The State-Contingent Properties of Stochastic Production Functions

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A series of papers (Chambers and Quiggin, 1996, 1997, 1998, 2001; Quiggin and Chambers, 1998a, 1998b) and a monograph (Chambers and Quiggin, 2000) have explored the fundamental theory of state-contingent production correspondences. This paper develops the state-contingent properties of the most frequently used representations of stochastic production in the agricultural-economics literature. Particular attention is paid to the cases of multiplicative and additive uncertainty and to the convolution of multiplicative and additive uncertainty embodied in the Just-Pope stochastic production function.

In what follows, we first present a brief review of state-contingent technologies and their associated cost functions. Then we apply that theory to the standard representation of production uncertainty in the agricultural economics literature–the stochastic production function. After that we consider in turn the state-contingent properties of additive production uncertainty, multiplicative production uncertainty, and the Just-Pope stochastic production function. We propose a generalization of the Just-Pope technology that has desirable state-contingent characteristics. Our substantive results close with a comparison of cost functions based upon state-contingent technologies with cost functions based on a parametrized distribution representation of production uncertainty.

1 State-Contingent Technologies[1]

Following Chambers and Quiggin (1996, 1997, 2000), the stochastic technology is represented by a state-contingent input correspondence. To maximize comparability with existing agricultural-economic research on production under uncertainty, we restrict ourselves to the case of a single stochastic output. One can extend the current analysis to the case of multiple stochastic outputs.

The crucial concept, due originally to Arrow and Debreu, is that production under uncertainty can be represented by differentiating outputs according to the state of nature in which they are realized. This approach is analogous to the Arrow-Debreu treatment of goods differentiated by time and place of delivery as a particular form of multi-output production.

The set of states of nature may be either finite and discrete or infinite and continuous.

Chambers and Quiggin (1998) focus on the continuous case to demonstrate that the cost function is well-defined under uncertainty, and that duality is, therefore, applicable. For ease of exposition, however, attention in the present paper is confined to the discrete case. One can extend the results derived below to the case of continuous probability distributions and to mixtures of discrete and continuous distributions.

Let the states of nature be given by the set $\Omega = \{1, 2, ..., S\}$, let $\mathbf{x} \in \Re_+^N$ be a vector of inputs committed prior to the resolution of uncertainty, and let $\mathbf{z} \in \Re_+^S$ be a vector of state-contingent outputs. So, if state $s \in \Omega$ is realized (picked by 'Nature'), and the producer has chosen the *ex ante* input-output combination $(\mathbf{x}, \mathbf{z})$, then the realized or *ex post* output is z_s corresponding to the sth element of $\mathbf{z}$.

In most analyses of production under uncertainty, it is assumed that the state-contingent vector of outputs is generated by a vector of inputs directly controlled by the producer and a random variable that it beyond the control of the producer. Let the random variable be denoted by $\boldsymbol{\varepsilon} \in \Re^S$. Then the *stochastic production function specification* requires that stochastic output be related to inputs by the production function $f : \Re_+^N \times \Re \rightarrow \Re_+$

$$z_s \leq f(\mathbf{x}, \varepsilon_s), \quad s \in \Omega. \tag{1}$$

For simplicity, f is taken as increasing in ε_s.[2] This case will be examined in the present paper. But it is important to note that it is a restrictive representation of the options available to producers facing uncertainty.

More generally, the technology can be represented by a continuous input correspondence, $X : \Re_+^S \rightarrow \Re_+^N$, which maps state-contingent outputs into input sets that are capable of producing that state-contingent output vector. Formally,

$$X(\mathbf{z}) = \{\mathbf{x} \in \Re_+^N : \mathbf{x} \text{ can produce } \mathbf{z}\}.$$

We impose the following axioms on $X(\mathbf{z})$:

X.1 $X(\mathbf{0}_S) = \Re_+^N$ (no fixed costs), and $\mathbf{0}_N \notin X(\mathbf{z})$ for $\mathbf{z} \geq \mathbf{0}_S$ and $\mathbf{z} \neq \mathbf{0}_S$ (no free lunch).

X.2 $\mathbf{z}' \leq \mathbf{z} \Rightarrow X(\mathbf{z}) \subset X(\mathbf{z}')$.

X.3 $\mathbf{x}' \geq \mathbf{x} \in X(\mathbf{z}) \Rightarrow \mathbf{x}' \in X(\mathbf{z})$.

X.4 $X(\mathbf{z})$ is closed for all $\mathbf{z} \in \Re_+^S$.

The first part of X.1 says that doing nothing is feasible, while the second part of X.1 says that realizing a positive output in any state of nature requires the committal of some inputs. X.2, free disposability of state-contingent outputs, says that if an input combination can produce a particular vector of state-contingent outputs then it can always be used to produce a smaller vector of state-contingent outputs. X.3 implies that inputs have non-negative marginal productivity. X.4 is a technical assumption that ensures the existence of the cost function that we develop below.

Dual to the input correspondence is the cost function, defined, for the case where all inputs are purchased at competitive prices $\mathbf{w}$, as

$$c(\mathbf{w},\mathbf{z}) = \min_{\mathbf{x}}\{\mathbf{wx} : \mathbf{x} \in X(\mathbf{z})\} \quad \mathbf{w} \in \Re^N_{++}$$

if there exists an $\mathbf{x} \in X(\mathbf{z})$ and ∞ otherwise. Mathematically, $c(\mathbf{w},\mathbf{z})$ is equivalent to the multi-product cost function familiar from non-stochastic production theory (Chambers, 1988; Färe, 1988). Under the presumption that the input correspondence satisfies properties X then the cost function satisfies:

C.1. $c(\mathbf{w},\mathbf{z})$ is continuous on $\Re^S_+$ and positively linearly homogeneous, non-decreasing, concave, and continuous on $\Re^N_{++}$;

C.2. Shephard's Lemma;

C.3. $c(\mathbf{w},\mathbf{z}) \geq 0$, $c(\mathbf{w},0_S) = 0$, and $c(\mathbf{w},\mathbf{z}) > 0$ for $\mathbf{z} \geq 0_S, \mathbf{z} \neq 0_S$;

C.4. $\mathbf{z}^o \geq \mathbf{z} \Rightarrow c(\mathbf{w},\mathbf{z}^o) \geq c(\mathbf{w},\mathbf{z})$.

Moreover, by standard duality theorems (Färe, 1988)

$$X(\mathbf{z}) = \cap_{\mathbf{w}>\mathbf{0}}\{\mathbf{x} : \mathbf{wx} \geq c(\mathbf{w},\mathbf{z})\}.$$

Following Chambers and Quiggin (2000, Chapter 4), for $\mathbf{z} \in \Re^S_+$, we define the *cost certainty equivalent output,* denoted by $e^c(\mathbf{z}) \in \Re_+$, as the maximum non-stochastic output that can be produced at cost $c(\mathbf{w},\mathbf{z})$, that is,

$$e^c(\mathbf{z}) = \sup\{e : c(\mathbf{w},e\mathbf{1}^S) \leq c(\mathbf{w},\mathbf{z})\},$$

where $\mathbf{1}^S$ denotes an $S-$dimensional vector of ones. Let $\bar{z}$ denote the mean of $\mathbf{z}$. The *absolute production risk premium,* $p^c(\mathbf{z}) \in \Re$, is defined implicitly by:

$$(2) \qquad c(\mathbf{w}, \mathbf{z}) = c\left(\mathbf{w}, (\bar{z} - p^c(\mathbf{z}))\mathbf{1}^S\right) = c\left(\mathbf{w}, e^c(\mathbf{z})\mathbf{1}^S\right)$$

so that

$$p^c(\mathbf{z}) = \bar{z} - e^c(\mathbf{z}).$$

The absolute production risk premium measures the cost of removing production uncertainty for a cost minimizer. If $p^c(\mathbf{z}) > 0$, it is more costly to produce the mean of $\mathbf{z}$ with certainty than it is to produce $\mathbf{z}$. Typically, one thinks of technologies as always involving an absolute risk premium that is positive. More generally this need not be the case. The technology is *inherently risky* at $\mathbf{z}$ if the absolute production risk premium is positive there, and *not inherently risky* otherwise.

An alternative measure of the cost of removing uncertainty is the *relative production risk premium*

$$r^c(\mathbf{z}) = \frac{\bar{z}}{e^c(\mathbf{z})}.$$

Properties C.1 and C.4 guarantee that $e^c(\mathbf{z})$ is a complete function representation of the cost function in the sense that

$$e^c(\mathbf{z}') \geq e^c(\mathbf{z}) \Leftrightarrow c(\mathbf{w}, \mathbf{z}') \geq c(\mathbf{w}, \mathbf{z}).$$

It is useful to have classes of technologies that are easily characterized in terms of either the production risk premium or the certainty equivalent. The state-contingent technology displays *constant absolute riskiness* if the cost certainty equivalent output responds to increasing output with certainty by increasing by that same amount, so that for all $\mathbf{z}$ and $t \in \Re$

$$(3) \qquad e^c\left(\mathbf{z} + t\mathbf{1}^S\right) = e^c(\mathbf{z}) + t.$$

Geometrically, constant absolute riskiness ensures that isocost contours are parallel to one another in the direction of the equal-output vector.

The technology displays *constant relative riskiness* if for all $\mathbf{z}, v \in \Re_+$:

(4) $$e^c(v\mathbf{z}) = ve^c(\mathbf{z})$$

and equivalently

$$p^c(v\mathbf{z}) = vp^c(\mathbf{z}), \quad v \in \Re_+.$$

By (3), (4), C.1, and C.4:

Lemma 1 The cost function satisfies constant absolute riskiness if and only if it is BD-translation homothetic in $\mathbf{z}$. The cost function satisfies constant relative riskiness if and only if it is homothetic in $\mathbf{z}$.[3]

2 Stochastic production functions as state-contingent technologies

The state-contingent input correspondence associated with (1) is

$$\begin{aligned} X(\mathbf{z}) &= \{\mathbf{x} : z_s \le f(\mathbf{x}, \varepsilon_s), \quad s \in \Omega\} \\ &= \cap_{s\in\Omega} \{\mathbf{x} : z_s \le f(\mathbf{x}, \varepsilon_s)\} \\ &= \cap_{s\in\Omega} \bar{X}(z_s; \varepsilon_s) \end{aligned}$$

where $\bar{X}(z_s; \varepsilon_s)$ may be interpreted as the *ex post* input set associated with the production function for a given realization of the random variable.

The dual cost structure for the stochastic production function specification defined,

$$c(\mathbf{w}, \mathbf{z}) = Min\left\{\mathbf{wx} : \mathbf{x} \in \cap_{s\in\Omega} \bar{X}(z_s; \varepsilon_s)\right\},$$

satisfies

(5) $$c(\mathbf{w}, \mathbf{z}) \ge Max\{\bar{c}(\mathbf{w}, z_1; \varepsilon_1), ..., \bar{c}(\mathbf{w}, z_S; \varepsilon_S)\}.$$

where $\bar{c}(\mathbf{w}, z_s; \varepsilon_s)$ is the *ex post* cost function dual to $\bar{X}(z_s; \varepsilon_s)$.There can exist instances where the inequality in (5) is strict (Chambers and Quiggin, 1998, 2000). In what follows, however, we focus on the special case

$$c(\mathbf{w},\mathbf{z}) = Max\left\{\bar{c}(\mathbf{w},z_1;\varepsilon_1),...,\bar{c}(\mathbf{w},z_S;\varepsilon_S)\right\}. \tag{6}$$

The imposition of restrictions such as constant absolute riskiness or constant relative riskiness may be necessary in order to yield either tractable estimation procedures or tractable analytic results. Hence, it is important to consider the implications of these restrictions for the familiar case of a stochastic production function. Here we want to determine the additional structure that imposing either constant absolute riskiness or constant relative riskiness imposes upon technologies described by (6). We have (the proof is in the appendix):

Result 1 Technologies of the class (6) satisfy constant absolute riskiness if and only if the *ex post* cost functions can be expressed as

$$\bar{c}(\mathbf{w},z_k;\varepsilon_k) = \hat{c}(\mathbf{w},z_k - \nu(\varepsilon_k)).$$

Technologies of the class (6) satisfy constant relative riskiness if and only if the *ex post* cost functions can be expressed as

$$\bar{c}(\mathbf{w},z_k;\varepsilon_k) = \tilde{c}\left(\mathbf{w},\frac{z_k}{\tau(\varepsilon_k)}\right).$$

Several additional observations should be made at this juncture. First, the stochastic production function specification involves a strong *a priori* functional restriction on the interaction between random factors and controllable inputs which does not appear to have been empirically validated. The most common justification for the stochastic error term is that it captures the effect of random inputs, such as rainfall and other climatic conditions, on an otherwise deterministic production process. Therefore, the error term conceptually can be viewed as an input aggregate consisting of these random inputs. This specification requires that the marginal rates of substitution between the stochastic inputs be independent of the controllable inputs. More formally, the stochastic inputs must be weakly separable from the controllable outputs.

Second, regardless of the degree of continuity or smoothness placed upon the stochastic production function, the cost function dual to this technology is not everywhere differentiable in outputs. Therefore, in most cases, it will be difficult to employ traditional methods of analysis, e.g., equating marginal cost to marginal benefit, which typically rely on smoothness with such technologies.

Third, and associated directly with the nondifferentiability in state-contingent outputs, is the fact that the output sets associated with the stochastic production function are cubes in state-contingent output space. Figure 1 illustrates in the case of two states of nature. Figure 1 implies that there is no substitutability between state-contingent outputs. Hence, the technology associated with the stochastic production function may be referred to as 'Leontief-in-outputs', 'fixed-output-proportions' or 'output-cubical'.

As is well known from the study of multi-input, multi-output technologies under certainty, Leontief-in-outputs technologies have somewhat pathological properties. Kohli has considered such technologies, which he refers to as input-price nonjoint, in detail and shown that the revenue functions dual to them are linear in output prices implying that revenue maximizing supplies are completely inelastic. In the case of decisionmaking under uncertainty, the parallel result is that changes in the distribution of the output price have no effect on the mix of state-contingent outputs as long as input use is held constant.

2.1 Additive uncertainty

Consider the special case

$$f\left(\mathbf{x},\varepsilon_s\right) = g\left(\mathbf{x}\right) + \varepsilon_s.$$

For this specification,

$$X\left(\mathbf{z}\right) = \cap_{s\in\Omega}\left\{\mathbf{x} : z_s - \varepsilon_s \leq g\left(\mathbf{x}\right)\right\}.$$

If $g\left(\mathbf{x}\right)$ is nondecreasing in each of the variable inputs under the producer's control,

$$\begin{aligned} X\left(\mathbf{z}\right) &= \left\{\mathbf{x} : Max\left\{z_1 - \varepsilon_1, ..., z_S - \varepsilon_S\right\} \leq g\left(\mathbf{x}\right)\right\} \\ &= X_g\left(Max\left\{z_1 - \varepsilon_1, ..., z_S - \varepsilon_S\right\}\right), \end{aligned}$$

where

$$X_g\left(q\right) = \left\{\mathbf{x} : g\left(\mathbf{x}\right) \geq q\right\}.$$

Therefore,

$$\bar{c}(\mathbf{w}, z_k; \varepsilon_k) = c_g(\mathbf{w}, z_k - \varepsilon_k),$$

and

$$c(\mathbf{w}, \mathbf{z}) = c_g(\mathbf{w}, Max\{z_1 - \varepsilon_1, ..., z_S - \varepsilon_S\})$$

where c_g is the cost function dual to X_g. This cost structure corresponds to the cost structure isolated in the first part of Result 1.[4]

Corollary 1.1 The stochastic production function with additive uncertainty is characterized by constant absolute riskiness.

As a consequence of Corollary 1.1, we may derive the well-known result that risk-averse individuals facing a stochastic production function with an additive error structure always choose the same input combination as risk-neutral individuals facing the same technology. This result is well-known for expected-utility preferences. Here we verify it for general preferences of the sort described by Yaari (1969) and Quiggin and Chambers (1998), which includes expected-utility as a special case. Let $W : \Re^S \to \Re$ be a continuous preference structure that is strictly increasing in state-contingent net returns. Suppose that there is no price uncertainty, and normalize the price of the stochastic input to one. Then the producer's objective function is

$$Max_{\mathbf{z}}\{W(z_1 - c_g(\mathbf{w}, Max\{z_1 - \varepsilon_1, ..., z_S - \varepsilon_S\}), ..., z_S - c_g(\mathbf{w}, Max\{z_1 - \varepsilon_1, ..., z_S - \varepsilon_S\}))\}.$$

A producer with this objective function will always operate at a point of economic and technical efficiency given by the kinked point on his optimal isocost contour (recall Figure 1). That is, he will choose the state-contingent outputs so that

$$z_1 - \varepsilon_1 = z_s - \varepsilon_s, \quad s \in \Omega.$$

If he did not operate in this fashion, he could always costlessly raise at least one *ex post* output. Visually, this implies that all producers, regardless of their risk preferences, share a common expansion path that is parallel to the non-stochastic production vector, i.e.,

$$z_s = z_1 + \varepsilon_s - \varepsilon_1.$$

Thus, the producer's objective function can be expressed solely in terms of the first state-contingent output as

$$Max_{z_1}\left\{W\left(z_1 - c_g\left(\mathbf{w},z_1 - \varepsilon_1\right), z_1 + \varepsilon_2 - \varepsilon_1 - c_g\left(\mathbf{w},z_1 - \varepsilon_1\right), ..., z_1 + \varepsilon_S - \varepsilon_1 - c_g\left(\mathbf{w},z_1 - \varepsilon_1\right)\right)\right\}.$$

Presuming that W and c_g are smoothly differentiable, therefore, yields the first-order condition

$$\left(1 - c_g^{'}\left(\mathbf{w},z_1 - \varepsilon_1\right)\right)\sum_{s\in\Omega} W_s\left(\mathbf{y}\right) \leq 0, \quad z_1 \geq 0.$$

For an interior solution, the producer's optimally chooses the first state-contingent output so that the marginal cost of producing it equals one. Recall that the additive structure of the production function in the random variable implies that any time the producer increases one state-contingent output, he increases all other state-contingent outputs by the same amount. Therefore, each time he increases one state-contingent output, he increases his *ex post* output with certainty by that same amount. Therefore, an increase in the first state-contingent output brings with it a sure return of a dollar, and the producer will continue to expand his output until the marginal cost of doing so reaches a dollar *regardless of his risk attitudes.* Risk neutral, risk averse, and risk lovers will all choose the same state-contingent output mix and correspondingly the same mix of inputs.

This property is not characteristic of the entire class of technologies exhibiting constant absolute riskiness. Consider,

$$c\left(\mathbf{w},\mathbf{z}\right) = \ln\left(\sum_{s\in\Omega} c_s\left(\mathbf{w}\right)\exp\left(z_s\right)\right).$$

This technology satisfies constant absolute riskiness, but a risk-averse individual with expected-utility preferences and a risk-neutral individual generally choose different optimal state-contingent output vectors. The 'irrelevance of risk attitudes' property emerges, therefore, not from the presence of constant absolute riskiness alone but from the convolution of constant absolute riskiness with the stochastic production function's inability to permit substitution between state-contingent outputs.

2.2 Multiplicative Uncertainty

Now consider

$$f(\mathbf{x},\varepsilon_s) = h(\mathbf{x})\,\varepsilon_s, \quad s \in \Omega,$$

where it is assumed that $\varepsilon_s \geq 0$ for all $s \in \Omega$. The associated state-contingent input correspondence is

$$\begin{aligned} X(\mathbf{z}) &= \cap_{s\in\Omega}\left\{\mathbf{x} : \frac{z_s}{\varepsilon_s} \leq h(\mathbf{x})\right\} \\ &= \left\{\mathbf{x} : Max\left\{\frac{z_1}{\varepsilon_1}, ..., \frac{z_S}{\varepsilon_S}\right\} \leq h(\mathbf{x})\right\} \\ &= X_h\left(Max\left\{\frac{z_1}{\varepsilon_1}, ..., \frac{z_S}{\varepsilon_S}\right\}\right). \end{aligned}$$

The associated cost function can be represented as

$$c(\mathbf{w}, \mathbf{z}) = c^h\left(\mathbf{w}, Max\left\{\frac{z_1}{\varepsilon_1}, ..., \frac{z_S}{\varepsilon_S}\right\}\right)$$

where c^h is the minimal cost function associated with the nonstochastic production function h. By Result 1:

Corollary 1.2 The stochastic production function with multiplicative uncertainty is characterized by constant relative riskiness.

As in the additive uncertainty case, an economically rational individual, who is also the residual claimant, will always locate at the outer vertex of her optimal isocost contour. Regardless of her risk preferences, her expansion path, therefore, is defined by

$$z_s = \frac{\varepsilon_s}{\varepsilon_1} z_1, \quad s \in \Omega.$$

The relative riskiness of the producer's optimal state-contingent production bundle, thus, is not the subject of economic choice.

Producer decisionmaking, therefore, reduces to choosing the optimal portfolio consisting of a safe asset and a risky asset that in state s always returns $\frac{\varepsilon_s}{\varepsilon_1}$ times what it would return in state 1. Geometrically, therefore, the producer's potential net returns vectors are spanned by the fixed vector given by

$$z_s = \frac{\varepsilon_s}{\varepsilon_1} z_1, \quad s \in \Omega \tag{7}$$

and the certain output vector. Figure 2 illustrates in the two state case. Committing inputs to produce the state-contingent output vector incurs a non-stochastic cost of $c(\mathbf{w}, z_1, z_2)$ and, assuming the price of the output is one, brings with it a stochastic return of (z_1, z_2). The cost, therefore, can be illustrated by point A in Figure 2, drawn on the negative reflection of the certainty output vector, while the stochastic return is illustrated by point B. The stochastic net return, in turn, is given by point C which is the vector sum of A and B. Trivially, all possible state-contingent net returns are spanned by these two vectors. The multiplicative uncertainty model is, therefore, just a simple application of basic portfolio analysis. Standard results from portfolio choice theory, therefore, apply directly.

For example, a basic result is that a risk-averse (regardless of the degree of risk aversion) individual with expected-utility preferences will always hold some of the risky asset if its expected return is even slightly larger than the safe asset's return. Here, the direct consequence of this theorem is that as long as marginal cost at the origin is suitably small, a risk averter (regardless of the degree of risk aversion) will produce a positive stochastic output. Other basic portfolio-choice results, for example, those relating to changes in the holding of the risky asset in response to wealth changes in wealth, also have exact analogues here in terms of production changes.

Because the more general state-contingent technology with constant relative riskiness does not require that stochastic outputs always be produced in the fixed proportions $\frac{\varepsilon_s}{\varepsilon_1}$, its analysis does not reduce to a special case of the standard portfolio model. For example, consider the case of technology whose state-contingent cost function is

$$c(\mathbf{w}, \mathbf{z}) = \vec{c}(\mathbf{w}) \sum_{s \in \Omega} \sum_{t \in \Omega} \beta_{st} (z_s z_t)^{\frac{1}{2}},$$

where $\vec{c}(\mathbf{w})$ is positively linearly homogeneous and concave. This technology satisfies constant absolute riskiness. It can be verified, for example, that optimal state-contingent output proportions are not fixed for a risk-neutral individual facing such a technology. The risk-averse producer, therefore, does more than just pick the size of his investment in the risky asset, he also chooses its composition in response to his subjective view of the world.

2.3 Just-Pope production technology

Recognizing the many shortcomings of the multiplicative and additive uncertainty specifications, Just and Pope suggested a specification of stochastic production that has become the model of choice for most agricultural economists investigating decisionmaking under production uncertainty. Although their arguments were not cast in state-contingent terms, they can be reinterpreted in those terms as recognizing the problems with the two technologies discussed in the preceding sections. In both cases, the producer's expansion paths in state-contingent output space are linear and independent of the producer's attitudes towards risk. When matched with the stochastic production function's inability to encompass the possibility of substitution between state-contingent outputs, this linearity predetermines stochastic decisionmaking, leading, as Just and Pope observed, to unrealistic restrictions on producer choice.

The Just-Pope technology can be recognized as the additive combination of a non-stochastic technology with a multiplicative uncertainty specification. The Just-Pope approach includes additive uncertainty and multiplicative uncertainty as special cases. It resolves the problems associated with linearity of the producer's expansion paths, but not the inability of the stochastic production function to permit substitutability between state-contingent outputs.

More formally, the Just-Pope formulation sets

$$f\left(\mathbf{x},\varepsilon_s\right) = g\left(\mathbf{x}\right) + h\left(\mathbf{x}\right)\varepsilon_s.$$

Hence, in the case where $g\left(\mathbf{x}\right) = 0$, the Just-Pope formulation reduces to the multiplicative uncertainty and constant relative riskiness special case, while in the case where $h\left(\mathbf{x}\right) = h$ (a constant function) it reduces to the additive uncertainty case.

The state-contingent input correspondence associated with the Just-Pope specification is

$$X\left(\mathbf{z}\right) = \cap_{s\in\Omega}\left\{\mathbf{x} : g\left(\mathbf{x}\right) + h\left(\mathbf{x}\right)\varepsilon_s \geq z_s\right\}.$$

Let

$$\begin{aligned}
c^{h}(\mathbf{w},h) &= Min_{\mathbf{x}}\left\{\mathbf{wx} : h(\mathbf{x}) = h\right\} \\
c^{g}(\mathbf{w},g) &= Min_{\mathbf{x}}\left\{\mathbf{wx} : g(\mathbf{x}) \geq g\right\} \\
c^{hg}(\mathbf{w},h,g) &= Min_{\mathbf{x}}\left\{\mathbf{wx} : h(\mathbf{x}) = h, g(\mathbf{x}) \geq g\right\}
\end{aligned}$$

and note that

$$c^{hg}(\mathbf{w},h,g) \geq Max\left\{c^{h}(\mathbf{w},h), c^{g}(\mathbf{w},g)\right\}.$$

The *ex post* cost functions for the Just-Pope technology are derived by

$$\begin{aligned}
\bar{c}(\mathbf{w},z_s;\varepsilon_s) &= Min_{\mathbf{x}}\left\{\mathbf{wx} : g(\mathbf{x}) + h(\mathbf{x})\varepsilon_s \geq z_s\right\} \\
&= Min_{h,g,\mathbf{x}}\left\{\mathbf{wx} : g + h\varepsilon_s \geq z_s, h(\mathbf{x}) = h, g(\mathbf{x}) \geq g\right\} \\
&= Min_{h,g}\left\{Min_{x}\left\{\mathbf{wx} : h(\mathbf{x}) = h, g(\mathbf{x}) \geq g\right\} : g + h\varepsilon_s \geq z_s\right\} \\
&= Min_{h,g}\left\{c^{hg}(\mathbf{w},h,g) : g + h\varepsilon_s \geq z_s\right\} \\
&= Min_{h}\left\{c^{hg}(\mathbf{w},h,z_s - h\varepsilon_s)\right\}.
\end{aligned}$$

Because $c^{hg}(\mathbf{w},h,g)$ need not be everywhere differentiable in h and g, the *ex post* cost functions associated with the Just-Pope technology need not be smoothly differentiable in the *ex post* output.

The potential nondifferentiability of the Just-Pope cost function and of cost functions associated with the stochastic production function, in general, is important for a number of reasons. Most obviously, it implies that, although Shephard's lemma applies for the Just-Pope specification, one cannot easily rely on it to characterize factor demands and costs econometrically. Notice, in particular, that econometric estimation of $c^{gf}(\mathbf{w},h,g)$ and the associated factor demands can require highly nonlinear econometric techniques. Proper identification of the parameters of the production structure requires identifying both $c^{h}(\mathbf{w},h)$ and $c^{g}(\mathbf{w},g)$ in estimation.

2.4 Generalized Just-Pope technologies

When viewed from a state-contingent perspective, the Just-Pope technology seems intractable. In this section, we offer an alternative specification of the state-contingent technology that captures the essence of the Just-Pope contribution, but which is tractable in state-contingent terms.

The Just-Pope specification is a mix of a non-stochastic technology and a technology that satisfies constant relative riskiness. Thus, we want to specify a technology which is a mixture of a non-stochastic technology and one satisfying constant relative riskiness, but which also leads to a differentiable cost function.

The best place to start the search for such a technology is with an alternative representation of the state-contingent technology. The state-contingent output correspondence, $Z : \Re^N_+ \to \Re^S_+$, dual to the state-contingent input correspondence is defined by

$$Z(\mathbf{x}) = \{\mathbf{z} : \mathbf{x} \in X(\mathbf{z})\}.$$

For the Just-Pope specification, we, therefore, have

$$\begin{aligned} Z(\mathbf{x}) &= \{\mathbf{z} : z_s \leq g(\mathbf{x}) + h(\mathbf{x})\varepsilon_s, \quad s \in \Omega\} \\ &= \times_{s \in \Omega}\{z_s : z_s \leq g(\mathbf{x}) + h(\mathbf{x})\varepsilon_s\} \end{aligned}$$

which can be visualized as an S-dimensional cube in state-contingent output space. Visually, it is easiest to consider the Just-Pope technology in two dimensions. Notice that it consists of two parts: the first is associated with the non-stochastic part of the technology given by $g(\mathbf{x})$. Under the assumption that the random vector $\boldsymbol{\varepsilon}$ has mean zero, this corresponds to mean output, and we represent this by point A in Figure 3. The other component, assuming $h \geq 0$ and $\boldsymbol{\varepsilon}$ has zero mean, can be represented by point B in Figure 3. The slope of the ray connecting the origin and point B is given by $\frac{\varepsilon_1}{\varepsilon_2}$. Then the $Z(\mathbf{x})$ that corresponds to these two points is arrived at by taking their vector sums to end at the point C. Everything falling within the rectangle with vertices at C and the origin then belongs to the state-contingent output set.

Now to see how the output set responds to a change in $\mathbf{x}$, notice that increasing the input vector increases the mean output. This can be visualized as the mean sliding out from A to A' along the certain-output vector. Similarly we can visualize increasing the input vector as sliding the stochastic, constant relative risky, output component out from B to B'. The new state-contingent output set is given by everything falling in the rectangle with outer vertex at point C'. As is apparent there is no reason to expect the resulting expansion path to be linear. However, because the state-contingent output set continues to be described by a cube, the associated cost function will not be everywhere differentiable in state-contingent outputs.

The nondifferentiability of the cost function emerges from the role that inputs are forced to play in this production process. The production function formulation does not allow input activity to be allocable toward different state-contingent outputs. Moreover, the Just-Pope formulation, because it is essentially the sum of two production functions, manifests this nondifferentiability problem by requiring that the mean output and the random output are both determined by the same input level. In short, inputs cannot be differentially allocated to solely mean increasing activities and variance controlling activities. The model may be made more realistic, but also more complicated by a finer specification of inputs. Therefore, it seems simpler to make explicit the possibility of allocating inputs to different state-contingent outputs.

Our proposed generalization of the Just-Pope formulation effectively mixes a technology exhibiting constant relative riskiness and the ability to substitute state-contingent outputs with a technology which is not inherently risky at all possible state-contingent outputs. More formally, we have

$$Z(\mathbf{x}) = \left\{ Z^o(\mathbf{x}^o) + z\left(\mathbf{x}^1\right) Z^1 : f(\mathbf{x}^o, \mathbf{x}^1) \leq \mathbf{x} \right\}.$$

Here $Z^1 \subset \Re^S_+$ is a fixed reference set in state-contingent output space and $z : \Re^N_+ \rightarrow \Re_{++}$ is a real-valued function that scales the reference set up or down depending upon input utilization. Output correspondences assuming the form, $z(\mathbf{x}^1) Z^1$, are referred to as output homothetic (Färe and Primont, 1995; Chambers and Quiggin, 2000). $Z^o(\mathbf{x}^o)$ is a generalized Schur-convex (see next paragraph) state-contingent output correspondence. The input quantities $\mathbf{x}^o$ and $\mathbf{x}^1$ represent the effective allocation of inputs to the activities represented by the technologies Z^o and zZ^1. In the Just-Pope model, inputs are not allocable so that

$$f(\mathbf{x}^o, \mathbf{x}^1) = \max\{\mathbf{x}^o, \mathbf{x}^1\}.$$

The opposite polar case is that of allocable inputs, where

$$f(\mathbf{x}^o, \mathbf{x}^1) = \mathbf{x}^o + \mathbf{x}^1.$$

In the remaining analysis, we will focus on this latter case. However, the analysis is easily extended to general families of technologies such as the CES class for which the allocable (linear) and Just-Pope (maximin) specifications are polar cases, or to general differentiable f.

Let $\boldsymbol{\pi} \in \Re_+^S$ be a vector of probabilities. For this vector of probabilities, let $\preceq_\pi$ denote a partial ordering of $\Re_+^S$ that orders state-contingent output vectors with the same mean. The notation

$$\mathbf{z} \preceq_\pi \mathbf{z}'$$

means that $\Sigma_{s\in\Omega}\pi_s z_s = \Sigma_{s\in\Omega}\pi_s z_s^{'}$, and that $\mathbf{z}$ is less risky in the Rothschild-Stiglitz sense than $\mathbf{z}'$. A state-contingent output correspondence, $Z(\mathbf{x})$, is said to be generalized Schur-convex (Chambers and Quiggin, 1997) if

$$\mathbf{z} \preceq_\pi \mathbf{z}' \in Z(\mathbf{x}) \Rightarrow \mathbf{z} \in Z(\mathbf{x}).$$

The special case of Schur convexity applies if all the π_s are equal. Schur-convexity may be visualised by thinking of a symmetric output set which has the fair-odds line tangent to the boundary of $Z(\mathbf{x})$ where $Z(\mathbf{x})$ intersects the certainty output vector. A non-stochastic technology, which in state-contingent terms is a cube with its vertex on the certainty output vector, trivially satisfies this tangency condition.

The cost function associated with the generalized Just-Pope technology can be derived as

$$\begin{aligned}
c(\mathbf{w}, \mathbf{z}) &= Min_{\mathbf{x},\mathbf{x}^o,\mathbf{x}^1} \left\{ \mathbf{wx} : \mathbf{z} \in Z^o(\mathbf{x}^o) + z\left(\mathbf{x}^1\right) Z^1, \mathbf{x}^o + \mathbf{x}^1 \leq \mathbf{x} \right\} \\
&= Min_{\mathbf{x},\mathbf{x}^o,\mathbf{x}^1,\mathbf{z}^o,\mathbf{z}^1} \left\{ \mathbf{wx} : \mathbf{z}^o \in Z^o(\mathbf{x}^o), \mathbf{z}^1 \in z\left(\mathbf{x}^1\right) Z^1, \mathbf{x}^o + \mathbf{x}^1 \leq \mathbf{x}, \mathbf{z}^o + \mathbf{z}^1 = \mathbf{z} \right\} \\
&= Min_{\mathbf{z}^0,\mathbf{z}^1} \left\{ Min_{\mathbf{x}^o} \left\{ \mathbf{wx}^0 : \mathbf{z}^o \in Z^o(\mathbf{x}^o) \right\} + Min_{\mathbf{x}^1} \left\{ \mathbf{wx}^1 : \mathbf{z}^1 \in z\left(\mathbf{x}^1\right) Z^1 \right\} : \mathbf{z}^o + \mathbf{z}^1 = \mathbf{z} \right\} \\
&= Min_{\mathbf{z}^o,\mathbf{z}^1} \left\{ c^o(\mathbf{w}, \mathbf{z}^o) + c^1\left(\mathbf{w}, m\left(\mathbf{z}^1\right)\right) : \mathbf{z}^o + \mathbf{z}^1 = \mathbf{z} \right\} \\
&= Min_{\mathbf{z}^o} \left\{ c^o(\mathbf{w}, \mathbf{z}^o) + c^1(\mathbf{w}, m(\mathbf{z} - \mathbf{z}^o)) \right\}
\end{aligned}$$

In this derivation, c^o is the cost function dual to the generalized Schur convex output correspondence. It has the property that $\mathbf{z}^o \preceq_\pi \mathbf{z}^{o\prime}$ implies $c^o(\mathbf{w}, \mathbf{z}^o) \le c^o(\mathbf{w}, \mathbf{z}^{o\prime})$. To see why, recognize that by the definition of generalized Schur convexity if an input combination can produce $\mathbf{z}^{o\prime}$, then it can also produce $\mathbf{z}^o$. Therefore, the minimal cost of producing the latter can never be greater than the minimal cost of producing the former.

c^1 is the cost function dual to $z(\mathbf{x}^1) Z^1$. It has been written as a function of a function $m : \Re^S_+ \to \Re_+$ that is positively linearly homogeneous in the state-contingent inputs, i.e., $m(\lambda \mathbf{z}^1) = \lambda m(\mathbf{z}^1)$, $\lambda > 0$. This is a standard property of output-homothetic production correspondences (Chambers and Quiggin, 2000, Chapter 4).

We close this section by demonstrating that the generalized Just-Pope technology is, indeed, a combination of a technology that exhibits constant relative riskiness and a technology that involves no production uncertainty. To do so, we show that a risk-neutral individual facing such a technology will always choose $\mathbf{z}^o$ to be on the certainty-output vector and $\mathbf{z}^1$ to be on a ray along which the relative risk premium (relative to $c^1(\mathbf{w}, m(\mathbf{z}^1))$) does not change. A risk-neutral individual facing this technology solves

$$Max \left\{ \sum_{s \in \Omega} \pi_s \left(z_s^o + z_s^1 \right) - c^o(\mathbf{w}, \mathbf{z}^o) - c^1\left(\mathbf{w}, m\left(\mathbf{z}^1\right)\right) \right\}.$$

Clearly, the problem is separable so that a risk-neutral individual solves two independent problems

$$\sum_{s \in \Omega} \pi_s z_s^o - c^o(\mathbf{w}, \mathbf{z}^o) \tag{8}$$

and

$$\sum_{s \in \Omega} \pi_s z_s^1 - c^1\left(\mathbf{w}, m\left(\mathbf{z}^1\right)\right). \tag{9}$$

For any $\mathbf{z}^o$, generalized Schur convexity implies that the point $\left(\sum_{s \in \Omega} \pi_s z_s^o\right) \mathbf{1}_S$ can be no more costly than $\mathbf{z}^o$ because it has the same mean and is less risky than $\mathbf{z}^o$ in the Rothschild-Stiglitz sense. Therefore, $\left(\sum_{s \in \Omega} \pi_s z_s^o\right) \mathbf{1}_S$ always at least weakly dominates $\mathbf{z}^o$ for a risk-neutral individual.

Now consider (9). Since the cost function is homothetic in state-contingent outputs, the solution, say $\mathbf{z}^1$, will always lie on a fixed ray from the origin. The cost certainty equivalent output for that point (in terms of $c^1(\mathbf{w}, m(\mathbf{z}^1))$) is given by

$$\begin{aligned} e^c\left(\mathbf{z}^1\right) &= \sup\left\{c : cm(1) \leq m\left(\mathbf{z}^1\right)\right\} \\ &= \frac{m(\mathbf{z}^1)}{m(1)}, \end{aligned}$$

where we have exploited the positive linear homogeneity of m. Now consider any other point on this same ray, call it $\lambda\mathbf{z}^1$. Its cost certainty equivalent output is given by

$$\lambda\frac{m(\mathbf{z}^1)}{m(1)},$$

Hence, the technology exhibits constant relative riskiness as claimed.

To construct a generalized Just-Pope technology, all that is needed is combine a constant relative risky technology along with a generalized Schur convex technology along the lines suggested above. For example, the technology associated with

$$c(\mathbf{w}, \mathbf{z}) = \min_{\mathbf{z}^o}\left\{c^o(\mathbf{w})\sum_{s\in\Omega}\sum_{t\in\Omega}\beta_{st}\left(z_s^o z_t^o\right)^{\frac{1}{2}} + c^1(\mathbf{w})\sum_{s\in\Omega}\pi_s \exp\left(z_s - z_s^o\right)\right\}$$

is generalized Just-Pope. Suitable choice of functional specifications permit approximation of a generalized Just-Pope technology to an arbitrary order.

2.5 The parametrized distribution approach to stochastic technologie

A common response to the difficulties associated with stochastic production function technologies is to ignore the technology and to focus attention instead on the cumulative probability distribution of output that is jointly generated from the stochastic production function and the distribution of the error structure. This is the approach pioneered by Mirrlees. Hart and Holmstrom refer to it as the *parametrized distribution approach.*

The first requirement for this approach is the specification of a probability distribution over the states of nature $\boldsymbol{\pi} = (\pi_1, \pi_2 \ldots \pi_S)$. For a given probability vector π, and a random output $\mathbf{z}$, define the *cumulative distribution function* $F(\bullet; \mathbf{z}, \boldsymbol{\pi}) : \Re \rightarrow [0, 1]$ by

$$F(t; \mathbf{z}, \boldsymbol{\pi}) = \sum_{\{s: y_s \leq t\}} \pi_s$$

and assume that the producer cares only about the cumulative distribution function of output and not about the particular states of nature in which high and low outputs occur.

Neither the general state-contingent technology or the general preference structures described above require the existence of well-defined subjective probabilities. Hence, the requirement that they exist and are objectively known to the researcher *a priori* restricts the range of technologies and problems which can be studied using the parametrized distribution formulation. The more general state-contingent technology, however, makes no use of probabilities in its definition.

The cumulative distribution can be characterized by the central moments of the distribution

$$\begin{aligned} \mu_1 &= E[\mathbf{z}] \\ \mu_m &= (E[(\mathbf{z}-\mu_1)^m]) \end{aligned}$$

In general, the sequence of moments required to characterize a cumulative distribution function is infinite. However, finite-dimensional specifications can be obtained in several ways. If the cumulative distribution function is derived from a discrete probability distribution over S states, it can be characterized uniquely by its first S moments. Alternatively, if parametric assumptions about the distribution are made (normal, uniform, triangular, and so on), a finite set of moments may provide sufficient statistics to characterize the distribution. Finally, structure can be imposed on producer preferences to ensure that they are expressible in terms of a finite subset of the moments (e.g. mean and standard deviation). This alternative, in essence, requires *a priori* that there be a conjugacy between the moments chosen and the producer's preferences over state-contingent returns.[6]

For any input correspondence, $X(\mathbf{z})$, and any finite set of moments,$\mu^1, ...\mu^M$, we may consider the parametrized-distribution representation of the cost function for the moments

$$c^M(\mathbf{w},\mu^1, ...\mu^M) = \min_{\mathbf{x}\in X(\mathbf{z})} \{\mathbf{wx} : \mu^m(\mathbf{z}) \geq \mu^m, m = 1, ..., M\} \tag{10}$$

The choice between the state-contingent representation $c(\mathbf{w}, \mathbf{z})$ and the parametrized-distribution representation $c^M(\mathbf{w},\mu^1, ...\mu^M)$ is logically independent of whether the associated technology is representable as a Leontief-in-outputs stochastic production function or as a more general state-contingent production set. The relationship between the state-contingent and parametrized-distribution representations of production under uncertainty is similar to that between structural and reduced forms in econometrics. Moreover, just as in econometrics, there is an inherent identification problem. For example, in the case of a stochastic production function, multiple stochastic production functions and error structures can correspond to a single parametrized distribution representation (Mirrlees). The structural information that is lost in the passage to the reduced forms includes any relevant information about the state of nature, such as the level of rainfall and average temperature. Empirically, this means that generally it will be impossible to identify estimates of characteristics of the technology, for example, marginal rates of substitution between inputs, from estimates of the parameters of the parametrized distribution.

We may develop this analogy further. Since the number of potentially distinguishable states of nature is very large, empirical modelling of state-contingent production requires the imposition of structural restrictions. The hypotheses of constant absolute riskiness and constant relative riskiness, discussed in this paper, are examples of such restrictions. Another set of possible restrictions may be generated by consideration of the econometric literature on lag structures where, as in the present case, it is necessary to represent a potentially infinite-dimensional set of possible structures with a finite set of parameters. Or alternatively, restrictions may be generated by considering the theoretical literature on nonlinear aggregation.

Within the parametrized-distribution representation, the choice of M, the number of moments, effectively determines the level of a nonlinear aggregation of the state-contingent outputs. If the selected moments, $\mu^1, ...\mu^M$, fully characterize the distribution of state-contingent output, then information about them is equivalent to having exact information about $\mathbf{z}$. Thus, in this case, (10) must correspond exactly to $c(\mathbf{w}, \mathbf{z})$ in the sense that

$$c(\mathbf{w}, \mathbf{z}) = c^M(\mathbf{w}, \mu^1(\mathbf{z}), ...\mu^M(\mathbf{z})), \tag{11}$$

for all $\mathbf{z}$. Hence, problems involved in either specification must be at least implicitly reflected in the other. On the other hand, if $\mu^1, ...\mu^M$ do not fully characterize the distribution, then requiring (11) to hold globally places restrictions on the underlying technology that can be identified using general results on nonlinear aggregation (Lau).

If the underlying state-contingent cost function $c(\mathbf{w}, \mathbf{z})$ is differentiable in $\mathbf{w}$ and $\mathbf{z}$, the parametrized-distribution representation $c^M(\mathbf{w}, \mu^1, ... \mu^M)$ will be differentiable in $\mathbf{w}$ and $\mu^1 ... \mu^M$. A superficially appealing feature of the parametrized-distribution representation is that, in some regions, $c^M(\mathbf{w}, \mu^1, ... \mu^M)$ may be locally differentiable even though $c(\mathbf{w}, \mathbf{z})$ is not everywhere differentiable.

The Just-Pope technology may be used to illustrate these points. Assuming without loss of generality $E[\varepsilon] = 0$, any output distribution $g(\mathbf{x}) + h(\mathbf{x})\varepsilon$ is characterized by its mean $\mu^1 = g(\mathbf{x})$ and standard deviation $\mu^2 = h(\mathbf{x})\sigma$, where σ is the standard deviation of ε. Hence,

$$\begin{aligned} c^2(\mathbf{w}, \mu^1, \mu^2) &= c^{hg}\left(\mathbf{w}, \mu_1, \frac{\mu_2}{\sigma}\right) \\ &\geq max\{c^g(\mathbf{w}, \mu_1), c^h(\mathbf{w}, \frac{\mu_2}{\sigma})\} \end{aligned}$$

where c^g, c^h and c^{hg} are as defined above.

Depending upon the prevailing input prices, the levels of the moments, and the forms of g and h, this cost structure can be locally differentiable. However, for the case of Just-Pope technology and one variable input, $c^2(\mathbf{w}, \mu^1, \mu^2)$ is not everywhere differentiable in moments. Even with multiple inputs, $c^2(\mathbf{w}, \mu^1, \mu^2)$ is not everywhere differentiable if the technology is input-separable in the sense that there exists a well-defined aggregate input.

Moreover, for a wide range of problems involving non-linear payment structures, including crop insurance and price support schemes, producer preferences will depend upon more than the just the first and second moments. Therefore, to reasonably approximate producer preferences it will generally be necessary to consider higher moments. For simplicity, consider the third moment μ^3, representing skewness. Denote the third moment of ε by ζ. The three-moment cost function for the Just-Pope technology is

$$\begin{aligned} c^3\left(\mathbf{w}, \mu^1, \mu^2, \mu^3\right) &= \min\left\{\mathbf{wx} : h\left(\mathbf{x}\right) \geq \mu^1, g\left(\mathbf{x}\right) = \frac{\mu^2}{\sigma}, g\left(\mathbf{x}\right) = \frac{\mu^3}{\zeta}\right\} \\ &\geq \max\left\{c^h\left(\mathbf{w}, \mu^1\right), c^g\left(\mathbf{w}, \frac{\mu^2}{\sigma}\right), c^g\left(\mathbf{w}, \frac{\mu^3}{\zeta}\right)\right\}. \end{aligned}$$

This cost function is not everywhere differentiable in its moments because the second and third moments cannot be varied independently.

Here an analogy with the literature on flexible functional forms is perhaps relevant. Flexible functional forms are often characterized in terms of second-order Taylor series approximations around a particular point. Second-order approximations are usually considered sound in economics because most comparative-static results of general interest involve only first or second partial derivatives of objective or indirect objective functions.

When $\boldsymbol{\varepsilon}$ has mean zero, the Just-Pope technology can be interpreted as a first-order Taylor series approximation of a general $f(\mathbf{x},\varepsilon_s)$ around the mean of $\boldsymbol{\varepsilon}$. This allows it to accomodate effects associated with the mean and dispersion. It is, therefore, possible to generalize the Just-Pope technology by extending the Taylor-series approximation to a higher order. However, if that order is M, a straightforward extension of the previous arguments shows that the associated cost function still retains the 'max' form above, and is never everywhere differentiable in the $(M+1)$th moment. More generally, with a Just-Pope representation such that $\mu^m = g_m(\mathbf{x})$, $m = 1, 2, ..., M$, with each g_m differentiable, a sufficient condition for $c^M(\mathbf{w},\mu^1, ...\mu^M)$ to be differentiable at an interior solution is that the associated Jacobian matrix $\nabla_{\mathbf{x}}\mathbf{g}$ have rank M.[7] Thus, as noted above, a finer disaggregation of inputs makes the stochastic production function more flexible, but empirically it also requires a close correspondence between the number of inputs and the number of moments considered. In the end, this remains an *ad hoc* substitute for an explicit specification of the state-contingent properties of the underlying technology.

Perhaps the most telling criticism of the moment-based approach to the specification of cost functions is also the simplest. It diverts attention away from the quantities of natural economic interest, which are the state-contingent outputs, and focuses attention instead on characteristics of the distribution, which for most, will have the vaguest of intuitive connections to the underlying economic phenomena. In essence, this argument reduces to which specification of the producer's objective function is the most informative and most in line with the general corpus of economic theory. Expected utility of net returns illustrates the point well. There are two alternatives

$$\max_{\mathbf{z}} \left\{ \sum_{s=1}^{S} \pi_s u \left(p_s z_s - c(\mathbf{w}, \mathbf{z}) \right) \right\},$$

and

$$\max_{\boldsymbol{\mu}} \left\{ \sum_{s=1}^{S} \pi_s u \left(p_s z_s \left(\mu^1, \ldots \mu^M \right) - c^M (\mathbf{w}, \mu^1, \ldots \mu^M) \right) \right\},$$

where $z_s \left(\mu^1, \ldots \mu^M \right)$ corresponds to the realization of the stochastic process when summarized in terms of its moments. Our position is that the former is the most informative to economists. Statisticians, on the other hand, may have a predilection for the latter.

3 Concluding comments

The central claim of Chambers and Quiggin (2000) is that the state-contingent production framework provides the most realistic and tractable representation of problems involving production under uncertainty. Because the state-contingent framework is more general than other approaches, to make this claim good, it remains to show that the concepts found useful in alternative frameworks, such as those based on stochastic production functions, may be transferred or generalized to the state-contingent framework. It is also necessary to consider whether the problems of inflexibility and non-differentiability associated with particular stochastic production functions may be resolved simply without losing the important insights that have been gained from them.

Taking the second issue first, we have discussed generalizations of the additive, multiplicative, and Just-Pope stochastic production functions. These are respectively the classes of constant absolute risky, constant relative risky, and generalized Just-Pope technologies. Each we feel preserves the essential character of its predecessor, and in the two former cases removes some of their more pathological properties in terms of economic decisionmaking.

The main purpose of this paper has been to generalize the concepts of additive and multiplicative uncertainty, which have been the source of useful insights, but also, on occasion, of considerable confusion, in the analysis of problems such as crop insurance and price stabilization. Considering the cost function as a general real-valued function, constant relative riskiness is simply homotheticity, while constant absolute riskiness is BD-translation homotheticity. These insights mean that a wide range of results from the general economic literature on demand, production, and inequality measurement are applicable to problems involving uncertainty. This suggests the possibility of resolving some of the tangle of confusion that surrounds the role of additive and multiplicative uncertainty in issues such as the analysis of price stabilization.

4 Appendix: Proof of Result 1

We demonstrate the result for constant absolute riskiness. A parallel argument establishes the result for constant relative riskiness. By the definition of the cost certainty equivalent output and the supposition that the cost function satisfies (6),

$$c\left(\mathbf{w},e^{c}\left(\mathbf{z}\right)\mathbf{1}^{S}\right)=\hat{c}\left(\mathbf{w},e^{c}\left(\mathbf{z}\right)\right)=Max\left\{\bar{c}\left(\mathbf{w},z_{1};\varepsilon_{1}\right),...,\bar{c}\left(\mathbf{w},z_{S};\varepsilon_{S}\right)\right\}. \tag{12}$$

If the technology satisfies constant absolute riskiness then

$$e^{c}\left(\mathbf{z}+\delta\mathbf{1}^{S}\right)=e^{c}\left(\mathbf{z}\right)+\delta.$$

Set $\mathbf{z}=\mathbf{0}^{S}$ to obtain

$$\hat{c}\left(\mathbf{w},e^{c}\left(\mathbf{0}^{S}\right)\right)=\bar{c}\left(\mathbf{w},0;Min\left\{\varepsilon_{1},...,\varepsilon_{S}\right\}\right).$$

Without loss of generality, set $\varepsilon_{k}=Min\left\{\varepsilon_{1},...,\varepsilon_{S}\right\}$ to obtain

$$\hat{c}\left(\mathbf{w},e^{c}\left(\mathbf{0}^{S}\right)\right)=\bar{c}\left(\mathbf{w},0;\varepsilon_{k}\right).$$

Thus, the cost certainty equivalent for $\mathbf{0}^{S}$ must be a monotonic transformation of ε_{k}. Call it $-\nu\left(\varepsilon_{k}\right)$. Now set $\mathbf{z}=z_{k}\mathbf{1}^{S}$ in (12) to obtain after using constant absolute riskiness

$$\hat{c}\left(\mathbf{w},z_{k}-\nu\left(\varepsilon_{k}\right)\right)=\bar{c}\left(\mathbf{w},z_{k};\varepsilon_{k}\right).$$

This establishes necessity. Now to go the other way. Suppose that the technology assumes this form. Then by the monotonicity of cost in output,

$$\begin{aligned}
c\left(\mathbf{w},\mathbf{z}\right) &= Max\left\{\hat{c}\left(\mathbf{w},z_{1}-\nu\left(\varepsilon_{1}\right)\right),...,\hat{c}\left(\mathbf{w},z_{S}-\nu\left(\varepsilon_{S}\right)\right)\right\} \\
&= \hat{c}\left(\mathbf{w},\,Max\left\{z_{1}-\nu\left(\varepsilon_{1}\right),...,z_{S}-\nu\left(\varepsilon_{S}\right)\right\}\right),
\end{aligned}$$

and the cost certainty equivalent output is

$$e^c(\mathbf{z}) = Max\{z_1 - \nu(\varepsilon_1), ..., z_S - \nu(\varepsilon_S)\} + MiM\{\nu(\varepsilon_1), ..., \nu(\varepsilon_S)\}$$

which exhibits constant absolute riskiness.

Notes

[1]This section draws heavily from Chapters 2 and 4 of Chambers and Quiggin (2000) with the important exception that their discussion of the cost certainty equivalent is in terms of stochastic revenue for a multi-product technology.

[2]This is done to streamline the proof of Result 1.

[3]A function $m : \Re_{+}^{S} \rightarrow \Re$ is BD (Blackorby-Donaldson) translation homothetic if and only if it can be expressed

$$m = m^{1} \circ m^{o}$$

where $m^{1} : \Re \rightarrow \Re$ and is monotonic, and $m^{o} : \Re_{+}^{S} \rightarrow \Re$ and satisfies

$$m^{o}\left(\mathbf{z}+\delta\mathbf{1}_{s}\right) = m^{o}\left(\mathbf{z}\right)+\delta, \quad \delta \in \Re.$$

This definition is due to Chambers and Färe. Blackorby and Donaldson refer to this property as unit translatability. A function is homothetic if it can always be expressed as a monotonic transformation of a positively linearly homogeneous function.

[4]This can also be seen directly by applying the definition of the cost certainty equivalent. Notice that the associated absolute production risk premium is independent of input prices.

[5]We thank an anonymous reviewer for suggesting the addition of this section to us.

[6]Pope and Chavas study the interaction between the structure of preferences and cost functions of this general form. In a state-contingent framework, their largely negative results can be interpreted as reflecting the restrictions on the producer preferences required for them to be expressible as functions of a finite number of moments or, more generally, a finite number of functions of the inputs.

[7]We thank a referee for pointing this out.