocities with arbitrary rational slope.

Proof. The points of Λ are the integer linear combinations,

$$(\tau, \xi) \;=\; n_1(1,1,0) + n_2(0,1,0) + n_3(0,3,4) \;=\; \Big(n_1\,,\, n_1 + n_2 + 3n_3\,,\, 4n_3\Big)\,. \tag{11.6.2}$$

This (τ, ξ) belongs to $\{\tau = |\xi|\}$ if and only if

$$n_1^2 \;=\; (n_1 + n_2 + 3n_3)^2 + (4n_3)^2, \qquad \text{and}, \qquad n_1 > 0\,.$$

Dividing by n_1^2 and setting

$$q_2 \;:=\; \frac{n_2}{n_1}, \qquad q_3 \;:=\; \frac{n_3}{n_1}\,, \tag{11.6.3}$$

shows that $q_j \in \mathbb{Q}$ satsify,

$$1 \;=\; (1 + q_2 + 3q_3)^2 + (4q_3)^2\,. \tag{11.6.4}$$

For any rational r, the line $q_2 = rq_3$ intersects the ellipse (11.6.4) when

$$1 = (1 + (r+3)q_3)^2 + (4q_3)^2, \quad \text{equivalently}, \quad q_3\Big(\big(4^2 + (r+3)^3\big)q_3 + 2(r+3)\Big) \;=\; 0\,.$$

Therefore, $q_3 = -2(r+3)/(4^2 + (r+3)^2)$ is a solution.
Mulitplying by the greatest common multiple of the denominators of the q_j gives an integer solution n. Multiplying by ± 1 gives the desired solution with $n_1 > 0$. ∎

§11.6.2. Construction of the profiles

We construct a solution of the profile equation (11.2.1) satisfying the conditions of Theorem 11.2.1. Introduce $g \in C^\infty(S^1)$ all of whose Fourier coefficients are nonzero,

$$g(\theta) := \sum_{n \in \mathbb{Z}} g_n\, e^{in\theta}\,, \qquad g_n \neq 0\,. \tag{11.6.5}$$

with g_n rapidly decreasing..

Let $U^0(T, X_1, X_2)$ be the solution of the linearized Euler equation consisting of three plane waves and given by

$$\begin{aligned} U^0(T,X) \;&:= |1,0,-1/\underline{f}\rangle\, g\big((1,1,0).(T,X_1,X_2)\big) \\ &\quad + |0,1,0\rangle\, g\big((0,1,0).(T,X_1,X_2)\big) + |4,-3,0\rangle\, g\big((0,3,4).(T,X_1,X_2)\big)\,. \end{aligned}$$

The spectrum of $U^0(T, X)$ is exactly equal to the integer multiples of the covectors in (11.6.1). Theorem 9.5.1 implies that there is a unique local solution $U = \sum U_\alpha(t) e^{i\alpha.(T,X)}$ of the profile equations (11.2.1) (equivalently (11.2.8)) with initial value $U(0, T, X) = U^0(T, X)$. Theorem 9.5.2 shows that $\operatorname{spec} U(t) \subset \Lambda \cap \operatorname{Char} L(\partial)$. As in §11.1, define $a_\alpha(t)$ by $U_\alpha(t) = a_\alpha(t) r_\alpha$.

The next result is stronger than that proved in [Joly, Métivier, Rauch, 1998]. The new proof also smooths a rough part of the earlier demonstration. The proof of Theorem 11.2.1 from Theorem 11.6.2 is presented afterward.

Theorem 11.6.2. *If*

$$(\tau, \xi) \;=\; n_1(1,1,0) + n_2(0,1,0) + n_3(0,3,4) \in \Lambda \cap \{\tau = |\xi|\}$$

with

$$n_1 n_2 n_3 \neq 0, \qquad n_2 + 5n_3 \neq 0, \qquad n_2 - 5n_3 \neq 0, \qquad \text{and} \qquad n_2 + 3n_3 \neq 0\,,$$

then,

$$\frac{d^2 U_{(\tau,\xi)}(0)}{dt^2} \;\neq\; 0\,, \qquad U_{(\tau,\xi)}(0) = \frac{dU_{(\tau,\xi)}(0)}{dt} \;=\; 0\,.$$

Proof. The dynamics is given by

$$\frac{da_\delta}{dt} \;=\; \sum_{\mu+\nu=\delta} K(\alpha,\beta)\, a_\alpha\, a_\beta\,.$$

Consider

$$\delta \;=\; (\tau,\xi) \;=\; n_1(1,1,0) + n_2((0,1,0) + n_3(0,3,4) := n_1\alpha + n_2\beta + n_3\gamma, \qquad \text{with} \qquad n_1 n_2 n_3 \neq 0\,.$$

This is the unique representation of δ as a linear combination of α, β, γ. Thus δ is never equal to a linear combination of only two of the vectors which proves that $da_\delta(0)/dt = 0$.
Compute,

$$\begin{aligned} \frac{d^2 a_\delta}{dt^2} \;&=\; \sum_{\alpha+\beta=\delta} K(\alpha,\beta)\Big((\partial_t a_\alpha) a_\beta + a_\alpha \partial_t a_\beta\Big) \\ &= \sum_{\alpha+\beta=\delta} K(\alpha,\beta)\Big(a_\beta \sum_{\mu+\nu=\alpha} K(\mu,\nu) a_\mu\, a_\nu\big) + a_\alpha \sum_{\kappa+\lambda=\beta} K(\kappa,\lambda) a_\kappa\, a_\lambda\Big) \\ &= \sum_{\substack{\alpha+\beta=\delta \\ \mu+\nu=\alpha}} K(\alpha,\beta)\, K(\mu,\nu)\; a_\beta a_\mu\, a_\nu \;\; + \sum_{\substack{\alpha+\beta=\delta \\ \kappa+\lambda=\beta}} K(\alpha,\beta)\, K(\kappa,\lambda)\; a_\alpha\, a_\kappa\, a_\lambda \\ &= \sum_{\beta+\mu+\nu=\delta} K(\mu+\nu,\beta)\, K(\mu,\nu)\, a_\beta\, a_\mu\, a_\nu \;+ \sum_{\alpha+\kappa+\lambda=\delta} K(\alpha,\kappa+\lambda)\, K(\kappa,\lambda)\, a_\alpha\, a_\kappa\, a_\lambda \\ &= \sum_{\beta+\mu+\nu=\delta} K(\mu+\nu,\beta)\, K(\mu,\nu)\, a_\beta\, a_\mu\, a_\nu \;+ \sum_{\alpha+\kappa+\lambda=\delta} K(\alpha,\kappa+\lambda)\, K(\kappa,\lambda)\, a_\alpha\, a_\kappa\, a_\lambda\,. \end{aligned}$$

Since K is symmetric, the two sums are equal. Therefore

$$\frac{d^2 a_\delta}{dt^2} \;=\; \sum_{\alpha+\mu+\nu=\delta} J(\alpha,\mu,\nu)\, a_\alpha\, a_\mu\, a_\nu\,, \qquad J(\alpha,\mu,\nu) \;:=\; 2\,K(\alpha\,,\,\mu+\nu)\,K(\mu\,,\,\nu)\,.$$

The interaction coefficient, $J(\alpha,\mu,\nu)$, is homogeneous of degree two in α,μ,ν and symmetric in μ,ν.

Then $d^2 a_\delta(0)/dt^2$ is the sum of six terms. Each is the product of

$$a_{n_1(1,1,0)}\, a_{n_2(0,1,0)}\, a_{n_3(0,3,4)}\big|_{t=0} \;=\; g_{n_1}\, g_{n_2}\, g_{n_3}\,.$$

and an interaction coefficient. A typical coefficient is

$$J\Big(n_1(1,1,0)\,,\, n_2(0,1,0)\,,\, n_3(0,3,4)\Big) \;=\; n_1^2\, J\Big((1,1,0)\,,\, q_2(0,1,0)\,,\, q_3(0,3,4)\Big)\,, \tag{11.6.6}$$

with q_j from (11.6.3). The other five terms come from permuting the arguments of J. We need to compute the sum of $J(\alpha,\beta,\gamma)$ and the five other terms which arise by permutation of the arguments.

Fortunately, four of the six summands vanish, and the other two are equal. Since $K(\kappa,\lambda)$ vanishes if λ is not characterisitic, J vanishes whenever the sum of its last two arguments is noncharacteristic. Of the six summands that leaves only

$$J(\alpha,\beta,\gamma) + J(\alpha,\gamma,\beta) \;=\; 2\,J(\alpha,\beta,\gamma) \;=\; 2\,K(\alpha,\beta+\gamma)\,K(\beta,\gamma) \qquad (11.6.7)\,.$$

Formula (11.5.4) implies that

$$K(\beta,\gamma)=0 \quad\Longleftrightarrow\quad \|\beta\|^2 = \|\alpha\|^2 \quad\Longleftrightarrow\quad q_2 \;=\; \pm\, 5q_3\,. \tag{11.6.8}$$

Formula (11.5.5) implies that

$$K(\alpha,\beta+\gamma)=0 \quad\Longleftrightarrow\quad (1,0)\perp\beta+\gamma \quad\Longleftrightarrow\quad q_2+3\,q_3=0\,. \tag{11.6.9}$$

Summarizing, at all points of the ellipse (11.6.4) except the intersection with the three lines in (11.6.8)-(11.6.9), the interaction coefficient is not equal to zero. This completes the proof. ∎

Proof of Theorem 11.2.1. The points (τ,ξ) of Theorem 11.6.2 have last components

$$(\xi_1\,,\,\xi_2) \;=\; n_1(1+q_2+3q_3\,,\,4q_3)\,.$$

If $r\neq 0$ is rational, to have slope r is equivalent to the equation

$$4q_3 \;=\; r(1+q_2+3q_3)\,,$$

which is a line in $\mathbb{R}^2_q$.

Intersecting with the ellipse (11.6.4) yields

$$1 \;=\; (1+r^2)(1+q_2+3q_3)^2\,.$$

For this to have a rational solution q requires that $(1+r^2) = \rho^{-2}$ be the square of a rational number which is a hypothesis of Theorem 11.2.1. Then $\rho\leq 1$, and, there are exactly two solutions q determined by

$$1+q_2+3q_2=\rho^2, \qquad 1=\rho^2+(4q_3)^2\,.$$

The three exceptional slopes come from $q_2=\pm 5q_3$ and $q_2+3q_3=0$. ∎

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