

CONSEQUENCES OF CAUCHY'S THEOREM

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In this handout, we meet some basic consequences of Cauchy's theorem in group theory. These consequences do not depend on the proof of Cauchy's theorem, but only on the conclusion of the theorem.

1. QUICK APPLICATIONS

Theorem 1.1. *For a finite group G and a prime p , $\#G$ is a power of p if and only if all elements of G have p -power order.*

What is special about prime powers for this theorem is that factors of a power of p are again powers of p .

Proof. If $\#G$ is a power of p , then the order of any element of G is a power of p since the order of any element divides the size of the group.

Conversely, assume all elements of G have p -power order. To show $\#G$ is a power of p , suppose it is not, so $\#G$ is divisible by a prime $q \neq p$. Then, by Cauchy, G has an element of order q , and that's a contradiction of our assumption. \square

Theorem 1.2. *If all non-identity elements of G have the same order, this order is a prime p and $\#G$ is a power of p .*

Proof. If $\#G$ has two prime factors, say p and q , then G contains elements of orders p and q by Cauchy, which contradicts the hypothesis. Thus $\#G$ has only one prime factor, say $\#G = p^m$ for a prime p . The orders of a non-identity element could be $\{p, p^2, \dots, p^m\}$. However, by Cauchy some $g \in G$ has order p , so the hypothesis tells us every non-identity element of G has order p . \square

Example 1.3. Abelian groups fitting the hypothesis of Theorem 1.2 are easy to write down, e.g., $(\mathbf{Z}/(p))^n$ where p is any prime. Every non-zero element has order p . For a non-abelian example, consider the Heisenberg group over $\mathbf{Z}/(p)$ when p is an odd prime. Every non-identity element has order p . Can you find a non-abelian example when $p = 2$?

Corollary 1.4. *The size of any finite field is a prime power.*

Proof. Let F be a finite field. Its size is at least 2 (since $1 \neq 0$ in F). We are going to think about F as an additive group.

Any two non-zero elements a and b in F have the same additive order, since the function $f(x) = (b/a)x$ is additive and invertible and sends a to $f(a) = b$. Thus $\#F$ is a prime power by Theorem 1.2. \square

Theorem 1.5. *Let G be a finite group and $k \in \mathbf{Z}$. The function $f(x) = x^k$ on G is a bijection if and only if $(k, \#G) = 1$.*

When G is non-abelian, the k th power map is usually not a homomorphism. Nevertheless, one can ask if it is a bijection or not.

Proof. Set $n = \#G$. First assume $(k, n) = 1$. Then we can solve $k\ell \equiv 1 \pmod n$. Writing $k\ell = 1 + nm$, every $x \in G$ satisfies $x^{k\ell} = xx^{nm} = x$. This shows the function $x \mapsto x^k$ on G has inverse function $x \mapsto x^\ell$, since $(x^k)^\ell = x$ and $(x^\ell)^k = x$ for $x \in G$. In particular, since the k th power map on G admits an inverse function it is a bijection. Notice we did not use Cauchy's theorem for this direction.

Now assume $(k, n) > 1$. We will show the k -th power map on G is not a bijection. Since k and n have a non-trivial common factor, they have a common prime factor, say p . Since $p|n$, Cauchy's theorem says G has an element of order p , say g . Then, since $p|k$, we have $g^k = (g^p)^{k/p} = e^{k/p} = e$. (Why does it matter that k/p is an integer?) Thus the k th power map is not a bijection, since it is not one-to-one: g and e both have k th power e . \square

Example 1.6. Check that cubing on D_7 is a bijection while squaring is not, and raising to the 5-th power is a bijection on A_4 .

2. DECOMPOSING ABELIAN GROUPS

As a more involved application of Cauchy's theorem, we describe how to decompose any finite abelian group into subgroups of prime power size.

Theorem 2.1. *Let G be finite abelian with $\#G = ab$, where $(a, b) = 1$. Then G is isomorphic to a direct product $A \times B$, where $\#A = a$ and $\#B = b$.*

Proof. If either a or b is 1, the result is clear: take A or B to be the trivial group.

Since $\#G = ab$, every $g \in G$ satisfies $g^{ab} = e$. For any $m \geq 1$ the elements g satisfying $g^m = e$ form a subgroup. Set

$$A = \{g \in G : g^a = e\}, \quad B = \{g \in G : g^b = e\}.$$

These are both subgroups of G . (Note A is *not* the set of a -th powers in G . It is the elements whose a -th power is the identity.)

We now show $G = AB$, *i.e.*, every element of G is the product of an element of A and an element of B . By Bezout, the relative primality of a and b lets us write

$$1 = ar + bs$$

for some r and s in \mathbf{Z} . Therefore any $g \in G$ can be written as

$$g = g^{ar+bs} = (g^r)^a (g^s)^b.$$

Notice $(g^r)^a \in B$ since $((g^r)^a)^b = (g^r)^{ab} = e$, and similarly $(g^s)^b \in A$. Therefore, re-ordering the terms as

$$g = (g^s)^b (g^r)^a$$

exhibits any $g \in G$ as the product of something in A and something in B , so $G = AB$.

Now we show G is isomorphic to the direct product $A \times B$. Let $f: A \times B \rightarrow G$ by multiplication: $f(x, y) = xy$. This is a homomorphism, since

$$f((x, y)(x', y')) = f(xx', yy') = xx'yy' = (xy)(x'y') = f(x, y)f(x', y').$$

Note we used commutativity in the third equation. The previous paragraph showed $G = AB$, so f is onto. To see f is one-to-one, we check the kernel is trivial. Suppose $f(x, y) = e$. Then $xy = e$, so $x = y^{-1}$. This equation shows x and y lie in $A \cap B$. The intersection $A \cap B$ is trivial, since every element of A has order dividing a , every element of B has order dividing b , and $(a, b) = 1$. Thus x and y are trivial, so $(x, y) = (e, e)$ is the identity element of $A \times B$. That proves f has trivial kernel.

Now that we know f is an isomorphism of $A \times B$ with G , we can show $\#A = a$ and $\#B = b$. Because f is a bijection, $\#(A \times B) = \#G$, so

$$(2.1) \quad \#A\#B = ab.$$

We now show $(\#A, b) = 1$. For this we bring in (at last!) Cauchy's theorem. If $(\#A, b) \neq 1$ then some prime p dividing b will divide $\#A$. Then Cauchy tells us A has an element of order p . But every element of A has order dividing a , and p does not divide a (since $p|b$ and $(a, b) = 1$). Thus $(\#A, b) = 1$. Similarly, $(\#B, a) = 1$. From (2.1), $\#A|ab$, so in fact $\#A|a$. Arguing similarly, $\#B|b$. Therefore, by (2.1), $\#A = a$ and $\#B = b$. \square

Corollary 2.2. *Any finite abelian group is isomorphic to a direct product of finite abelian groups with prime-power size.*

Proof. Let G be a finite abelian group, with size $n = p_1^{e_1} \cdots p_r^{e_r}$. We can suppose $n > 1$ and each e_i is non-zero. Taking $a = p_1^{e_1}$ and $b = n/a$, we have $(a, b) = 1$, so Theorem 2.1 tells us G is isomorphic to a direct product $A \times B$, where $\#A = a = p_1^{e_1}$ and $\#B = n/a$. By induction on the size of the finite abelian group, B is isomorphic to a direct product of groups of size $p_2^{e_2}, \dots, p_r^{e_r}$. \square

3. GROUPS OF SIZE pq

The rest of this handout provides a deeper application of Cauchy's theorem. We will classify, up to isomorphism, all groups having size pq , where p and q are different primes.

Theorem 3.1. *Let p and q be distinct primes, with $p < q$. If $q \not\equiv 1 \pmod p$, then all groups of size pq are cyclic. In particular, all groups of size pq are isomorphic. If $q \equiv 1 \pmod p$, then up to isomorphism there are two groups of size pq .*

Example 3.2. Any group of size 15 is cyclic.

Example 3.3. When $p = 2$ and $q = 3$, we already know two non-isomorphic groups of size 6: $\mathbf{Z}/(6)$ and S_3 . The first is abelian and the second is not. Theorem 3.1 says any group of size 6 is isomorphic to one of these. Here is a table listing four examples each of abelian and non-abelian groups of order 6. The groups in each column are isomorphic to each other.

Abelian	Non-abelian
$\mathbf{Z}/(6)$	S_3
$\mathbf{Z}/(2) \times \mathbf{Z}/(3)$	D_3
$(\mathbf{Z}/(9))^\times$	$\text{Aff}(\mathbf{Z}/(3))$
μ_6	$\text{GL}_2(\mathbf{Z}/(2))$

Our proof of Theorem 3.1 will use the following lemma.

Lemma 3.4. *Let G be a group of size pq , where p and q are prime with $p < q$. There is only one subgroup of G with size q .*

Notice the hypotheses are not symmetric in p and q ; the lemma is about subgroups with the larger prime order.

Proof. By Cauchy, there is a subgroup of G with size q . This subgroup has prime size and therefore is cyclic. Write it as $\langle g \rangle$.

In order to show $\langle g \rangle$ is the only subgroup with size q , assume otherwise. Then there is some $h \in G$ having order q such that $h \notin \langle g \rangle$. The only power of h which lies in $\langle g \rangle$ is the identity: if $h^i = g^j$ and $i \not\equiv 0 \pmod q$, then raising both sides to the inverse of i modulo q

shows h itself is a power of g , which is not the case. Thus if $h^i \in \langle g \rangle$ then $i \equiv 0 \pmod q$, so $h^i = e$.

Consider the left cosets of $\langle g \rangle$ represented by powers of h :

$$(3.1) \quad \langle g \rangle, h\langle g \rangle, h^2\langle g \rangle, \dots, h^{q-1}\langle g \rangle.$$

There are q cosets in this list. The total number of different left $\langle g \rangle$ -cosets is $[G : \langle g \rangle] = (pq)/q = p$. Since $p < q$, two of the cosets in (3.1) must be equal, say $h^i\langle g \rangle = h^{i'}\langle g \rangle$ with $0 \leq i < i' \leq q-1$. Then $h^{i'-i} \in \langle g \rangle$, so we have a power of h other than the identity equal to a power of g . This contradicts the previous paragraph, so h can't exist. That is, every element of G with order q lies in $\langle g \rangle$, so $\langle g \rangle$ is the only subgroup with order q . \square

Example 3.5. Consider the group D_5 , of size 10. Here $q = 5$. There is one subgroup of size 5: the rotations.

Now we are ready to prove part of Theorem 3.1.

Theorem 3.6. *Let p, q be primes where $p < q$. Any abelian group of size pq is cyclic. If $q \not\equiv 1 \pmod p$, any group of size pq is abelian, and thus is cyclic.*

Examples satisfying the condition $q \not\equiv 1 \pmod p$ are $15 = 3 \cdot 5$, $35 = 5 \cdot 7$, 33 , 65 , 77 , and 95 . Any group with these sizes is cyclic.

Proof. Let $\#G = pq$. By Cauchy's theorem, G has an element a of order p and an element b of order q . If G is abelian then ab has order pq , so G is cyclic.

The rest of the proof is devoted to showing that, when $q \not\equiv 1 \pmod p$, a and b must commute even if we don't assume in advance that G is abelian, so again we get that ab has order pq and G is cyclic.

Write the desired condition $ab = ba$ as: $aba^{-1} = b$. This is what we will show. Since aba^{-1} is a conjugate of b , it has order q . Thus, by Lemma 3.4, aba^{-1} is a power of b , say

$$aba^{-1} = b^k$$

for some integer k . By induction, $a^m b a^{-m} = b^{k^m}$ for all $m \geq 1$. Taking $m = p$ gives

$$b = b^{k^p}.$$

Since b has order q , this equality implies $k^p \equiv 1 \pmod q$, so $k \pmod q$ has order either 1 or p in $(\mathbf{Z}/(q))^\times$, a group of size $q-1$. If the order is p , then $p|(q-1)$, so $q \equiv 1 \pmod p$. But $q \not\equiv 1 \pmod p$ by hypothesis (aha). Thus the order of k in $(\mathbf{Z}/(q))^\times$ is 1, so $k \equiv 1 \pmod q$ and

$$aba^{-1} = b^k = b^1 = b,$$

which means $ab = ba$. \square

So far we have proved Theorem 3.1 for $q \not\equiv 1 \pmod p$ and partially if $q \equiv 1 \pmod p$. It remains to look at nonabelian groups of size pq for primes $p < q$ if $q \equiv 1 \pmod p$.

There is always a non-abelian group of size pq . Here is one, constructed as a subgroup of the affine group $\text{Aff}(\mathbf{Z}/(q))$:

$$(3.2) \quad H = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in \text{Aff}(\mathbf{Z}/(q)) : x^p \equiv 1 \pmod q \right\}.$$

The reader can check this is a subgroup of $\text{Aff}(\mathbf{Z}/(q))$. To count its size, note there are q choices for y , since we imposed no constraints on y . How many choices are there for x ? Since $x \in (\mathbf{Z}/(q))^\times$ and $p|(q-1)$, Cauchy's theorem tells us there is an element of order

p in $(\mathbf{Z}/(q))^\times$, and thus its powers give at least p solutions to $x^p = 1$ in $\mathbf{Z}/(q)$. There can be no more than p solutions (see Theorem A.1), so there are exactly p solutions. Thus, the solutions to $x^p = 1$ in $(\mathbf{Z}/(q))^\times$ are a cyclic group of size p . Therefore $\#H = pq$.

The group H is non-abelian, since the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ don't commute, where x is any element of order p in $(\mathbf{Z}/(q))^\times$.

Now we are ready to complete the proof of Theorem 3.1 with an analysis of the non-abelian case when $q \equiv 1 \pmod p$.

Theorem 3.7. *Let p and q be primes with $p < q$ and $q \equiv 1 \pmod p$. Up to isomorphism, the group H in (3.2) is the only non-abelian group of size pq .*

Proof. Let G be a non-abelian group with size pq . By Cauchy, there is $a \in G$ with order p and $b \in G$ with order q . If a and b commute, then ab has order pq , so G is cyclic. As we are assuming G is non-abelian, a and b do not commute. Since $\langle a, b \rangle$ has subgroups of size p and q , $\langle a, b \rangle$ is divisible by pq , so $\langle a, b \rangle = G$.

By Lemma 3.4, all elements of order q in G are powers of b , so $aba^{-1} = b^t$ for some t . We have $t \not\equiv 1 \pmod q$, as otherwise $aba^{-1} = b$, which would imply $ab = ba$, but a and b do not commute.

For any $k \geq 0$, conjugating b by a a total of k times gives

$$(3.3) \quad a^k b a^{-k} = b^{t^k}.$$

Taking $k = p$, $b = b^{t^p}$, so $t^p \equiv 1 \pmod q$. Therefore t has order p in $(\mathbf{Z}/(q))^\times$. Moreover, raising both sides of (3.3) to an arbitrary integer power ℓ yields

$$a^k b^\ell a^{-k} = b^{\ell t^k},$$

so

$$(3.4) \quad a^k b^\ell = b^{\ell t^k} a^k.$$

This means any (non-negative) power of a to the left of a power of b can be placed on the right side at the cost of changing the exponent of b . Thus,

$$(3.5) \quad G = \langle a, b \rangle = \{b^n a^m : m, n \geq 0\}.$$

Now we look at the group H in (3.2). The upper left entry of a matrix in this group is an element of $(\mathbf{Z}/(q))^\times$ with order dividing p . Because the solutions to $x^p = 1$ in $(\mathbf{Z}/(q))^\times$ are a subgroup of size p and t has order p in $(\mathbf{Z}/(q))^\times$, these solutions are exactly the powers of t . Therefore a typical matrix in H has the following form, where $m, n \geq 0$:

$$\begin{aligned} \begin{pmatrix} t^m & n \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^m & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}^m. \end{aligned}$$

This suggests, since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has order q (like b) and $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ has order p (like a), the following function $f: H \rightarrow G$:

$$f \begin{pmatrix} t^m & n \\ 0 & 1 \end{pmatrix} = b^n a^m.$$

This function f is well-defined, since m and n in $\begin{pmatrix} t^m & n \\ 0 & 1 \end{pmatrix}$ only matter modulo p and q , respectively.

To check f is an isomorphism, we must show it is a homomorphism ($f(AB) = f(A)f(B)$) for any A and B in H), one-to-one, and onto.

For the homomorphism property, pick $A = \begin{pmatrix} t^m & n \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} t^{m'} & n' \\ 0 & 1 \end{pmatrix}$ in H . Then

$$\begin{aligned} f(AB) &= f\left(\begin{pmatrix} t^m & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{m'} & n' \\ 0 & 1 \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} t^{m+m'} & t^m n' + n \\ 0 & 1 \end{pmatrix}\right) \\ &= b^{t^m n' + n} a^{m+m'} \\ &= b^n b^{t^m n'} a^m a^{m'} \\ &= b^n a^m b^{n'} a^{m'} \quad \text{by (3.4)} \\ &= f(A)f(B). \end{aligned}$$

The reason f is onto is (3.5), which shows every element of G is a value of f .

To see that f is one-to-one, we check the kernel of f is trivial. Suppose $f\left(\begin{pmatrix} t^m & n \\ 0 & 1 \end{pmatrix}\right)$ is trivial. Then $b^n a^m = 1$, so

$$a^m = b^{-n} \in \langle a \rangle \cap \langle b \rangle.$$

This intersection is trivial since $\langle a \rangle$ and $\langle b \rangle$ have relatively prime sizes, so $a^m = 1$ and $b^n = 1$. Thus $p|m$ and $q|n$, so $\begin{pmatrix} t^m & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which shows f has trivial kernel and therefore is one-to-one. \square

Corollary 3.8. *For any odd prime q , any group of order $2q$ is either cyclic or dihedral: it is isomorphic to $\mathbf{Z}/(2q)$ or to D_q .*

Proof. Taking $p = 2$ in Theorem 3.7, the subgroup H of $\text{Aff}(\mathbf{Z}/(q))$ is one of the matrix models for D_q . \square

Remark 3.9. Recalling the convention that $D_2 = \mathbf{Z}/(2) \times \mathbf{Z}/(2)$, Corollary 3.8 is true at $q = 2$ since any group of order 4 is isomorphic to $\mathbf{Z}/(4)$ or $\mathbf{Z}/(2) \times \mathbf{Z}/(2)$. So Corollary 3.8 is valid for all primes q .

APPENDIX A. ROOT COUNTING

We prove here a result on polynomials that is used to count the size of the group H in (3.2).

Theorem A.1. *Let $f(x)$ be a non-constant polynomial with coefficients in $\mathbf{Z}/(p)$, of degree d . Then $f(x)$ has at most d roots in $\mathbf{Z}/(p)$.*

To prove Theorem A.1, we use a preliminary lemma connecting roots and linear factors.

Lemma A.2. *Let $f(x)$ be a non-constant polynomial with coefficients in $\mathbf{Z}/(p)$. For a in $\mathbf{Z}/(p)$, $f(a) = 0$ if and only if $x - a$ is a factor of $f(x)$.*

Proof. If $x - a$ is a factor of $f(x)$, then $f(x) = (x - a)h(x)$ and substituting a for x shows $f(a) = 0$.

Conversely, suppose $f(a) = 0$. Write the polynomial as

$$(A.1) \quad f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0,$$

where $c_j \in \mathbf{Z}/(p)$ and $c_n \neq 0$. Then

$$(A.2) \quad 0 = c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0.$$

Subtracting (A.2) from (A.1), the terms c_0 cancel and we get

$$(A.3) \quad f(x) = c_n(x^n - a^n) + c_{n-1}(x^{n-1} - a^{n-1}) + \cdots + c_1(x - a).$$

Since

$$x^j - a^j = (x - a)(x^{j-1} + ax^{j-2} + \cdots + a^i x^{j-1-i} + \cdots + a^{j-2}x + a^{j-1}),$$

each term on the right side of (A.3) has a factor of $x - a$. Factor this out of each term in (A.3) and we obtain $f(x) = (x - a)g(x)$ where $g(x)$ is another polynomial with coefficients in $\mathbf{Z}/(p)$. \square

Now we prove Theorem A.1.

Proof. We induct on the degree d of $f(x)$. Note $d \geq 1$.

A polynomial of degree 1 has the form $f(x) = ax + b$, where a and b are in $\mathbf{Z}/(p)$ and $a \neq 0$. This has exactly one root in $\mathbf{Z}/(p)$, namely $-b/a$, and thus *at most* one root in $\mathbf{Z}/(p)$. That settles the theorem for $d = 1$.

Now assume the theorem is true for all polynomials with coefficients in $\mathbf{Z}/(p)$ of degree d . We verify the theorem for all polynomials with coefficients in $\mathbf{Z}/(p)$ of degree $d + 1$.

A polynomial of degree $d + 1$ is

$$(A.4) \quad f(x) = c_{d+1}x^{d+1} + c_d x^d + \cdots + c_1 x + c_0,$$

where $c_j \in \mathbf{Z}/(p)$ and $c_{d+1} \neq 0$. If $f(x)$ has no roots in $\mathbf{Z}/(p)$, then we're done, since $0 \leq d + 1$. If $f(x)$ has a root in $\mathbf{Z}/(p)$, say r , then Lemma A.2 tells us $f(x) = (x - r)g(x)$, where $g(x)$ is a polynomial with coefficients in $\mathbf{Z}/(p)$ of degree d . We can therefore apply the inductive hypothesis to $g(x)$ and conclude that $g(x)$ has at most d roots in $\mathbf{Z}/(p)$. Since $f(a) = (a - r)g(a)$ for any $a \in \mathbf{Z}/(p)$ and a product of numbers in $\mathbf{Z}/(p)$ is 0 only when one of the factors is 0 (this would be *false* if our modulus was composite rather than prime!), we see that any root of $f(x)$ in $\mathbf{Z}/(p)$ is either r or is a root of $g(x)$. Thus, $f(x)$ has at most $d + 1$ roots in $\mathbf{Z}/(p)$. As $f(x)$ was an arbitrary polynomial of degree $d + 1$ with coefficients in $\mathbf{Z}/(p)$, we are done with the inductive step. \square