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The interaction between the equity premium and the risk-free rate*

Abstract

Explanations based on the existence of undiversifiable background risk have been offered for both the equity premium puzzle and the risk-free puzzle. We show that the theoretical equivalent to the observed equity premium arises from the interaction between atemporal risk premium for equity, the risk-free rate of intertemporal substitution and the impact of risk on the precauctionary motive for saving. Depending on parameter values, the equity premium may either be increased or reduced by the presence of undiversifiable background risk.

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1 Introduction

The behavior of asset prices differs in significant ways from the predictions of models where prices are determined by the decisions of agents with plausible risk preferences interacting through perfect capital markets. The most notable deviations are the 'equity premium puzzle' (Mehra and Prescott, 1985) and the 'risk-free rate puzzle' (Weil, 1992). Explanations of these phenomena based on the existence of undiversifiable background risk have been offered for both the equity premium puzzle (Mankiw, 1986) and the risk-free rate puzzle (Weil, 1992). Grant and Quiggin (1999), in an analysis of options for funding social security liabilities, present a simple two-period model with polar cases capturing the key features of the models proposed by Mankiw and Weil. However, since the main focus of the Grant and Quiggin (1999) is on policy implications, the properties of the models are not analysed in detail.

The purposes of this note is to compare and contrast the Mankiw and Weil models within the Grant-Quiggin two-period framework. Importantly, we show that the equity premium derived in the Mankiw model is not the theoretical equivalent of the observed equity premium, which is the difference between rates of return to equity and debt observed over time. Rather it is an atemporal measure of the risk premium for equity. The theoretical variable equivalent to the observed equity premium arises from the interaction between the atemporal risk premium for equity, the risk-free rate of intertemporal substitution and the impact of risk on the precautionary motive for saving. Depending on parameter values, the equity premium may be either increased or reduced by the presence of undiversifiable background risk.

2 The risk-free rate and the equity premium

Following Weil (1992) we introduce a two-period Lucas (1978) style economy in which there are a continuum of ex ante identical consumers defined over the interval $[0, 1]$. They are assumed to be expected utility maximizes having tastes over two-period random consumption streams represented by the additive utility index,

$$u(c_1) + v(c_2) \tag{1}$$

defined over non-random consumption streams of the single consumption good (c_1, c_2). Both $u(\cdot)$ and $v(\cdot)$ are strictly increasing and strictly concave.

For each i in $[0, 1]$, consumer i receives a known amount of 'labor' income $y_1 > 0$ in the current period and a random second-period labor income $Y_i > 0$.[1] In addition, all consumers are also endowed at birth with the same number (normalized to one) of shares

[1] Throughout, capital letters will denote random variables and lower case letters will denote realizations.

of the market portfolio of tradeable equity. The current dividend $d_1 \geq 0$ is known, but the dividend payable in the second period, D, is random. Consumers may also buy and sell a risk-free bond which pays unconditionally one unit of the consumption good in period 2. All consumers are endowed with zero units of the risk-free bond.

As Weil notes, since agents are risk-averse and identical ex ante (although not ex post) they will not trade with each other in equilibrium. Hence the characterization of the equilibrium simply involves finding asset prices that support the consumers' initial endowment.

Each consumer i faces the constraints:

$$c_{1i} + px_i + qb_i = p + d_1 + y_1 \tag{2}$$

$$C_i = Dx_i + Y_i \tag{3}$$

$$c_{1i}, C_i \geq 0 \tag{4}$$

where p and q denote the prices of equity and bonds, respectively and (x_i, b_i) denotes the portfolio of equity and bonds chosen by consumer i, and C_i denotes his or her random second period consumption.

The first-order conditions for the optimum holdings of equity and bonds are given by, for each i in $[0, 1]$

$$pu'(c_{1i}) = \mathrm{E}\left[Dv'(C_i)\right] \tag{5}$$

$$qu'(c_{1i}) = \mathrm{E}\left[v'(C_i)\right] \tag{6}$$

where E is the mathematical expectations operator.

Given the agents are risk averse and ex ante identical, the equilibrium holdings of equity and bonds for each consumer i must be

$$x_i = 1 \tag{7}$$

$$b_i = 0 \tag{8}$$

Scaling $u(\cdot)$ so that, without loss of generality,

$$u'(d_1 + y_1) = 1$$

the equilibrium prices for equity and bonds may be expressed as

$$p = \mathrm{E}\left[Dv'(D + Y_i)\right] \tag{9}$$

$$q = \mathrm{E}\left[v'(D + Y_i)\right] \tag{10}$$

Letting R_e (respectively, R_b) denote the (gross) return to holding equity (respectively, a bond) it readily follows from (9) and (10) that

$$\mathrm{E}[R_e] = \mathrm{E}\left[\frac{D}{p}\right] = \frac{\mathrm{E}[D]}{\mathrm{E}[Dv'(D+Y_i)]}$$
$$\mathrm{E}[R_b] = \mathrm{E}\left[\frac{1}{q}\right] = \frac{1}{\mathrm{E}[v'(D+Y_i)]}$$

and thus, the equilibrium equity premium π may in turn be expressed as

$$\begin{aligned}\pi &\equiv \mathrm{E}[R_e - R_b] \\ &= \frac{-\mathrm{Cov}[D, v'(D+Y_i)]}{\mathrm{E}[Dv'(D+Y_i)]\,\mathrm{E}[v'(D+Y_i)]} \\ &= \frac{-\mathrm{Cov}[D, v'(D+Y_i)]}{pq}\end{aligned} \tag{11}$$

Notice that as our numeraire is a unit of current (that is, period 1) consumption, the equity premium π measures the excess expected return from forgoing one unit (that is, one dollar's worth) of current consumption and investing that dollar in equity rather than in bonds.

Weil (1992) and Mankiw (1986) express their equity premium in terms of a unit of (uncontingent) consumption in period 2. Formally, their (ratio) premium, Π, is defined as

$$\begin{aligned}\Pi \equiv \frac{\mathrm{E}[R_e]}{\mathrm{E}[R_b]} &= \frac{\mathrm{E}[D]\,\mathrm{E}[v'(D+Y_i)]}{\mathrm{E}[Dv'(D+Y_i)]} = \frac{\mathrm{E}[D]}{p/q} \\ &= \frac{\mathrm{E}[Dv'(D+Y_i)] - \mathrm{Cov}[D, v'(D+Y_i)]}{\mathrm{E}[Dv'(D+Y_i)]} \\ \Pi - 1 &= \frac{-\mathrm{Cov}[D, v'(D+Y_i)]}{p}\end{aligned} \tag{12}$$

In contrast to π, and what has been reported in the empirical literature (such as Mehra and Prescott, 1985), $\Pi - 1$ measures the additional second period consumption that can be expected by exchanging a bond for p/q units of equity. From (11) and (12), it follows that the same condition, namely, second period dividends and the second period marginal utility of consumption are negatively correlated (i.e. $\mathrm{Cov}[D, v'(D+Y_i)] < 0$) is necessary and sufficient for both premium measures to be positive (that is, for both $\pi > 0$ and $\Pi > 1$).

In Weil's setting, D and Y_i are assumed to be statistically independent which means that risk aversion (that is, $v'' > 0$) is sufficient to ensure that $\mathrm{Cov}[D, v'(D+Y_i)] < 0$. In Mankiw's setting a single measure of aggregate (or systemic risk) is concentrated on a small proportion of the population. This can be naturally interpreted in Weil's framework with an individual facing both aggregate systemic risk and a personal or idiosyncratic risk associated with his or her labor income, by the requirement that the distribution of labor income across

the population improves in the sense of second-order 'stochastic' dominance for higher values of the second period dividend. More formally, for all pairs of states, ω and ω', and for all increasing concave functions f

$$\left[\int_0^1 \left[f\left(Y_i\left(\omega\right)\right) - f\left(Y_i\left(\omega'\right)\right)\right] di\right]\left[D\left(\omega\right) - D\left(\omega'\right)\right] \geq 0. \tag{13}$$

One may interpret (13) as saying that for any concave funtion, f, the pair of random variables $\int_0^1 f\left(Y_i\right) di$ and D are *co-monotonic.*[2]

Weil's assumption that Y and D are statistically independent may be viewed as the special case of (13) in which the distribution of labor income across the population is invariant to the realization of the second period dividend, thus leading yielding for all pairs of states, ω and ω', and for all functions f

$$\left[\int_0^1 \left[f\left(Y_i\left(\omega\right)\right) - f\left(Y_i\left(\omega'\right)\right)\right] di\right] = 0.$$

The statistical relationship between Y_i and D, embodied in (13), along with risk aversion and prudence (i.e. $v''\left(\cdot\right) < 0$ and $v'''\left(\cdot\right) > 0$), is sufficient to ensure that both measures of the equity premia are non-negative.

Proposition 1 *If* $v''\left(\cdot\right) < 0$, $v'''\left(\cdot\right) > 0$ *and* (13) *holds then* $\mathrm{Cov}\left[D, v'\left(D + Y_i\right)\right] \leq 0$.

Proof. ¿From (13) and the fact that $-v'$ is an increasing concave function, it follows that $\int_0^1 -v'\left(D + Y_i\right) di$ and D are co-monotonic. Hence,

$$\begin{aligned} & \mathrm{Cov}\left[D, \textstyle\int_0^1 -v'\left(D + Y_i\right) di\right] \geq 0 \\ \Leftrightarrow \quad & \textstyle\int_0^1 \mathrm{Cov}\left[D, -v'\left(D + Y_i\right)\right] di \geq 0 \\ \Leftrightarrow \quad & \mathrm{Cov}\left[D, v'\left(D + Y_i\right)\right] \leq 0 \end{aligned}$$

as required. ■

3 Mis-calibration

Both Mankiw and Weil wished to consider the effect on the estimated equilibrium equity premium from an outside observer (calibrator) failing to take into account the personal or idiosyncratic risk that individuals face in addition to the systemic risk. That is, suppose the calibrator presumes that all risk (including that associated with labor income) is diversifiable so that the equilibrium allocation will be the equal treatment, risk-pooling allocation

[2] Two random variables, X and Y, are co-monotonic, if for any pair of states, ω and ω',

$$\left[X\left(\omega\right) - X\left(\omega'\right)\right]\left[Y\left(\omega\right) - Y\left(\omega'\right)\right] \geq 0.$$

$\left(d_1 + c_1, D + \overline{Y}\right)$, where $\overline{Y} \equiv \int_0^1 Y_i di$ is the random variable second period per capita labor income.

By a similar line of reasoning to that used in the previous section, it is straightforward to show that the calibrator will 'predict' that the equilibrium prices of equity and bonds are

$$\widehat{p} = \mathrm{E}\left[Dv'\left(D + \overline{Y}\right)\right] \tag{14}$$

$$\widehat{q} = \mathrm{E}\left[v'\left(D + \overline{Y}\right)\right] \tag{15}$$

Leading to

$$\mathrm{E}\left[\widehat{R}_e\right] = \frac{\mathrm{E}[D]}{\mathrm{E}\left[Dv'\left(D + \overline{Y}\right)\right]}$$

$$\mathrm{E}\left[\widehat{R}_b\right] = \frac{1}{\mathrm{E}\left[v'\left(D + \overline{Y}\right)\right]}$$

for the predicted expected rates of returns on equity and bonds, respectively. Hence for this mis-calibrated model, the equilibrium equity premium $\widehat{\pi}$ becomes

$$\widehat{\pi} = \frac{-\operatorname{Cov}\left[D, v'\left(D + \overline{Y}\right)\right]}{\widehat{p}\widehat{q}} \tag{16}$$

and the Mankiw-Weil ratio version of the equity premium is now:

$$\widehat{\Pi} = \frac{\mathrm{E}\left[\widehat{R}_e\right]}{\mathrm{E}\left[\widehat{R}_b\right]} = \frac{\mathrm{E}[D]}{\widehat{p}/\widehat{q}}$$

$$\widehat{\Pi} - 1 = \frac{-\operatorname{Cov}\left[D, v'\left(D + \overline{Y}\right)\right]}{\widehat{p}} \tag{17}$$

Assuming the consumers exhibit 'prudence' (that is, $v'''(\cdot) > 0$) and hence engage in precautionary saving, then as is well known, neglecting the idiosyncratic uncertainty each individual faces associated with his or her second period labor income leads the analyst to underpredict the demand for both equity and bonds. Since aggregate and thus per-capita first-period consumption is fixed in equilibrium, this results in underpredicting the prices of both equity and bonds. Thus we obtain the standard result:

Proposition 2 *Suppose $v'''(\cdot) > 0$. A calibrator who is not aware of undiversifiable labor income risk will overpredict the magnitudes of the risk-free and equity premium returns.*[3]

Proof. If $v'''(\cdot) > 0$, it follows by Jensen's inequality that $1/\widehat{R}_b = \mathrm{E}\left[v'\left(D + \overline{Y}\right)\right] = \mathrm{E}\left[v'\left(D + \int_0^1 Y_i di\right)\right] < \mathrm{E}\left[\int_0^1 v'\left(D + Y_i\right) di\right] = \int_0^1 \mathrm{E}\left[v'\left(D + Y_i\right)\right] di = 1/R_b$. The same argument applied to $\widehat{p} = \mathrm{E}\left[Dv'\left(D + \overline{Y}\right)\right]$ implies $\widehat{R}_e > R$ ■

[3] This result unlike Weil's Proposition 2 does not rely on any statistical assumption (such as independence) relationship between Y_i and D.

Since both returns are overpredicted the change in the equity premium (either measured in terms of current or second period consumption) is not immediately apparent. Indeed (11), (12), (16) and (17) all illustrate that the equity premia depend on the covariance between the second period dividend and the marginal utility of second period consumption. As the next proposition shows the difference between $[D, v'(D+Y_i)]$ and $\text{Cov}\left[D, v'\left(D+\overline{Y}\right)\right]$ prevents us from immediately determining the sign of the bias in the equity premium resulting from this misspecification of the model.

Proposition 3 *Suppose* $v''(\cdot) < 0$, $v'''(\cdot) > 0$ *and* (13) *holds. Then* $\text{Cov}\left[D, v'\left(D+\overline{Y}\right)\right] < \text{Cov}\left[D, v'(D+Y_i)\right]$.

Proof. ¿From (13) and the fact that $-v'$ is an increasing concave function, it follows that $\int_0^1 -v'(D+Y_i)\,di$ and D are co-monotonic, as so are $-v'\left(D+\overline{Y}\right)$ and D, and thus both $\text{Cov}\left[D, \int_0^1 -v'(D+Y_i)\,di\right]$ and $\text{Cov}\left[D, -v'\left(D+\overline{Y}\right)\right]$ are non-negative. But it also follows from the strict concavity of $-v'$ that

$$\text{Cov}\left[D, -v'\left(D+\overline{Y}\right)\right] < \text{Cov}\left[D, \int_0^1 -v'(D+Y_i)\,di\right]$$

$$= -\int_0^1 \text{Cov}\left[D, v'(D+Y_i)\right]di = -\,\text{Cov}\left[D, v'(D+Y_i)\right],$$

as required. ■

So we have from (11), (12), (16) and (17)

$$\pi - \widehat{\pi} = \frac{\widehat{p}\widehat{q}\left(-\,\text{Cov}\left[D, v'(D+Y_i)\right]\right) - pq\left(-\,\text{Cov}\left[D, v'\left(D+\overline{Y}\right)\right]\right)}{pq\widehat{p}\widehat{q}} \tag{18}$$

and

$$\Pi - \widehat{\Pi} = \frac{\widehat{p}\left(-\,\text{Cov}\left[D, v'(D+Y_i)\right]\right) - p\left(-\,\text{Cov}\left[D, v'\left(D+\overline{Y}\right)\right]\right)}{p\widehat{p}} \tag{19}$$

The signs of these expressions are not determinate as we have only established $\widehat{p} < p$, $\widehat{q} < q$ and $\text{Cov}\left[D, v'(D+Y_i)\right] < \text{Cov}\left[D, v'\left(D+\overline{Y}\right)\right]$.

The ambiguity of the effect of background risk on the equity premium π arises from the fact that background risk in period 1 income affects both the risk-free rate $\frac{1}{q}$ (through the precautionary saving motive) and the price $\frac{p}{q}$ of equity in terms of bonds (because of prudence). These effects work in opposite directions. However, for sufficiently severe background risk, the precautionary saving effect must dominate. This is because the price of equity, relative to a bond with a unit payout in every state of nature is bounded below by the dividend in the worst state of nature. More formally,

$$\frac{p}{q} = \frac{\text{E}\left[Dv'(C_i)\right]}{\text{E}\left[v'(C_i)\right]}$$

$$\geq \min(D)$$

is bounded. By contrast, given prudence and sufficiently severe background risk, the bond price

$$q = E\left[v'\left(D + Y_i\right)\right]$$

is unbounded. Now consider the equity premium

$$\begin{aligned}\pi \equiv \mathrm{E}\left[R_e - R_b\right] &= \mathrm{E}\left[\frac{R_e}{R_b} - 1\right] R_b \\ &= \mathrm{E}\left[D\frac{q}{p} - 1\right]\frac{1}{q} \leq \left(\frac{\mathrm{E}\left[D\right]}{\min(D)} - 1\right)\frac{1}{q}\end{aligned}$$

which take arbitrarily small values for appropriately severe background risk.Since background risk has no effect on miscalibrated estimates of the equilibrium equity premium, the premium will be overestimated for sufficiently severe background risk.

4 Example

Suppose

$$u\left(c_1\right) + v\left(c_2\right) = \begin{cases} \frac{c_1^{1-\alpha}}{1-\alpha} + \beta\frac{c_2^{1-\alpha}}{1-\alpha} & \text{if } \alpha \neq 1 \\ \ln c_1 + \beta \ln c_2 & \text{if } \alpha = 1 \end{cases}$$

That is,

$$u'\left(c_1\right) = c_1^{-\alpha},\ v'\left(c_2\right) = \beta c_2^{-\alpha}$$

Further take $d_1 = 0.3$, $y_1 = 0.7$ (i.e. $c_1 = 1$).

First in Weil type framework where D and Y are independent, the following table summarises the economy in the four welfare events:

	Event			
	Recession		Boom	
	R & u/e	R & e	B & u/e	B & e
Prob.	1/6	1/3	1/6	1/3
D	0.3	0.3	0.6	0.6
Y_i	0	1.05	0	1.05
$\overline{Y}$	0.7	0.7	0.7	0.7
C_i	0.3	1.35	0.6	1.65
$\widehat{C}_i$	1	1	1.3	1.3
$v'(C_i)$	$\frac{\beta}{(0.3)^{\alpha}}$	$\frac{\beta}{(1.35)^{\alpha}}$	$\frac{\beta}{(0.6)^{\alpha}}$	$\frac{\beta}{(1.65)^{\alpha}}$
$v'\left(\widehat{C}_i\right)$	β	β	$\frac{\beta}{(1.3)^{\alpha}}$	$\frac{\beta}{(1.3)^{\alpha}}$

¿From this table we can calculate the elements of the following table, taking $\beta = 0.5$:

α	1	2	3	4
p	0.26	0.48	1.20	3.50
q	0.64	1.31	3.58	11.00
π	0.14	0.17	0.10	0.04
Π	1.09	1.23	1.34	1.4
$\widehat{p}$	0.19	0.16	0.14	0.13
$\widehat{q}$	0.44	0.40	0.36	0.34
$\widehat{\pi}$	0.10	0.23	0.39	0.57
$\widehat{\Pi}$	1.05	1.09	1.14	1.19
$\pi - \widehat{\pi}$	0.04	-0.06	-0.30	-0.53
$\Pi - \widehat{\Pi}$	0.05	0.13	0.20	0.22

The incorrect assumption of risk pooling always leads to an underestimate of the equilibrium risk premium for equity II. However, the effect on estimates of the equity premium is ambiguous. For small values of α, π is underestimated, but for large values, π is overestimated.

Similarly, for a Mankiw type setup we have

	Event		
	Recession		Boom
	R & u/e	R & e	
Prob.	1/6	1/3	1/2
D	0.3	0.3	0.6
Y_i	0	1.05	1.4
$\overline{Y}$	0.7	0.7	1.4
C_i	0.3	1.35	2
$\widehat{C}_i$	1	1	2
$v'(C_i)$	$\frac{\beta}{(0.3)^{\alpha}}$	$\frac{\beta}{(1.35)^{\alpha}}$	$\frac{\beta}{2^{\alpha}}$
$v'\left(\widehat{C}_i\right)$	β	β	$\frac{\beta}{2^{\alpha}}$

and again taking $\beta = 0.5$:

α	1	2	3	4
p	0.20	0.34	0.96	3.11
q	0.53	1.08	3.19	10.35
π	0.40	0.39	0.15	0.05
Π	1.21	1.42	1.49	1.50
$\widehat{p}$	0.15	0.11	0.09	0.08
$\widehat{q}$	0.38	0.31	0.28	0.27
$\widehat{\pi}$	0.33	0.80	1.24	1.57
$\widehat{\Pi}$	1.13	1.25	1.35	1.42
$\pi - \widehat{\pi}$	0.07	-0.41	-1.09	-1.52
$\Pi - \widehat{\Pi}$	0.09	0.17	0.14	0.08

As before, the incorrect assumption of risk pooling leads to an underestimate of the equilibrium risk premium for equity Π, but the effect on estimates of the equity premium π is ambiguous.

5 Concluding comments

The risk-free rate puzzle and the equity premium puzzle have a natural explanation in terms of missing markets and undiversifiable background risk. By integrating the models of Mankiw and Weil into a simple two-period framework it is possible to examine the interaction between the changes in the intertemporal relative prices of risk-free consumption and the changes in state-contingent relative prices at the point when uncertainty is resolved. Considered separately, these changes in prices may be signed unambiguously, but their combined effects on the equity premium are ambiguous. We thus find that the risk-free rate puzzle is somewhat more robust than the equity-premium puzzle.

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